Community Detection Using Time-Dependent Personalized PageRank Supplementary Material

6. Appendix: Pseudocode

Input: Graph G, sparse seed vector $\mathbf{s} \in \mathbb{R}^{|V|}$, α, γ and ϵ.

Choose N, as explained in subsection 3.3.
Compute $\boldsymbol{\Xi}_1^+$ and $\mathbf{u}$ as defined in subsection 3.2.3.

$violating \leftarrow empty\_queue$

Initialize non-zero functions (i.e., indexes with non-zero seed value)
for each u s.t. $s_u \neq 0$: $\mathbf{y}_u \leftarrow s_u \mathbf{1}_{N+1}$, $\mathbf{r}_i \leftarrow -\alpha s_u \mathbf{1}_{N+1}$

# Initialize neighbors of seeds that are not seeds
for $u\ s.t.\ s_u \neq 0$ **do**
    for each $v \in G.neighbors(u)\ s.t.\ s_v = 0$: $\mathbf{y}_v \leftarrow \mathbf{0}_{N+1}$, $\mathbf{r}_v \leftarrow \mathbf{0}_{N+1}$
end for

# Update residual based on seeds
for $u\ s.t.\ (\mathbf{s})_u \neq 0$ **do**
    for each $v \in G.neighbors(u)$: $\mathbf{r}_v \leftarrow \mathbf{r}_v + \alpha \mathbf{y}_u / G.degree(u)$
end for
for each initialized $\mathbf{u}$ s.t. $\|\mathbf{r}_u\|_\infty \geq \frac{(1-\alpha)\cdot G.degree(u)\cdot\epsilon}{(1-\exp((\alpha-1)\gamma))(1+\frac{2}{\pi}\log N)}$: $violating.push(u)$

# Main loop
while $violating$ is not empty **do**
    $u \leftarrow violating.pop()$
    $\mathbf{d} \leftarrow \boldsymbol{\Xi}_1^+ \mathbf{r}_u$
    $\mathbf{y}_u \leftarrow \mathbf{y}_u + \mathbf{d}$
    $\mathbf{r}_u \leftarrow (\mathbf{u}^\mathrm{T}\mathbf{d})\mathbf{u}$
    for $v \in G.neighbors(u)$ **do**
        if $\mathbf{y}_v$ and $\mathbf{r}_v$ have not been initialized yet: $\mathbf{y}_v \leftarrow \mathbf{0}_{N+1}$, $\mathbf{r}_v \leftarrow \mathbf{0}_{N+1}$
        $\mathbf{r}_v \leftarrow \mathbf{r}_v + \alpha \mathbf{d} / G.degree(u)$
        unless v is already in $violating$: **if** $\|\mathbf{r}_v\|_\infty \geq \frac{(1-\alpha)\cdot G.degree(v)\cdot\epsilon}{(1-\exp((\alpha-1)\gamma))(1+\frac{2}{\pi}\log N)}$: $violating.push(v)$
    end for
end while

return the first coordinate of y_u for u's that have been initialized and for which the value $\neq 0$.

7. Appendix: Proofs

7.1. Proof of Proposition 1

Let $\mathbf{e}(\cdot) \equiv \mathbf{x}(\cdot) - \mathbf{y}(\cdot)$. $\mathbf{e}(\cdot)$ is the solution to the following initial value problem:

$$\mathbf{e}'(t) = -(I - \alpha P)\mathbf{e}(t) + \mathbf{r}(t), \quad \mathbf{e}(0) = 0, \quad t \in [0, \gamma] .$$

It follows that (Botchev et al., 2013)

$$\begin{aligned} \mathbf{e}(t) &= \int_0^t \exp(-(t-s)(\mathbf{I} - \alpha\mathbf{P}))\mathbf{r}(s)ds \\ &= \int_0^t \exp(s-t)\exp((t-s)\alpha\mathbf{P})\mathbf{r}(s)ds . \end{aligned}$$

(The last inequality follows from the fact that $\exp(t\mathbf{A})\exp(t\mathbf{B}) = \exp(t(\mathbf{A}+\mathbf{B})) \iff \mathbf{AB} = \mathbf{BA}$.)

For any ω, we have

$$\begin{aligned} \mathbf{D}^{-1}\exp(\omega\mathbf{P}) &= \mathbf{D}^{-1}\exp(\omega\mathbf{A}\mathbf{D}^{-1}) \\ &= \mathbf{D}^{-1}\sum_{k=0}^{\infty}\frac{\omega^k}{k!}(\mathbf{A}\mathbf{D}^{-1})^k \\ &= \left(\sum_{k=0}^{\infty}\frac{\omega^k}{k!}(\mathbf{D}^{-1}\mathbf{A})^k\right)\mathbf{D}^{-1} \\ &= \exp(\omega\mathbf{P}^{\mathrm{T}})\mathbf{D}^{-1} . \end{aligned}$$

It follows that

$$\begin{aligned} \mathbf{D}^{-1}\mathbf{e}(t) &= \int_0^t \exp(s-t)\mathbf{D}^{-1}\exp((t-s)\alpha\mathbf{P})\mathbf{r}(s)ds \\ &= \int_0^t \exp(s-t)\exp((t-s)\alpha\mathbf{P}^{\mathrm{T}})\mathbf{D}^{-1}\mathbf{r}(s)ds . \end{aligned}$$

Recalling that $\mathbf{P}$ is row-stochastic, we can now bound

$$\begin{aligned} \|\mathbf{D}^{-1}\mathbf{e}(t)\|_\infty &\leq \int_0^t \exp(s-t)\cdot\|\exp((t-s)\alpha\mathbf{P}^{\mathrm{T}})\mathbf{D}^{-1}\mathbf{r}(s)\|_\infty ds \\ &\leq \int_0^t \exp(s-t)\cdot\|\exp((t-s)\alpha\mathbf{P}^{\mathrm{T}})\|_\infty\cdot\|\mathbf{D}^{-1}\mathbf{r}(s)\|_\infty ds \\ &= \int_0^t \exp((1-\alpha)(s-t))\|\mathbf{D}^{-1}\mathbf{r}(s)\|_\infty ds \end{aligned}$$

The last equality is because $\mathbf{P}$ is row-stochastic: for $\omega \geq 0$ we have

$$\begin{aligned} \|\exp(\omega\mathbf{P}^{\mathrm{T}})\|_\infty &= \|\sum_{k=0}^{\infty}\frac{\omega^k}{k!}(\mathbf{P}^{\mathrm{T}})^k\|_\infty \\ &= \sum_{k=0}^{\infty}\frac{\omega^k}{k!}\|(\mathbf{P}^{\mathrm{T}})^k\|_\infty \\ &= \sum_{k=0}^{\infty}\frac{\omega^k}{k!} \\ &= \exp(\omega) . \end{aligned}$$

The second equality is due to the fact that $\mathbf{P}^{\mathrm{T}}$ has only positive values, and the third is because $\mathbf{P}$ is row-stochastic.

Clearly if

$$\frac{1-\exp((\alpha-1)\gamma)}{1-\alpha}\|d_i^{-1}r_i(\cdot)\|_\infty < \epsilon$$

for all i then condition (3) holds for all $t \in [0,\gamma]$. This is equivalent to having

$$\|r_i(\cdot)\|_\infty < \frac{(1-\alpha)d_i\epsilon}{1-\exp((\alpha-1)\gamma)}$$

for all i.

7.2. Proof of Proposition 2

Let $\mathbb{S}_N^{-1}$ be the inverse operator of $\mathbb{S}_N$ on $C([0,\gamma])$, the set of continuous functions on $[0,\gamma]$. That is, $\mathbb{S}_N^{-1}$ maps a vector in $\mathbb{R}^{N+1}$ to the unique interpolating polynomial. In particular, $\mathbb{S}_N^{-1}[\mathbb{S}_N[p(\cdot)]] = p(\cdot)$. Let Π_N be the mapping of a continuous function $[0,\gamma]$ to $\mathbb{P}_N$ by sampling at $t_0,\ldots,t_N$ and interpolating, that is $\Pi_N[f(\cdot)] \equiv \mathbb{S}_N^{-1}[\mathbb{S}_N[f(\cdot)]]$. Let

$$\Lambda_N \equiv \sup_{f(\cdot)\in C([0,\gamma])} \frac{\|\Pi_N[f(\cdot)]\|_\infty}{\|f(\cdot)\|_\infty}\,.$$

Λ_N is the *Lebesgue constant* associated with $t_0,\ldots,t_N$, and it is well known that $\Lambda_N \le 1+\frac{2}{\pi}\log N$. Now let $f(\cdot)$ be a piece-wise linear interpolation of $p(t_0),\ldots,p(t_N)$. Since $\mathbb{S}_N[f(\cdot)] = \mathbb{S}_N[p(\cdot)]$ it follows that $\Pi_N[f(\cdot)] = \Pi_N[p(\cdot)] = p(\cdot)$, so

$$\|p(\cdot)\|_\infty = \|\Pi_N[f(\cdot)]\|_\infty \le \Lambda_N\|f(\cdot)\|_\infty \le (1+\frac{2}{\pi}\log N)\|\mathbb{S}_N[p(\cdot)]\|_\infty\,.$$

7.3. Proof of Lemma 3

We have

$$\|(\mathbf{\Xi}+\mathbf{I}_{N+1})\mathbf{1}_{N+1}\|_\infty \le \|(\mathbf{\Xi}+\mathbf{I}_{N+1})\mathbf{1}_{N+1}\|_2 = \min_{\mathbf{x}}\|\mathbf{\Xi}_1\mathbf{x}-\mathbf{1}_{N+1}\|_2\,,$$

so it suffices to show there exists a vector $\mathbf{y}\in\mathbb{R}^{N+1}$ such that

$$\|(\mathbf{\Xi}+\mathbf{I}_{N+1})\mathbf{y}-\mathbf{1}_{N+1}\|_2\,,$$

under the constraint that $y_{N+1}=0$.

We use the following expansion of $\exp(\cdot)$ on $[0,\gamma]$ in the basis of the Chebyshev polynomials (Abramowitz & Stegun, 1964):

$$\exp(-t) = a_0 + \sum_{i=1}^{\infty} a_i T_i\left(\frac{2t-\gamma}{\gamma}\right)$$

$$a_0 = \exp(-\gamma/2)I_0(-\gamma/2) \quad a_i = 2\exp(-\gamma/2)I_i(-\gamma/2),\ k\ge 1$$

where $T_0(\cdot), T_1(\cdot),\ldots$ are the Chebyshev polynomials of the first kind, and $I_0(\cdot), I_1(\cdot),\ldots$ are the modified Bessel functions of the first kind.

Let

$$p(t) = 1 - a_0 - \sum_{i=1}^{N} a_i T_i\left(\frac{2t-\gamma}{\gamma}\right) + c$$

where c is a constant selected so that $p(0)=0$. We now set $\mathbf{y}=\mathbb{S}_N[p(\cdot)]$. Note that $y_{N+1}=0$, as required.

$p(\cdot)$ is a polynomial of degree N, so $\mathbb{S}_N[p'(\cdot)] = \mathbf{\Xi}\mathbb{S}_N[p(\cdot)]$. Therefore,

$$\begin{aligned}\|(\mathbf{\Xi}+\mathbf{I}_N)\mathbf{y}-\mathbf{1}_N\|_2 &\le \sqrt{N}\|(\mathbf{\Xi}+\mathbf{I}_N)\mathbf{y}-\mathbf{1}_N\|_\infty\\ &\le \sqrt{N}\|p(\cdot)+p'(\cdot)-1\|_\infty\end{aligned}$$

For $t \in [0, \gamma]$

$$\begin{aligned}
|p(t) + p'(t) - 1| &= |p(t) - 1 - \exp(-t) + p'(t) + \exp(-t)| \\
&\leq |p(t) - 1 - \exp(-t)| + |p'(t) + \exp(-t)| \\
&= \left| - \sum_{i=N+1}^{\infty} a_i T_i \left(\frac{2t - \gamma}{\gamma} \right) + c \right| + |p'(t) + \exp(-t)| \\
&\leq \left| \sum_{i=N+1}^{\infty} a_i T_i \left(\frac{2t - \gamma}{\gamma} \right) \right| + |c| + |p'(t) + \exp(-t)| \\
&\leq \sum_{i=N+1}^{\infty} |a_i| + |c| + |p'(t) + \exp(-t)|
\end{aligned}$$

where the last inequality is due to the fact that $\|T_i(\cdot)\|_\infty = 1$ for all i.

Since $\exp(0) = 1$ we have

$$1 = a_0 + \sum_{i=1}^{\infty} a_i T_i(-1)$$

which implies

$$\begin{aligned}
-c &= p(0) - c \\
&= 1 - a_0 - \sum_{i=1}^{N} a_i T_i(0) \\
&= \sum_{i=N+1}^{\infty} a_i T_i(0)
\end{aligned}$$

which leads to the bound

$$|c| \leq \sum_{i=N+1}^{\infty} |a_i| \ .$$

Using the fact that $T_i^{'}(t) = iU_{i-1}(t)$, where $U_0(\cdot), U_1(\cdot), \ldots$ are the Chebyshev polynomials of the second kind, we find that

$$p'(t) = \frac{2}{\gamma} \sum_{i=1}^{N} i a_i U_{i-1} \left(\frac{2t - \gamma}{\gamma} \right) \ .$$

Since $(\exp(-t))' = -\exp(-t)$ we have

$$\exp(-t) = -\frac{2}{\gamma} \sum_{i=1}^{\infty} i a_i U_{i-1} \left(\frac{2t - \gamma}{\gamma} \right) \ .$$

Combining the last two equalities we find

$$\begin{aligned}
|p'(t) + \exp(-t)| &= \left| \frac{2}{\gamma} \sum_{i=N+1}^{\infty} i a_i U_{i-1} \left(\frac{2t - \gamma}{\gamma} \right) \right| \\
&\leq \frac{2}{\gamma} \sum_{i=N+1}^{\infty} i^2 |a_i|
\end{aligned}$$

where the last inequality is due to the fact that $\|U_i(\cdot)\|_\infty = i + 1$ for all i.

We now find that

$$\begin{aligned} |p(t)+p'(t)-1| &\leq \sum_{i=N+1}^{\infty}(2+\frac{2}{\gamma}i^2)\,|a_i| \\ &= 4\sum_{i=N+1}^{\infty}\left|(1+\frac{1}{\gamma}i^2)\exp(-\gamma/2)I_i(-\gamma/2)\right| \\ &= 4\exp(-\gamma/2)\sum_{i=N+1}^{\infty}(1+\frac{1}{\gamma}i^2)I_i(\gamma/2)\,, \end{aligned}$$

where last equality is due to the fact that for every integer i, $|I_i(x)| = I_i(|x|)$.

The following two inequalities are known (Paris, 1984; Laforgia, 1991):

$$\frac{I_\nu(x)}{I_\nu(y)} \leq \left(\frac{x}{y}\right)^\nu, \quad \nu > -\frac{1}{2},\ 0 < x < y$$

$$I_{\nu+1}(x) \leq I_\nu(x), \quad \nu > \frac{1}{2}\,.$$

This now leads to

$$|p(t)+p'(t)-1| \leq 4\exp(-\gamma/2)I_{N+1}(\gamma)\sum_{i=N+1}^{\infty}(1+\frac{1}{\gamma}i^2)2^{-i}\,.$$

Provided $\gamma \geq 1$ and $i \geq 11$ we have $(1+i^2/\gamma)2^{-i} \leq (4/5)^{-i}$, so

$$|p(t)+p'(t)-1| \leq 4\exp(-\gamma/2)I_{N+1}(\gamma)\sum_{i=N+1}^{\infty}(4/5)^{-i} = 20\exp(-\gamma/2)I_{N+1}(\gamma)(4/5)^{N+1}\,.$$