Supplementary Material for
Approval Voting and Incentives in Crowdsourcing

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A. Proofs

A.1. Proof of Theorem 4.1

We assume that there indeed exists some incentive-compatible payment function f, and prove a contradiction.

Let us first consider the special case of $N = G = 1$ and $B = 2$. Since $N = G = 1$, there is only one question. Let $p_1 > 0.5$ be the probability, according to the belief of the worker, that option 1 is correct; the worker then believes that option 2 is correct with probability $(1 - p_1)$.

When $p_1 = 1$, we need the worker to select option 1 alone. Thus we need

$$f(1) > f(2).$$

When $p_1 \in (0.5, 1)$, we require the worker to select options 1 and 2, as opposed to selecting option 1 alone. For this we need

$$p_1 f(1) + (1 - p_1) f(-1) < f(2)$$

It follows that we need

$$(1 - p_1)(f(1) - f(-1)) > f(1) - f(2). \tag{5}$$

However, the inequality (5) is satisfied only when $f(1) > f(-1)$ and $(1 - p_1) > \frac{f(1)-f(2)}{f(1)-f(-1)}$. Thus for any given payment function f, a worker with belief $(1 - p_1) \in (0, \frac{f(1)-f(2)}{f(1)-f(-1)})$ will not be incentivized to select the support of her belief. This yields a contradiction.

We now move on to the general case of $N \geq G \geq 1$ and $B \geq 2$. Consider a worker who is clueless about questions 2 through N (i.e., her belief is uniform across all options for these questions). Suppose this worker selects all B options for these questions as desired. For the first question, suppose that the worker is sure that options $3, \ldots, B$ are incorrect. We are now left with the first question and the first two options for this question. Letting X denote a random variable representing the evaluation of the worker's response to the first question, the expected payment then is

$$\frac{G}{N}\mathbb{E}[f(X, B, \ldots, B)] + (1 - \frac{G}{N}) f(B, \ldots, B).$$

The expectation in the first term is taken with respect to the randomness in X. Defining

$$\tilde{f}(X) := \frac{G}{N} f(X, B, \ldots, B) + (1 - \frac{G}{N}) f(B, \ldots, B),$$

and applying the same arguments to $\tilde{f}$ as those for f for the case of $N = G = 1$, $B = 2$ above gives the desired contradiction. This thus completes the proof of impossibility.

Remark 1. *We can use the techniques developed in this section to prove a stronger result: Consider any value $T \in [0, 1]$. A natural goal is to design a mechanism such that for any question, the worker is incentivized to select the smallest subset of items whose combined belief is T or larger; if there are multiple such smallest subsets then the worker should be incentivized to select the subset among these that has the largest combined belief.*

The case of $T = 1$ degenerates to Theorem 4.1, the case of $T \in [0, \frac{1}{B}]$ degenerates to the traditional single-selection setting, and the case of $B = 2$, $T \in (\frac{1}{B}, 1)$ degenerates to the skip-based setting of Shah & Zhou (2014). We can show that for the remaining parameter space, i.e., for any value of $B \geq 3$, $T \in (\frac{1}{B}, 1)$, and $N \geq G \geq 1$, there exists no incentive-compatible mechanism.

A.2. Proof of Lemma 6.3

Consider some $\rho_0 \in (\rho, \frac{1}{B})$. Consider a worker such that for every question $i \in \mathcal{I}$, her belief is ρ_0 for the first option and $\frac{1-\rho_0}{y_i-1}$ for each of the last $(y_i - 1)$ options. For every question $i \notin \mathcal{I}$, her belief is uniformly distributed among the first y_i options. Now, if the worker selects precisely the support of her beliefs for every question then her expected payment $\$_1$ is

$$\$_1 = \frac{1}{\binom{N}{G}} \sum_{(j_1,\ldots,j_G)\subseteq[N]} f(y_{j_1},\ldots,y_{j_G}). \tag{6}$$

We will compare the aforementioned action to another action, where for each question $i \in \mathcal{I}$, the worker selects only the last $(y_i - 1)$ options but not the first option; for each question $i \notin \mathcal{I}$, the worker selects the support of her belief. Under this action, the expected payment $\$_2$ is

$$\$_2 = \frac{1}{\binom{N}{G}} \sum_{\substack{(j_1,\ldots,j_G)\\ \subseteq[N]}} \sum_{\substack{(\epsilon_1,\ldots,\epsilon_G)\\ \in\{-1,1\}^G}} \mathbf{1}\{\{j_i \mid \epsilon_i = -1\} \subseteq \mathcal{I}\}(1-\rho_0)^{|\mathcal{I}\cap\{j_i|\epsilon_i=1\}|}\rho_0^{|\mathcal{I}\cap\{j_i|\epsilon_i=-1\}|} f(\epsilon_1 y'_{j_1},\ldots,\epsilon_G y'_{j_G}). \tag{7}$$

In the expression (7), the outer summation represents the expectation over the random choice of the G gold standard questions among the N questions. The inner summation represents the expectation with respect to the correctness or incorrectness of the answers to the G gold standard questions: for any question i, $\epsilon_i = 1$ captures the event where the i^{th} question in the gold standard is answered correctly and $\epsilon_i = -1$ represents the event of this question being answered incorrectly. The term $\mathbf{1}\{\{j_i \mid \epsilon_i = -1\} \subseteq \mathcal{I}\}$ ensures that only the questions in $\mathcal{I}$ can be wrong, since it is only these questions for which the worker has selected a subset of her belief's support.

Since $f(x) \geq 0$ for all x, we can lower bound $\$_2$ as

$$\$_2 \geq \frac{1}{\binom{N}{G}} \sum_{(j_1,\ldots,j_G)\subseteq[N]} (1-\rho_0)^{|\mathcal{I}\cap\{j_1,\ldots,j_G\}|} f(y'_{j_1},\ldots,y'_{j_G}). \tag{8}$$

An incentive compatible mechanism must incentivize the worker to perform the first action (over the second), i.e, must have $\$_1 > \$_2$. Thus from (6) and (8), we get

$$\frac{1}{\binom{N}{G}} \sum_{(j_1,\ldots,j_G)\subseteq[N]} f(y_{j_1},\ldots,y_{j_G}) > \frac{1}{\binom{N}{G}} \sum_{(j_1,\ldots,j_G)\subseteq[N]} (1-\rho_0)^{|\mathcal{I}\cap\{j_1,\ldots,j_G\}|} f(y'_{j_1},\ldots,y'_{j_G}). \tag{9}$$

Note that (9) must hold for all $\rho_0 > \rho$. The left hand side of (9) does not involve ρ_0 whereas the right hand side is continuous in ρ_0. It follows that

$$\frac{1}{\binom{N}{G}} \sum_{(j_1,\ldots,j_G)\subseteq[N]} f(y_{j_1},\ldots,y_{j_G}) \geq \frac{1}{\binom{N}{G}} \sum_{(j_1,\ldots,j_G)\subseteq[N]} (1-\rho)^{|\mathcal{I}\cap\{j_1,\ldots,j_G\}|} f(y'_{j_1},\ldots,y'_{j_G}). \tag{10}$$

This proves the first part of the lemma.

We now move on to the second part of the lemma, concerning equality in (10). Suppose $f(\epsilon_1 y'_{j_1},\ldots,\epsilon_G y'_{j_G})$ is strictly positive for any $(j_1,\ldots,j_G) \subseteq [N]$, $\{(\epsilon_1,\ldots,\epsilon_G) \in \{-1,1\}^G \backslash \{1\}^G \mid \epsilon_i = 1$ whenever $j_i \notin \mathcal{I}\}$. Then (10) will necessarily be a strict inequality. The claimed necessary condition for equality is thus established.

A.3. Proof of Theorem 6.2

Consider any incentive compatible mechanism f such that $f(1,\ldots,1) = \alpha$ and $f(B,\ldots,B) = (1-\rho)^{G(B-1)}\alpha$. We will show that this payment mechanism must be identical to the mechanism described in Algorithm 1.

We consider the set of evaluations x whose elements are non-decreasing, i.e., $x_1 \geq x_2 \geq \cdots \geq x_G$; The proof for any other ordering follows in an identical manner.

First consider any x such that $x_G > 0$.

- Let $\gamma(x)$ denote the number of distinct entries in x:

$$\gamma(x) := 1 + \sum_{i=1}^{G-1} \mathbf{1}\{x_i \neq x_{i+1}\}$$

- Let $\sigma(x)$ denote the size of the last jump in x:

$$\sigma(x) := x_j - x_{j+1} \qquad \text{where } j = \underset{i \in [G-1]}{\arg\max}\, x_i \neq x_{i+1}$$

- Let $\beta(x)$ denote the numeric value of x in a B-ary number system:

$$\beta(x) := \sum_{i=1}^{G} B^{G-i}(x_i - 1).$$

For example, if $B = 5$, $G = 5$ and $x = (5, 5, 4, 1, 1)$ then $\gamma(x) = |\{5, 4, 1\}| = 3$, $\sigma(x) = 4 - 1 = 3$ (where $j = 3$), and $\beta(x) = 4 \cdot 5^4 + 4 \cdot 5^3 + 3 \cdot 5^2 + 0 \cdot 5^1 + 0 \cdot 5^0 = 3075$. The proof involves three nested levels of induction: on γ, on σ and then on β.

We first induct on γ. The base case is the set $\{x|\gamma(x) = 1\}$, i.e., the set of vectors which have the same value for all its components. Consider any $x_0 \in [B-1]$. Applying Lemma 6.3 with $y = (x_0+1, \ldots, x_0+1)$ and $y' = (x_0, \ldots, x_0)$ gives

$$f(x_0 + 1, \ldots, x_0 + 1) \geq (1 - \rho)^G f(x_0, \ldots, x_0).$$

Since this inequality is true for every $x_0 \in [B - 1]$, we have

$$f(B, \ldots, B) \geq (1 - \rho)^{(B-x_0)G} f(x_0, \ldots, x_0) \geq (1 - \rho)^{(B-1)G} f(1, \ldots, 1).$$

Setting $f(B, \ldots, B) = (1 - \rho)^{(B-1)}\alpha$ and $f(1, \ldots, 1) = \alpha$ proves the base case.

Now suppose our hypothesis is true for all $\{x|\gamma(x) \leq \gamma_0 - 1\}$ for some $\gamma_0 \in \{2, \ldots, B\}$. We will now prove that the hypothesis is also true for all $\{x|\gamma(x) \leq \gamma_0\}$. Towards this goal, we will now induct on σ. The set of all $\{x|\gamma(x) = \gamma_0 - 1\}$ can be treated as a base case for our induction, with this base case corresponding to $\sigma = 0$. Due to the induction hypothesis on γ, the base case of $\sigma = 0$ is already proven.

Now suppose that the hypothesis is true for all $\{x|\gamma(x) = \gamma_0,\ \sigma(x) \leq \sigma_0 - 1\}$ for some $\sigma_0 \in [B-1]$. We will prove that the hypothesis remains true for all $\{x|\gamma(x) = \gamma_0,\ \sigma(x) = \sigma_0\}$. To this end, we will induct on β.

Recall that we have restricted our attention to those x which have their elements in a descending order. Observe that the element with the minimum value of β in the set $\{x|\gamma(x) = \gamma_0,\ \sigma(x) = \sigma_0\}$ is $(\gamma_0+\sigma_0-1, \ldots, \sigma_0+1, 1, \ldots, 1)$. We will prove the hypothesis for this element as the base case for our induction on β. Applying Lemma 6.3 with $y = (\gamma_0+\sigma_0-1, \ldots, \sigma_0+2, \sigma_0+1, 1, \ldots, 1)$ and $y' = (\gamma_0+\sigma_0-1, \ldots, \sigma_0+2, \sigma_0, 1, \ldots, 1)$ gives the inequality

$$\begin{aligned}
&c_1 f(\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 2, \sigma_0 + 1, 1, \ldots, 1) + c_1' f(\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 2, 1, 1, \ldots, 1) \\
&\quad + \sum_{s \subsetneq \{\gamma_0+\sigma_0-1, \ldots, \sigma_0+2\}} (c_s f(s, 1, 1, \ldots, 1) + c_s' f(s, \sigma_0 + 1, 1, \ldots, 1)) \\
&\geq c_1(1-\rho) f(\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 2, \sigma_0, 1, \ldots, 1) + c_1' f(\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 2, 1, 1, \ldots, 1) \\
&\quad + \sum_{s \subsetneq \{\gamma_0+\sigma_0-1, \ldots, \sigma_0+2\}} (c_s f(s, 1, 1, \ldots, 1) + c_s'(1-\rho) f(s, \sigma_0, 1, \ldots, 1)), \qquad (11)
\end{aligned}$$

for some positive constants c_1, c_1', c_s, c_s' (which represent the probabilities of the respective set of G questions being chosen as the G gold standard questions). Now, for any $s \subsetneq \{\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 2\}$, observe that $\gamma(s, \sigma_0 + 1, 1, \ldots, 1) \leq \gamma_0 - 1$ and $\gamma(s, \sigma_0, 1, \ldots, 1) \leq \sigma_0 - 1$. Thus from our induction hypothesis, we have

$$f(s, \sigma_0 + 1, 1, \ldots, 1) = (1 - \rho) f(s, \sigma_0, 1, \ldots, 1). \qquad (12)$$

Also, $\gamma(\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 2, \sigma_0, 1, \ldots, 1) = \gamma_0$ and $\sigma(\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 2, \sigma_0, 1, \ldots, 1) = \sigma_0 - 1$. Consequently from our induction hypothesis, we have

$$f(\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 2, \sigma_0, 1, \ldots, 1) = (1 - \rho)^{\gamma_0+\sigma_0-2+\cdots+\sigma_0+1+\sigma_0-1}\alpha. \qquad (13)$$

Substituting (12) and (13) in (11) and canceling out common terms gives

$$f(\gamma_0+\sigma_0 - 1, \ldots, \sigma_0 + 2, \sigma_0 + 1, 1, \ldots, 1) \geq (1 - \rho)^{\gamma_0+\sigma_0-2+\cdots+\sigma_0}\alpha.$$

We will now derive a matching upper bound on $f(\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 2, \sigma_0 + 1, 1, \ldots, 1)$. Applying Lemma 6.3 with $y = (\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 1, 2, \ldots, 2)$ and $y' = (\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 1, 1, \ldots, 1)$ gives

$$\begin{aligned} &c_1 f(\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 1, 2, \ldots, 2) + \sum_{s \subsetneq \{\gamma_0+\sigma_0-1,\ldots,\sigma_0+1\}} c_s f(s, 2, \ldots, 2) \\ &\geq c_1 (1-\rho)^{G-\gamma+1} f(\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 1, 1, \ldots, 1) + \sum_{s \subsetneq \{\gamma_0+\sigma_0-1,\ldots,\sigma_0+1\}} c_s (1-\rho)^{G-|s|} f(s, 1, \ldots, 1), \end{aligned} \tag{14}$$

for some positive constants $c_1,\ c_s$. Now, for any $s \subsetneq \{\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 2\}$, observe that $\gamma(s, 2, \ldots, 2) \leq \gamma_0 - 1$ and $\gamma(s, 1, \ldots, 1) \leq \sigma_0 - 1$. Thus from our induction hypothesis, we have

$$f(s, 2, \ldots, 2) = (1-\rho)^{G-|s|} f(s, 1, \ldots, 1). \tag{15}$$

Also, $\gamma(\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 1, 2, \ldots, 2) \leq \gamma_0$ and $\sigma(\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 1, 2, \ldots, 2) = \sigma_0 - 1$. Consequently from our induction hypothesis,

$$f(\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 1, 2, \ldots, 2) = (1-\rho)^{\gamma_0+\sigma_0-2+\ldots+\sigma_0+G-\gamma+1} \alpha. \tag{16}$$

Substituting these values in (14) and canceling out common terms gives

$$f(\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 2, \sigma_0 + 1, 1, \ldots, 1) \leq (1-\rho)^{\gamma_0+\sigma_0-2+\cdots+\sigma_0} \alpha.$$

We have thus proved that the hypothesis is true for $x = (\gamma_0 + \sigma_0 - 1, \ldots, \sigma_0 + 2, \sigma_0 + 1, 1, \ldots, 1)$, the base case for our induction on β.

Now consider some x^* such that $\gamma(x^*) = \gamma_0$, $\sigma(x^*) = \sigma_0$ and $\beta(x^*) = \beta_0$, for some β_0. Let us denote the components of x^* as $x^* = (x_1^*, \ldots, x_m^*, \underbrace{\sigma_0 + x_G^*, \ldots, \sigma_0 + x_G^*}_{m_1}, x_G^*, \ldots, x_G^*)$ with $x_1^* \geq x_2^* \geq \cdots \geq x_m^* > \sigma_0 + x_G^*$ for some $m \geq 0$, $m_1 \geq 1$, $m + m_1 < G$. Suppose the hypothesis is true for all $\{x | \gamma(x) = \gamma_0,\ \sigma(x) = \sigma_0,\ \beta(x) \leq \beta_0 - 1\}$. Applying Lemma 6.3 with $y = (x_1^*, \ldots, x_m^*, \underbrace{\sigma_0 + x_G^*, \ldots, \sigma_0 + x_G^*}_{m_1}, x_G^*, \ldots, x_G^*)$ and $y' = (x_1^*, \ldots, x_m^*, \underbrace{\sigma_0 + x_G^* - 1, \ldots, \sigma_0 + x_G^* - 1}_{m_1}, x_G^*, \ldots, x_G^*)$ gives the inequality

$$\begin{aligned} &c_1 f(x_1^*, \ldots, x_m^*, \underbrace{\sigma_0 + x_G^*, \ldots, \sigma_0 + x_G^*}_{m_1}, x_G^*, \ldots, x_G^*) \\ &\qquad + \sum_{s \subsetneq \{x_1^*, \ldots, x_m^*, \underbrace{\sigma_0+x_G^*, \ldots, \sigma_0+x_G^*}_{m_1}\}} c_s f(s, x_G^*, \ldots, x_G^*) \\ &\geq c_1 (1-\rho)^{m_1} f(x_1^*, \ldots, x_m^*, \underbrace{\sigma_0 + x_G^* - 1, \ldots, \sigma_0 + x_G^* - 1}_{m_1}, x_G^*, \ldots, x_G^*) \\ &\qquad + \sum_{s \subsetneq \{x_1^*, \ldots, x_m^*, \underbrace{\sigma_0+x_G^*-1, \ldots, \sigma_0+x_G^*-1}_{m_1}\}} c_s (1-\rho)^{\sum_i \mathbf{1}\{s_i = \sigma_0 + x_G^* - 1\}} f(s, x_G^*, \ldots, x_G^*), \end{aligned} \tag{17}$$

for some positive constants $c_1,\ c_s$. Observe that

$$\gamma(x_1^*, \ldots, x_m^*, \underbrace{\sigma_0 + x_G^* - 1, \ldots, \sigma_0 + x_G^* - 1}_{m_1}, x_G^*, \ldots, x_G^*) = \begin{cases} \gamma_0 - 1 & \text{if } \sigma_0 = 1 \\ \gamma_0 & \text{otherwise,} \end{cases}$$

and the induction hypothesis is satisfied in the first case. In the second case,

$$\sigma(x_1^*, \ldots, x_m^*, \underbrace{\sigma_0 + x_G^* - 1, \ldots, \sigma_0 + x_G^* - 1}_{m_1}, x_G^*, \ldots, x_G^*) = \sigma_0 - 1,$$

and hence the induction hypothesis is satisfied in the second case as well. Thus

$$f(x_1^*,\dots,x_m^*,\underbrace{\sigma_0+x_G^*-1,\dots,\sigma_0+x_G^*-1}_{m_1},x_G^*,\dots,x_G^*)$$
$$=(1-\rho)^{\sum_{i=1}^m (x_i^*-1)+m_1(\sigma_0+x_G^*-2)+(G-m_1-m)(x_G^*-1)}\alpha. \tag{18}$$

For any for any $s\subsetneq\{x_1^*,\dots,x_m^*,\underbrace{\sigma_0+x_G^*-1,\dots,\sigma_0+x_G^*-1}_{m_1}\}$, define $\mathfrak{m}_1(s):=\sum_i \mathbf{1}\{s_i=\sigma_0+x_G^*-1\}$. Observe that if $\mathfrak{m}_1(s)>0$ then either $\gamma((s,x_G^*,\dots,x_G^*))\le\gamma_0-1$ or $\sigma((s,x_G^*,\dots,x_G^*))\le\sigma_0-1$; if $\mathfrak{m}_1(s)=0$ then $\gamma((s,x_G^*,\dots,x_G^*))\le\gamma_0-1$. For any $s\subsetneq\{x_1^*,\dots,x_m^*,\underbrace{\sigma_0+x_G^*,\dots,\sigma_0+x_G^*}_{m_1}\}$, define $\tilde{\mathfrak{m}}_1(s):=\sum_i\mathbf{1}\{s_i=\sigma_0+x_G^*\}$. Observe that if $\tilde{\mathfrak{m}}_1(s)>0$ then either $\gamma((s,x_G^*,\dots,x_G^*))\le\gamma_0-1$ or $\beta((s,x_G^*,\dots,x_G^*))\le\beta_0-1$; if $\tilde{\mathfrak{m}}_1(s)=0$ then $\gamma((s,x_G^*,\dots,x_G^*))\le\gamma_0-1$. Consequently from our induction hypothesis we have

$$\sum_{s\subsetneq\{x_1^*,\dots,x_m^*,\underbrace{\sigma_0+x_G^*,\dots,\sigma_0+x_G^*}_{m_1}\}} c_s f(s,x_G^*,\dots,x_G^*)$$
$$=\sum_{s\subsetneq\{x_1^*,\dots,x_m^*,\underbrace{\sigma_0+x_G^*-1,\dots,\sigma_0+x_G^*-1}_{m_1}\}} c_s(1-\rho)^{\sum_i\mathbf{1}\{s_i=\sigma_0+x_G^*-1\}}f(s,x_G^*,\dots,x_G^*). \tag{19}$$

Substituting (18) and (19) in (17) and canceling out common terms gives

$$f(x_1^*,\dots,x_m^*,\underbrace{\sigma_0+x_G^*,\dots,\sigma_0+x_G^*}_{m_1},x_G^*,\dots,x_G^*)$$
$$\ge(1-\rho)^{m_1}f(x_1^*,\dots,x_m^*,\underbrace{\sigma_0+x_G^*-1,\dots,\sigma_0+x_G^*-1}_{m_1},x_G^*,\dots,x_G^*)$$
$$=(1-\rho)^{\sum_{i=1}^m(x_i^*-1)+m_1(\sigma_0+x_G^*-1)+(G-m_1-m)(x_G^*-1)}\alpha.$$

We will now employ Lemma 6.3 again to derive a matching lower bound. Setting $y=(x_1^*,\dots,x_m^*,\underbrace{\sigma_0+x_G^*,\dots,\sigma_0+x_G^*}_{m_1},x_G^*+1,\dots,x_G^*+1)$ and $y'=(x_1^*,\dots,x_m^*,\underbrace{\sigma_0+x_G^*,\dots,\sigma_0+x_G^*}_{m_1},x_G^*,\dots,x_G^*)$ in Lemma 6.3 yields the inequality

$$c_1f(x_1^*,\dots,x_m^*,\underbrace{\sigma_0+x_G^*,\dots,\sigma_0+x_G^*}_{m_1},x_G^*+1,\dots,x_G^*+1)$$
$$+\sum_{s\subsetneq\{x_1^*,\dots,x_m^*,\underbrace{\sigma_0+x_G^*,\dots,\sigma_0+x_G^*}_{m_1}\}}c_sf(s,x_G^*+1,\dots,x_G^*+1)$$
$$\ge c_1(1-\rho)^{m_1}f(x_1^*,\dots,x_m^*,\underbrace{\sigma_0+x_G^*,\dots,\sigma_0+x_G^*}_{m_1},x_G^*,\dots,x_G^*)$$
$$+\sum_{s\subsetneq\{x_1^*,\dots,x_m^*,\underbrace{\sigma_0+x_G^*,\dots,\sigma_0+x_G^*}_{m_1}\}}c_s(1-\rho)^{G-|s|}f(s,x_G^*,\dots,x_G^*), \tag{20}$$

for some positive constants c_1, c_s. Observe that

$$\gamma(x_1^*,\dots,x_m^*,\underbrace{\sigma_0+x_G^*,\dots,\sigma_0+x_G^*}_{m_1},x_G^*+1,\dots,x_G^*+1)=\begin{cases}\gamma_0-1 & \text{if } \sigma_0=1\\ \gamma_0 & \text{otherwise,}\end{cases}$$

and that the induction hypothesis is satisfied in the first case. In the second case,

$$\sigma(x_1^*,\ldots,x_m^*,\underbrace{\sigma_0+x_G^*,\ldots,\sigma_0+x_G^*}_{m_1},x_G^*+1,\ldots,x_G^*+1)=\sigma_0-1,$$

and hence the induction hypothesis is satisfied in the second case as well. Thus

$$\begin{aligned} f(x_1^*,\ldots,x_m^*,&\underbrace{\sigma_0+x_G^*,\ldots,\sigma_0+x_G^*}_{m_1},x_G^*+1,\ldots,x_G^*+1) \\ &=(1-\rho)^{\sum_{i=1}^m(x_i^*-1)+m_1(\sigma_0+x_G^*-1)+(G-m_1-m)(x_G^*-2)}\alpha. \end{aligned} \tag{21}$$

Now consider any $s \subsetneq \{x_1^*,\ldots,x_m^*,\underbrace{\sigma_0+x_G^*,\ldots,\sigma_0+x_G^*}_{m_1}\}$, and recall our notation of $\tilde{\mathfrak{m}}_1(s) := \sum_i \mathbf{1}\{s_i = \sigma_0+x_G^*\}$. If $\sigma_0=1$ or if $\tilde{\mathfrak{m}}_1(s)=0$ then $\gamma((s,x_G^*+1,\ldots,x_G^*+1)) \le \gamma_0-1$; if $\sigma>1$ and $\tilde{\mathfrak{m}}_1(s)>0$ then $\gamma((s,x_G^*+1,\ldots,x_G^*+1)) \le \gamma_0$ and $\sigma(s,x_G^*+1,\ldots,x_G^*+1) \le \sigma_0-1$. If $\tilde{\mathfrak{m}}_1(s)=0$ then $\gamma((s,x_G^*,\ldots,x_G^*)) \le \gamma_0-1$, otherwise $\gamma((s,x_G^*,\ldots,x_G^*)) \le \gamma_0$, $\sigma((s,x_G^*,\ldots,x_G^*))=\sigma_0$ and $\beta((s,x_G^*,\ldots,x_G^*)) \le \beta_0-1$. These terms thus satisfy our induction hypothesis and hence

$$f(s,x_G^*+1,\ldots,x_G^*+1)=(1-\rho)^{G-|s|}f(s,x_G^*,\ldots,x_G^*). \tag{22}$$

Substituting (21) and (22) in (20) gives us our desired matching lower bound

$$f(x_1^*,\ldots,x_m^*,\underbrace{\sigma_0+x_G^*,\ldots,\sigma_0+x_G^*}_{m_1},x_G^*,\ldots,x_G^*) \le (1-\rho)^{\sum_{i=1}^m(x_i^*-1)+m_1(\sigma_0+x_G^*-1)+(G-m_1-m)(x_G^*-1)}\alpha.$$

This completes the proof for $\{x|x_i \ge 0 \,\forall\, i \in [G]\}$.

We will now show that $f(x)=0$ for all $\{x \mid \min_{i\in[G]} x_i < 0\}$. The arguments above for the case $\{x \mid \min_{i\in[G]} x_i > 0\}$ imply that for any incentive-compatible function f, the first part of Lemma A.1 must be satisfied with equality. This allows us to employ the second part of Lemma A.1. For $i \in [G]$, let $y_i = y_i' = x_i$ if $x_i > 0$, and $y_i - 1 = y_i' = |x_i|$ otherwise; set $y_i = y_i' = B$ for all $i \in \{G+1,\ldots,N\}$. Then the second part of Lemma 6.3 necessitates $f(x_1,\ldots,x_G)=0$, thus completing the proof.

A.4. Proof of Theorem 7.1

First consider the case of $N=G=1$. The mechanism of Algorithm 1 reduces to $f(x)=\alpha(1-\rho)^{(x_1-1)}\mathbf{1}\{x_1 \ge 0\}$. Suppose without loss of generality that the worker's beliefs for the B options are $p_1 \ge \cdots \ge p_B$ and suppose $m = \arg\max_z \left(\frac{p_{(z)}}{\sum_{i=1}^z p_{(i)}} > \rho\right)$. A mechanism that is incentive compatible will strictly maximize the worker's expected payment when she selects the options $\{1,\ldots,m\}$.

Suppose a worker decides to select some ℓ of the B options, say options $\{o_1,\ldots,o_\ell\} \subseteq [B]$. Then it is easy to see that her expected payment,

$$\alpha\sum_{i=1}^{\ell} p_{o_i}(1-\rho)^{\ell-1},$$

is maximized when she selects options $\{1,\ldots,\ell\}$, i.e., the ℓ options that are most likely to be correct. It remains to show that among all choices of $\ell \in [B]$, the expected payment is maximized when the worker selects $\ell=m$. Let $\$_\ell$ denote the expected payment when the worker selects ℓ options:

$$\$_\ell = \alpha\sum_{i=1}^{\ell} p_i(1-\rho)^{\ell-1}.$$

Hence for any $\ell \in \{2,\ldots,B\}$, we have

$$\frac{\$_{\ell-1}}{\$_\ell} = \frac{\alpha\sum_{i=1}^{\ell-1} p_i(1-\rho)^{\ell-2}}{\alpha\sum_{i=1}^{\ell} p_i(1-\rho)^{\ell-1}} = \frac{1}{1-\rho}\left(1-\frac{p_\ell}{\sum_{i=1}^{\ell} p_i}\right).$$

We know that $\frac{p_\ell}{\sum_{i=1}^{\ell} p_i} < \rho$ whenever $\ell > m$, and $\frac{p_\ell}{\sum_{i=1}^{\ell} p_i} > \rho$ when $\ell = m$. Furthermore, since p_ℓ decreases with ℓ and $\sum_{i=1}^{\ell} p_i$ increases with ℓ, it must also be that $\frac{p_\ell}{\sum_{i=1}^{\ell} p_i} > \rho$ for all $\ell < m$. Thus we have $\frac{\$_\ell}{\$_{\ell-1}} > 1$ for all $\ell \leq m$ and $\frac{\$_\ell}{\$_{\ell-1}} < 1$ for all $\ell > m$, or in other words,

$$\cdots < \$_{m-2} < \$_{m-1} < \$_m > \$_{m+1} > \$_{m+2} > \cdots .$$

It follows that the worker will be incentivized to choose $\ell = m$.

Let us now consider the case of $N = G \geq 1$. By our assumption of the independence of the beliefs of the worker across the questions, the expected payment equals

$$\prod_{i=1}^{G} \mathbf{E}\left[\alpha(1-\rho)^{(x_i-1)}\mathbf{1}\{x_i \geq 0\}\right].$$

Since the payments are non-negative, if each individual component in the product is maximized then the product is also necessarily maximized. Each individual component simply corresponds to the setting of $N = G = 1$ discussed earlier. Thus calling upon our earlier result, we get that the expected payment for the case $N = G \geq 1$ is maximized when the worker acts as desired for every question.

Let us finally consider the general case of $N \geq G \geq 1$. Recall from (3) that the expected payment for the general case is a cascade of two expectations: the outer expectation is with respect to the uniformly random distribution of the G gold standard questions among the N total questions, while the inner expectation is taken over the worker's beliefs of the different questions conditioned on the choice of the gold standard questions and restricts attention to only these G questions. The arguments above for the case $N = G$ prove that every individual term in the inner expectation is maximized when the worker acts as desired. The outer expectation does not affect this argument. The expected payment is thus maximized when the worker acts as desired.

A.5. Proof of Theorem 7.2

The proof of this theorem employs some of the tools developed in (Shah & Zhou, 2014). We begin with a lemma deriving a condition that must necessarily be satisfied by any incentive-compatible mechanism. Note that we are not making the coarse belief assumption and supposing that workers can have arbitrary beliefs.

Lemma A.1. *Any incentive-compatible mechanism must satisfy*

$$\begin{aligned} &f(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_G) \\ &\qquad = (1-\rho) f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_G) + \rho f(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_G), \end{aligned}$$

for every $i \in [G]$ *and* $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_G) \in \{-(B-1), \ldots, -1, 1, \ldots, B\}^{G-1}$, $x_i \in [B-1]$.

Note that the lemma does *not* use the no-free-lunch condition. The proof of the lemma is provided at the end of this section. Using this lemma, we now complete the proof of the theorem.

Consider any incentive-compatible mechanism f that satisfies the no-free-lunch condition. We first show that the mechanism must necessarily make a zero payment when one more more questions in the gold standard are attempted incorrectly. To this end, observe that since $f \geq 0$ and $\rho \in (0, 1)$, the statement of Lemma A.1 necessitates that for every $i \in [G]$ and $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_G) \in \{-(B-1), \ldots, B\}^{G-1}$, $x_i \in [B-1]$:

$$\begin{aligned} &\text{If } f(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_G) = 0 \\ &\qquad \text{then } f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_G) = f(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_G) = 0. \end{aligned}$$

A repeated application of this argument implies:

$$\text{If } f(x_1, \ldots, x_{i-1}, B, x_{i+1}, \ldots, x_G) = 0 \quad \text{then } f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_G) = 0,$$

for all $x_i \in \{-(B-1), \ldots, -1, 1, \ldots, B-1\}$.

Now consider any evaluation $(x_1, \ldots, x_G)$ which has at least one incorrect answer. Suppose without loss of generality that the first question is the one answered incorrectly, i.e., $x_1 \leq -1$. The no-free-lunch condition then makes

$f(x_1, B, \ldots, B) = 0$. Applying our arguments from above we get that $f(x_1, x_2, \ldots, x_G) = 0$ for every value of $(x_2, \ldots, x_G) \in \{-(B-1), \ldots, -1, 1, \ldots, B\}$.

Substituting this necessary condition in Lemma A.1, we get that for every question $i \in \{1, \ldots, G\}$ and every $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_G) \in [B]^{G-1}$, $x_i \in [B-1]$,

$$f(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_G) = (1-\rho) f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_G).$$

Substituting $f(1, \ldots, 1) = \alpha$, we get the desired answer.

We now return to complete the proof of Lemma A.1.

Proof of Lemma A.1. First consider the case of $G = N$. Consider some $\eta, \gamma \in \{0, \ldots, G-1\}$ with $\eta + \gamma < G$. Suppose $i = \eta + \gamma + 1$, $x_1, \ldots, x_\eta \in [B-1]$, $x_{\eta+1}, \ldots, x_{\eta+\gamma} \in -[B-1]$ and $x_{\eta+\gamma+2}, \ldots, x_N = B$.

For every question $j \in [\eta + \gamma]$, suppose the worker's belief is $\delta_j \in (0, \rho)$ for the last option and $\frac{1-\delta_j}{|x_j|}$ each for the first $|x_j|$ options. One can verify that since $\delta_j < \rho < \frac{1}{B}$ and $|x_j| \leq B-1$, it must be that $\frac{1-\delta_j}{|x_j|} > \delta_j$, and that incentive-compatibility requires incentivizing the worker to select the first $|x_j|$ options. Suppose the worker does so. Now for every question $j' \in \{\eta + \gamma + 2, \ldots, N\}$, suppose the belief of the worker is uniform across all B options. The worker should be incentivized to select all B options in this case; suppose the worker does so. Finally, for question i, suppose the worker's belief is $\delta \in (\frac{\rho}{2}, \frac{3\rho}{2})$ for the last option and $\frac{1-\delta}{|x_i|}$ each for the first $|x_i|$ options. Then the worker must be incentivized to select the first $|x_i|$ options alone if $\delta < \rho$, and select the last option along with the first $|x_i|$ options if $\delta > \rho$.

Define $\{r_j\}_{j \in [\eta+\gamma]}$ as $r_j = \delta_j$ for $j \in [\eta]$, and $r_j = 1 - \delta_j$ for $j \in \{\eta + 1, \eta + \gamma\}$. Let $\boldsymbol{\epsilon} := \{\epsilon_1, \ldots, \epsilon_{\eta+\gamma}\} \in \{-1, 1\}^{\eta+\gamma}$. Incentive-compatibility for question i necessitates

$$\begin{aligned}
&(1-\delta) \sum_{\boldsymbol{\epsilon} \in \{-1,1\}^{\eta+\gamma}} \left(f(\epsilon_1 x_1, \ldots, \epsilon_\eta x_\eta, \epsilon_{\eta+1} x_{\eta+1}, \ldots, \epsilon_{\eta+\gamma} x_{\eta+\gamma}, x_i, B, \ldots, B) \prod_{j \in [\eta+\gamma]} r_j^{\frac{1-\epsilon_j}{2}} (1-r_j)^{\frac{1+\epsilon_j}{2}} \right) \\
&\quad + \delta \sum_{\boldsymbol{\epsilon} \in \{-1,1\}^{\eta+\gamma}} \left(f(\epsilon_1 x_1, \ldots, \epsilon_\eta x_\eta, \epsilon_{\eta+1} x_{\eta+1}, \ldots, \epsilon_{\eta+\gamma} x_{\eta+\gamma}, -x_i, B, \ldots, B) \prod_{j \in [\eta+\gamma]} r_j^{\frac{1-\epsilon_j}{2}} (1-r_j)^{\frac{1+\epsilon_j}{2}} \right) \\
&\underset{\delta<\rho}{\overset{\delta>\rho}{\lesseqgtr}} \sum_{\boldsymbol{\epsilon} \in \{-1,1\}^{\eta+\gamma}} \left(f(\epsilon_1 x_1, \ldots, \epsilon_\eta x_\eta, \epsilon_{\eta+1} x_{\eta+1}, \ldots, \epsilon_{\eta+\gamma} x_{\eta+\gamma}, x_i + 1, B, \ldots, B) \prod_{j \in [\eta+\gamma]} r_j^{\frac{1-\epsilon_j}{2}} (1-r_j)^{\frac{1+\epsilon_j}{2}} \right).
\end{aligned}$$

The left hand side of this expression is the expected payment if the worker chooses the first $|x_i|$ options for question $(\eta + \gamma + 1)$, while the right hand side is the expected payment if she chooses the first $|x_i|$ options as well as the last option. For any real-valued variable q, and for any real-valued constants a, b and c,

$$aq \underset{q>c}{\overset{q<c}{\lesseqgtr}} b \quad \Rightarrow \quad ac = b\,.$$

With $q = 1 - \delta$ in this argument, we get

$$\begin{aligned}
&(1-\rho) \sum_{\boldsymbol{\epsilon} \in \{-1,1\}^{\eta+\gamma}} \left(f(\epsilon_1 x_1, \ldots, \epsilon_\eta x_\eta, \epsilon_{\eta+1} x_{\eta+1}, \ldots, \epsilon_{\eta+\gamma} x_{\eta+\gamma}, x_i, B, \ldots, B) \prod_{j \in [\eta+\gamma]} r_j^{\frac{1-\epsilon_j}{2}} (1-r_j)^{\frac{1+\epsilon_j}{2}} \right) \\
&\quad + \rho \sum_{\boldsymbol{\epsilon} \in \{-1,1\}^{\eta+\gamma}} \left(f(\epsilon_1 x_1, \ldots, \epsilon_\eta x_\eta, \epsilon_{\eta+1} x_{\eta+1}, \ldots, \epsilon_{\eta+\gamma} x_{\eta+\gamma}, -x_i, B, \ldots, B) \prod_{j \in [\eta+\gamma]} r_j^{\frac{1-\epsilon_j}{2}} (1-r_j)^{\frac{1+\epsilon_j}{2}} \right) \\
&- \sum_{\boldsymbol{\epsilon} \in \{-1,1\}^{\eta+\gamma}} \left(f(\epsilon_1 x_1, \ldots, \epsilon_\eta x_\eta, \epsilon_{\eta+1} x_{\eta+1}, \ldots, \epsilon_{\eta+\gamma} x_{\eta+\gamma}, x_i + 1, B, \ldots, B) \prod_{j \in [\eta+\gamma]} r_j^{\frac{1-\epsilon_j}{2}} (1-r_j)^{\frac{1+\epsilon_j}{2}} \right) = 0.
\end{aligned} \tag{23}$$

The left hand side of (23) represents a polynomial in $(\eta+\gamma)$ variables $\{r_j\}_{j=1}^{\eta+\gamma}$ which evaluates to zero for all values of the variables within an $(\eta+\gamma)$-dimensional solid ball. Thus, the coefficients of the monomials in this polynomial must be zero. In particular, the constant term must be zero. The constant term appears when $\epsilon_j = 1 \; \forall \; j$ in the summations in (23). Setting the constant term to zero gives

$$(1-\rho)f(x_1,\ldots,x_{\eta+\gamma},x_{\eta+\gamma+1},B,\ldots,B)+\rho f(x_1,\ldots,x_{\eta+\gamma},-x_{\eta+\gamma+1},B,\ldots,B) \\ - f(x_1,\ldots,x_{\eta+\gamma},x_{\eta+\gamma+1}+1,B,\ldots,B)=0$$

as desired. Since the arguments above hold for any permutation of the N questions, this completes the proof for the case of $G=N$.

Now consider the case $G<N$. Let $g:\{-(B-1),\ldots,-1,1,\cdots,B\}^N \rightarrow \mathbb{R}_+$ represent the expected payment given an evaluation of all the N answers, when the identities of the gold standard questions are unknown. Here, the expectation is with respect to the (uniformly random) choice of the G gold standard questions. If $(x_1,\ldots,x_N) \in \{-(B-1),\ldots,-1,1,\cdots,B\}^N$ are the evaluations of the worker's answers to the N questions then the expected payment is

$$g(x_1,\ldots,x_N)=\frac{1}{\binom{N}{G}}\sum_{(i_1,\ldots,i_G)\subseteq\{1,\ldots,N\}} f(x_{i_1},\ldots,x_{i_G}). \tag{24}$$

Applying the same arguments to g as done to f above, gives

$$(1-\rho)g(x_1,\ldots,x_{\eta+\gamma},x_{\eta+\gamma+1},B,\ldots,B)+\rho g(x_1,\ldots,x_{\eta+\gamma},-x_{\eta+\gamma+1},B,\ldots,B) \\ - g(x_1,\ldots,x_{\eta+\gamma},x_{\eta+\gamma+1}+1,B,\ldots,B)=0. \tag{25}$$

The proof now proceeds via an induction on the quantity $(G-\eta-\gamma-1)$. We begin with the case of $(G-\eta-\gamma-1)=G-1$ which implies $\eta=\gamma=0$. In this case (23) simplifies to

$$(1-\rho)g(x_1,B,\ldots,B)+\rho g(-x_1,B,\ldots,B)=g(x_1+1,B,\ldots,B).$$

Applying the expansion of function g in terms of function f from (24) for some $x_1\in[B-1]$ gives

$$(1-\rho)\left(c_1 f(x_1,B,\ldots,B)+c_2 f(B,B,\ldots,B)\right)+\rho\left(c_1 f(-x_1,B,\ldots,B)+c_2 f(B,B,\ldots,B)\right) \\ = c_1 f(x_1+1,B,\ldots,B)+c_2 f(B,B,\ldots,B)$$

for constants $c_1>0$ and $c_2>0$ that respectively represent the probabilities that the first question is picked and not picked in the set of G gold standard questions. Cancelling out the common terms on both sides of the equation, we get the desired result

$$(1-\rho)f(x_1,B,\ldots,B)+\rho f(-x_1,B,\ldots,B)=f(x_1+1,B,\ldots,B).$$

Next, we consider the case when $(G-\eta-\gamma-1)$ questions are skipped in the gold standard, and assume that the result is true when more than $(G-\eta-\gamma-1)$ questions are skipped in the gold standard. In (25), the functions g decompose into a sum of the constituent f functions. These constituent functions f are of two types: the first where all of the first $(\eta+\gamma+1)$ questions are included in the gold standard, and the second where one or more of the first $(\eta+\gamma+1)$ questions are not included in the gold standard. The second case corresponds to situations where there are more than $(G-\eta-\gamma-1)$ questions skipped in the gold standard and hence satisfies our induction hypothesis. The terms corresponding to these functions thus cancel out in the expansion of (25). The remainder comprises only evaluations of function f for arguments in which the first $(\eta+\gamma+1)$ questions are included in the gold standard. Since the last $(N-\eta-\gamma-1)$ questions are skipped by the worker, the remainder evaluates to

$$(1-\rho)c_3 f(x_1,\ldots,x_{\eta+\gamma},x_i,B,\ldots,B)+\rho c_3 f(x_1,\ldots,x_{\eta+\gamma},-x_i,B,\ldots,B) \\ = c_3 f(x_1,\ldots,x_{\eta+\gamma},x_i+1,B,\ldots,B) \tag{26}$$

for some constant $c_3>0$. Dividing throughout by c_3 gives the desired result.

Finally, the arguments above hold for any permutation of the first G questions, thus completing the proof. □