1. Proof of Theorem 1

1.1. Proving Theorem 1 to Any Interval in $\mathcal{I}$

Proof (of Lemma 1) The proof is by induction on $t$. For $t = 1$, we have

$$\hat{W}_1 = \hat{w}_1([1, 1]) = 1 .$$

Next, we assume that the claim holds for any $t' \leq t$ and prove it for $t + 1$. Since $|\{[q, s] \in \mathcal{I} : q = t\}| \leq \lceil \log(t) \rceil + 1$ for all $t \geq 1$, we have

$$\hat{W}_{t+1} = \sum_{I=[q,s] \in \mathcal{I} : q \leq t} \hat{w}_{t+1}(I)$$

$$= \sum_{I=[q,s] \in \mathcal{I} : q \leq t} \hat{w}_{t+1}(I) + \sum_{I=[q,s] \in \mathcal{I} : q \leq t} \hat{w}_{t+1}(I)$$

$$\leq \log(t) + 1 + \sum_{I=[q,s] \in \mathcal{I} : q \leq t} \hat{w}_{t+1}(I) .$$

Next, according to the induction hypothesis, we have

$$\sum_{I=[q,s] \in \mathcal{I} : q \leq t} \hat{w}_{t+1}(I) = \sum_{I=[q,s] \in \mathcal{I} : q \leq t} \hat{w}_t(I)(1 + \eta_I \cdot I(t) \cdot r_t(I))$$

$$= \hat{W}_t + \sum_{I \in \mathcal{I}} \eta_I \cdot I(t) \cdot r_t(I) \cdot \hat{w}_t(I)$$

$$\leq t(\log(t) + 1) + \sum_{I \in \mathcal{I}} w_t(I) \cdot r_t(I) .$$

Hence,

$$\hat{W}_{t+1} \leq t(\log(t) + 1) + \log(t + 1) + 1 + \sum_{I \in \mathcal{I}} w_t(I) \cdot r_t(I)$$

$$\leq (t + 1)(\log(t + 1) + 1) + \sum_{I \in \mathcal{I}} w_t(I) \cdot r_t(I) .$$

We complete the proof by showing that $\sum_{I \in \mathcal{I}} w_t(I) \cdot r_t(I) = 0$. Since $x_t = x_{t,t}$ with probability $p_t(I)$ for every $I \in \mathcal{I}$, we obtain

$$\sum_{I \in \mathcal{I}} w_t(I) \cdot r_t(I) = W_t \sum_{I \in \mathcal{I}} p_t(I)(\ell_t(x_t) - \ell_t(x_t(I)))$$

$$= W_t(\ell_t(x_t) - \ell_t(x_t(I)))$$

$$= 0 .$$

Combining the above inequalities, we conclude the lemma.

Proof (of Lemma 2) Fix some $I = [q, s] \in \mathcal{I}$. We need to show that

$$\sum_{t=q}^s r_t(I) \leq 5 \log(s + 1) \sqrt{|I|} .$$

Since weights are non-negative, using Lemma 1, we obtain

$$\hat{w}_{s+1}(I) \leq \hat{W}_{s+1} \leq (s + 1)(\log(s + 1) + 1) ,$$

Hence,

$$\ln(\hat{w}_{s+1}(I)) \leq \ln(s + 1) + \ln(\log(s + 1) + 1) . \quad (1)$$

Next, we note that

$$\hat{w}_{s+1}(I) = \prod_{t=q}^s (1 + \eta_I \cdot I(t) \cdot r_t(I)) = \prod_{t=q}^s (1 + \eta_I \cdot r_t(I)) .$$

Noting that $\eta_I \in (0, 1/2)$ and using the inequality $\ln(1 + x) \geq x - x^2$ which holds for every $x \geq -1/2$, we obtain

$$\ln(\hat{w}_{s+1}(I)) = \sum_{t=q}^s \ln(1 + \eta_I \cdot r_t(I))$$

$$\geq \sum_{t=q}^s \eta_I \cdot r_t(I) - \sum_{t=q}^s (\eta_I \cdot r_t(I))^2$$

$$\geq \eta_I \sum_{t=q}^s r_t(I) - \eta_I |I| . \quad (2)$$

Combining Equation (2) and Equation (1) and dividing by
\( \eta_I \), we obtain
\[
\sum_{t=q}^{s} r_t(I) \leq \eta_I |I| + \eta_I^{-1} (\ln(s + 1) + \ln(\log(s + 1) + 1)) \\
\leq \eta_I |I| + \eta_I^{-1} (\ln(s + 1) + \log(s + 1)) \\
\leq \eta_I |I| + 2\eta_I^{-1} \log(s + 1) ,
\]
where the second inequality follows from the inequality \( x \geq \ln(1 + x) \). Substituting \( \eta_I := \min \left\{ \frac{1}{2}, \frac{1}{\sqrt{|I|}} \right\} \), we conclude the lemma.

1.2. Extending Theorem to Any Interval

In the next part we complete the proof of Theorem 1 by extending Lemma 2 to every interval.

Before proceeding, we set up an additional notation and also make some simple but useful observations regarding the properties of the set \( I \) (defined in Section 2).

For an interval \( J \subseteq \mathbb{N} \), we define the restriction of \( I \) to \( J \) by \( I|_J \). That is, \( I|_J = \{ I \in I : I \subseteq J \} \). We next list some useful properties of the set \( I \) that follow immediately from its definition (thus, we do not prove these claims).

**Lemma 1.1**

1. The size of every interval \( I \in \mathcal{I} \) is \( 2^j \) for some \( j \in \mathbb{N} \cup \{0\} \).
2. For every \( j \in \mathbb{N} \cup \{0\} \), the left endpoint of the leftmost interval \( I \) whose size is \( 2^j \) is \( 2^j \). Thus, the size of every interval which is located to the left of \( I \) is smaller than \( |I| = 2^j \).
3. Let \( I = [q, s] \in \mathcal{I} \) be an interval and let \( I' = [q', q-1] \) be another interval of size \( 2^j |I| \) for some \( j \leq 0 \). Then, \( I' \in \mathcal{I} \).
4. Let \( I = [q, s] \in \mathcal{I} \) be an interval and let \( I' = [s+1, s'] \) be a consecutive interval of size \( 2^j |I| \) for some \( j \leq 0 \). Then, \( I' \in \mathcal{I} \).
5. Let \( I = [q, s] \in \mathcal{I} \) be an interval of size \( 2^j \) for some \( j \in \mathbb{N} \cup \{0\} \). Then, (exactly) one of the intervals \( [q, q + 2^{j+1} - 1], [s+1, s+2^{j+1}] \) (whose size is \( 2^{j+1} \)) belongs to \( \mathcal{I} \).

The following lemma is a key tool for extending Lemma 2 to any interval.

**Lemma 1.2** Let \( I = [q, s] \subseteq \mathbb{N} \) be an arbitrary interval. Then, the interval \( I \) can be partitioned into two finite sequences of disjoint and consecutive intervals, denoted \( (I_{-k}, \ldots, I_0) \subseteq \mathcal{I}|_I \) and \( (I_1, I_2, \ldots, I_p) \subseteq \mathcal{I}|_I \), such that
\[
(\forall i \geq 1) \quad |I_{i-1}|/|I_{i+1}| \leq 1/2.
\]

The lemma is illustrated in Figure 1.2. We next prove the lemma. Whenever we mention Property 1, \ldots, 5, we refer to Property 1, \ldots, 5 of Lemma 1.1.

**Proof** Let \( b_0 = \max\{ |I'| : I' \in \mathcal{I}|_I \} \) be the maximal size of any interval \( I' \in \mathcal{I} \) that is contained in \( I \). Among all of these intervals, let \( I_0 \) be the leftmost interval, i.e., we define
\[
q_0 := \arg \min \{ q' : |I'| < \sqrt{|I|} \} \\
\]
\[
s_0 = q_0 + b_0 - 1 \\
I_0 = [q_0, s_0] .
\]

Starting from \( q_0 - 1 \), we define a sequence of disjoint and consecutive intervals (in a reversed order), denoted \( (I_{-1}, \ldots, I_{-k}) \), as follows:
\[
[q_{-1}, s_{-1}] := I_{-1} \\
= \arg \max_{I' = [q', s'] \in \mathcal{I}|_I : q' < q_0 - 1} |I'| \\
= \arg \max_{I' = [q', s'] \in \mathcal{I}|_I : q' < q_0 - 1} |I'| \\
\]
\[
[q_{-i}, s_{-i}] := I_{-i} \\
\]
\[
[s_{-i+1}, q_{-i+1}] := I_{-i+1} \\
\]
\[
|I_{-i}| \leq 2b_{-i} \\
\]
\[
|I_{-i}| \leq 2b_{-i} \\
\]

Clearly, this sequence is finite and the left endpoint of the leftmost interval, \( I_{-k} \), is \( q \). Denote the size of \( I_{-1} \) by \( b_{-1} \). We next prove that for every \( i \geq 1 \), \( b_{-i}/b_{-i+1} = 2^i \) for some \( j \leq -1 \). We note that according to Property 1, it suffices to show that \( b_{-i} < b_{-i+1} \) for every \( i \geq 1 \). We use induction. The base case follows from the minimality of \( I_0 \). We next assume that the claim holds for every \( i \in \{1, \ldots, k-1\} \) and prove for \( k \). Assume by contradiction that \( b_{-k} \geq b_{-k+1} \). Consider the interval \( I_{-k+1} \) which is obtained by concatenating a copy of \( I_{-k+1} \) to its left\(^1\). It follows that \( I_{-k+1} \) is an interval of size \( 2b_{-k+1} \) which is contained in \( [q, q_{-k+2} - 1] \) and its right endpoint is \( q_{-k+2} - 1 \). According to the induction hypothesis, \( |I_{-k+1}| = 2b_{-k+1} = 2^j \cdot b_{-k+2} \) for some \( j \leq 0 \). It follows from Property 3 that \( I_{-k+1} \in \mathcal{I}|_I \), contradicting the maximality of \( I_{-k+1} \).

Similarly, starting from \( s_0 + 1 \), we define a sequence of
\[1\text{Formally, } I_{-k+1} := [q_{-k+1} + b_{-k+1}, q_{-k+1} + 1] \cup I_{-k+1} .
\]
disjoint and consecutive intervals, denoted \((I_1, \ldots, I_p)\):

\[
[q_1, s_1] := I_1 \\
:= \arg \max_{i' = [q', s'] \in \mathcal{I}, q' = s_{i+1}} |I'_i| \\
\vdots \\
[q_i, s_i] := I_i \\
:= \arg \max_{i' = [q', s'] \in \mathcal{I}, q' = s_{i+1}} |I'_i| \\
\vdots
\]

Clearly, this sequence is finite and the right endpoint of the rightmost interval, \(I_p\), is \(s\). Denote the size of \(I_i\) by \(b_i\). We next prove that for every \(i \geq 2\), \(b_i/b_{i-1} = 2^j\) for some \(j \leq -1\). According to Property 1, it suffices to prove that \(b_i < b_{i-1}\) for every \(i \geq 2\). For this purpose, we first note that \(b_1 \leq b_0\); this follows immediately from the definition of \(b_0\). Hence, we may assume that \(b_i/b_{i-1} \in \{2^j : j \leq 0\}\) for every \(i \in \{1, \ldots, p-1\}\) and prove that \(b_p < b_{p-1}\). Assume by contradiction that \(b_p \geq b_{p-1}\). Consider the interval \(\hat{I}_{p-1}\) which is obtained by concatenating a copy of \(I_{p-1}\) to its right. It follows that \(\hat{I}_{p-1}\) is an interval of size \(2b_{p-1}\) which is contained in \([s_{p-2} + 1, s]\) and its left endpoint is \(s_{p-2} + 1\). According to the induction hypothesis, \(|\hat{I}_{p-1}| = 2b_{p-1} = 2^j \cdot b_{p-2}\) for some \(j \leq 1\). We need to consider the following two cases:

- Assume first that \(j \leq 0\) (thus, \(b_{p-1}/b_{p-2} \leq 1/2\)). Then, it follows from Property 4 that \(\hat{I}_{p-1} \in \mathcal{I}_j\), contradicting the maximality of \(I_{p-1}\).

- Assume that \(j = 1\) (i.e., \(b_{p-1} = b_{p-2}\)). Then, using Property 5, we obtain a contradiction to the maximality of \(I_{k-2}\).

We are now ready to complete the proof of Theorem 1.

**Proof (of Theorem 1)** Consider an arbitrary interval \(I = [q, s] \subseteq [T]\), and let \(I = \bigcup_{i=-k}^{p} I_i\) be the partition described in Lemma 1.2. Then,

\[
R_{\text{SAOL}}(I) \leq \sum_{i \leq 0} R_{\text{SAOL}}(I_i) \\
+ \sum_{i \geq 1} R_{\text{SAOL}}(I_i). 
\]  

We next bound the first term in the right-hand side of Equation (3). According to Lemma 2, we obtain that

\[
\sum_{i \leq 0} R_{\text{SAOL}}(I_i) \leq C \sum_{i \leq 0} |I_i|^\alpha \\
+ 5 \sum_{i \leq 0} \log(s_i + 1)|I_i|^{1/2} \\
\leq C \sum_{i \leq 0} |I_i|^\alpha \\
+ 5 \log(s + 1) \sum_{i \leq 0} |I_i|^{1/2}.
\]

According to Lemma 1.2,

\[
\sum_{i \leq 0} |I_i|^\alpha \leq \sum_{i = 0}^\infty (2^{-i} |I|)^\alpha \\
= \frac{2^\alpha}{2^\alpha - 1} |I|^\alpha \\
\leq \frac{2}{2^\alpha - 1} |I|^\alpha.
\]

Similarly, we have

\[
\sum_{i \leq 0} |I_i|^{1/2} \leq \frac{\sqrt{2}}{\sqrt{2} - 1} |I|^{1/2} \leq 4 |I|^{1/2}.
\]

Combining the last three inequalities, we obtain that

\[
\sum_{i \leq 0} R_{\text{SAOL}}(I_i) \leq \frac{2}{2^\alpha - 1} C |I|^\alpha + 20 \log(s + 1)|I|^{1/2}.
\]

The second term of the right-hand side of Equation (3) is bounded identically. Hence,

\[
R_{\text{SAOL}}(I) \leq \frac{4}{2^\alpha - 1} C |I|^\alpha + 40 \log(s + 1)|I|^{1/2}.
\]