Supplementary Material

A. Proof of Lemma 2

We begin by bounding, for any value of m, the distance between $\hat{G}$ and $\hat{G}^m$. Set m to any integer greater or equal to 1. Writing

$$\epsilon_1 = \frac{1}{n-1}\sum_{i=1}^{m-1} G_i - \mathbb{E}[G_i]$$

$$\text{and} \quad \epsilon_2 = \frac{1}{n-1}\sum_{i=m}^{n-1}(z_i - z_i^m)\tau(X_i, X_{i+1})^T - \mathbb{E}[(z_i - z_i^m)\tau(X_i, X_{i+1})^T],$$

we have

$$\begin{aligned}\frac{1}{n-1}\sum_{i=1}^{n-1} G_i - \mathbb{E}[G_i] &= \frac{1}{n-1}\sum_{i=m}^{n-1} G_i - \mathbb{E}[G_i] + \epsilon_1 \\ &= \frac{1}{n-1}\sum_{i=m}^{n-1} z_i\tau(X_i, X_{i+1})^T - \mathbb{E}[z_i\tau(X_i, X_{i+1})^T] + \epsilon_1 \\ &= \frac{1}{n-1}\sum_{i=m}^{n-1} z_i^m\tau(X_i, X_{i+1})^T - \mathbb{E}[z_i^m\tau(X_i, X_{i+1})^T] + (\epsilon_1 + \epsilon_2) \\ &= \frac{1}{n-1}\sum_{i=m}^{n-1}(G_i^m - \mathbb{E}[G_i^m]) + (\epsilon_1 + \epsilon_2). \end{aligned} \tag{23}$$

For all i, we have $\|z_i\|_\infty \leq \frac{L}{1-\lambda\gamma}$, $\|G_i\|_\infty \leq \frac{LL'}{1-\lambda\gamma}$, and $\|z_i - z_i^m\|_\infty \leq \frac{(\lambda\gamma)^m L}{1-\lambda\gamma}$. As a consequence—using $\|M\|_2 \leq \|M\|_F = \sqrt{d \times k}\|x\|_\infty$ for $M \in \mathbb{R}^{d\times k}$ with x the vector obtained by concatenating all M columns—, we can see that

$$\|\epsilon_1 + \epsilon_2\|_2 \leq \frac{2(m-1)\sqrt{d\times k}LL'}{(n-1)(1-\lambda\gamma)} + \frac{2(\lambda\gamma)^m\sqrt{d\times k}LL'}{(1-\lambda\gamma)} \tag{24}$$

By concatenating all its columns, the $d \times k$ matrix G_i^m may be seen a single vector U_i^m of size dk. Then, for all $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}\left(\left\|\frac{1}{n-m}\sum_{i=m}^{n-1}(G_i^m - \mathbb{E}[G_i^m])\right\|_2 \geq \epsilon\right) &\leq \mathbb{P}\left(\left\|\frac{1}{n-m}\sum_{i=m}^{n-1}(G_i^m - \mathbb{E}[G_i^m])\right\|_F \geq \epsilon\right) \\ &= \mathbb{P}\left(\left\|\frac{1}{n-m}\sum_{i=m}^{n-1}(U_i^m - \mathbb{E}[U_i^m])\right\|_2 \geq \epsilon\right). \end{aligned} \tag{25}$$

The process $(U_n^m)_{n\geq m}$, defined as a function of the process $(Z_n)_{n\geq m} = (X_{n-m+1}, X_{n-m+2}\ , \ldots, X_{n+1})_{n\geq m}$, is stationary. By using the next lemma, we can see that it inherits in some sense the β-mixing property of the process $(X_i)_{i\geq 1}$ (Assumption 2).

Lemma 5 (originally stated as Lemma 3)**.** *Let $(X_n)_{n\geq 1}$ be a β-mixing process, then $(Z_n)_{n\geq m} = (X_{n-m+1}, X_{n-m+2}\ , \ldots, X_{n+1})_{n\geq m}$ is a β-mixing process such that its i^{th} β mixing coefficient β_i^Z satisfies $\beta_i^Z \leq \beta_{i-m}^X$.*

Proof. Let $\Gamma = \sigma(Z_m, ..., Z_t)$, by definition we have

$$\Gamma = \sigma(Z_j^{-1}(B) : j \in \{m, ..., t\}, B \in \sigma(\mathcal{X}^{m+1})).$$

For all $j \in \{m, ..., t\}$ we have

$$Z_j^{-1}(B) = \{\omega \in \Omega, Z_j(\omega) \in B\}.$$

For $B = B_0 \times ... \times B_m$, we observe that

$$Z_j^{-1}(B) = \{\omega \in \Omega, X_{j-m+1}(\omega) \in B_0, ..., X_{j+1}(\omega) \in B_m\}.$$

Then we have

$$\Gamma = \sigma(X_j^{-1}(B) : j \in \{m, ..., t\}, B \in \sigma(\mathcal{X})) = \sigma(X_1, ..., X_{t+1}).$$

Similarly we can prove that $\sigma(Z_{t+i}^{\infty}) = \sigma(X_{t+i-m+1}^{\infty})$. Then let β_i^X be the i^{th} β-mixing coefficient of the process $(X_n)_{n\geq 1}$, we have

$$\beta_i^X = \sup_{t\geq 1} \mathbb{E}\left[\sup_{B\in\sigma(X_{t+i}^{\infty})} |P(B|\sigma(X_1, ..., X_t)) - P(B)|\right].$$

Similarly for the process $(Z_n)_{n\geq m}$ we can see that

$$\beta_i^Z = \sup_{t\geq m} \mathbb{E}\left[\sup_{B\in\sigma(Z_{t+i}^{\infty})} |P(B|\sigma(Z_m, ..., Z_t)) - P(B)|\right].$$

By applying what we developed above we obtain

$$\beta_i^Z = \sup_{t\geq m} \mathbb{E}\left[\sup_{B\in\sigma(X_{t+i-m+1}^{\infty})} |P(B|\sigma(X_1, ..., X_{t+1})) - P(B)|\right].$$

Denote $u = t + 1$ we have

$$\beta_i^Z = \sup_{u\geq m+1} \mathbb{E}\left[\sup_{B\in\sigma(X_{u+i-m}^{\infty})} |P(B|\sigma(X_1, ..., X_u)) - P(B)|\right]$$

Then for $i > m$

$$\beta_i^Z \leq \beta_{i-m}^X.$$

□

Now that we know that $(U_n^m)_{n\geq m}$ is a β-mixing stationary process, we shall use the decomposition technique proposed by Yu (1994) that consists in dividing the sequence $U_m^m, \dots, U_{n-1}^m$ into $2\mu_{n-m}$ blocks of length a_{n-m} (we assume here that $n - m = 2a_{n-m}\mu_{n-m}$). The blocks are of two kinds: those which contains the even indexes $E = \cup_{l=1}^{\mu_{n-m}} E_l$ and those with odd indexes $H = \cup_{l=1}^{\mu_{n-m}} H_l$. Thus, by grouping the variables into blocks we get

$$\begin{aligned}
&\mathbb{P}\left(\left\|\frac{1}{n-m}\sum_{i=m}^{n-1} U_i^m - \mathbb{E}[U_i^m]\right\|_2 \geq \epsilon\right) \\
\leq&\mathbb{P}\left(\left\|\sum_{i\in H} U_i^m - \mathbb{E}[U_i^m]\right\|_2 + \left\|\sum_{i\in E} U_i^m - \mathbb{E}[U_i^m]\right\|_2 \geq (n-m)\frac{\epsilon}{2}\right) && (26)\\
\leq&\mathbb{P}\left(\left\|\sum_{i\in H} U_i^m - \mathbb{E}[U_i^m]\right\|_2 \geq \frac{(n-m)\epsilon}{4}\right) + \mathbb{P}\left(\left\|\sum_{i\in E} U_i^m - \mathbb{E}[U_i^m]\right\|_2 \geq \frac{(n-m)\epsilon}{4}\right) && (27)\\
=&2\mathbb{P}\left(\left\|\sum_{i\in H} U_i^m - \mathbb{E}[U_i^m]\right\|_2 \geq \frac{(n-m)\epsilon}{4}\right) && (28)
\end{aligned}$$

where Equation (26) follows from the triangle inequality, Equation (27) from the fact that the event $\{X+Y \geq a\}$ implies $\{X \geq \frac{a}{2}\}$ or $\{Y \geq \frac{a}{2}\}$, and Equation (28) from the assumption that the process is stationary. Since $H = \cup_{l=1}^{\mu_{n-m}} H_l$ we have

$$\mathbb{P}\left(\left\|\frac{1}{n-m}\sum_{i=m}^{n-1} U_i^m - \mathbb{E}[U_i^m]\right\|_2 \geq \epsilon\right) \leq 2\mathbb{P}\left(\left\|\sum_{l=1}^{\mu_{n-m}}\sum_{i\in H_l} U_i^m - \mathbb{E}[U_i^m]\right\|_2 \geq \frac{(n-m)\epsilon}{4}\right)$$
$$= 2\mathbb{P}\left(\left\|\sum_{l=1}^{\mu_{n-m}} U(H_l) - \mathbb{E}[U(H_l)]\right\|_2 \geq \frac{(n-m)\epsilon}{4}\right) \tag{29}$$

where we defined $U(H_l) = \sum_{i\in H_l} U_i^m$. Now consider the sequence of identically distributed independent blocks $(U'(H_l))_{l=1,\dots,\mu_{n-m}}$ such that each block $U'(H_l)$ has the same distribution as $U(H_l)$. We are going to use the following technical result.

Lemma 6. *(Yu, 1994) Let $X_1,\dots,X_n$ be a sequence of samples drawn from a stationary β-mixing process with coefficients $\{\beta_i\}$. Let $X(H) = (X(H_1),\dots,X(H_{\mu_{n-m}}))$ where for all j $X(H_j) = (X_i)_{i\in H_j}$. Let $X'(H) = (X'(H_1),\dots,X'(H_{\mu_{n-m}}))$ with $X'(H_j)$ independent and such that for all j, $X'(H_j)$ has same distribution as $X(H_j)$. Let Q and Q' be the distribution of $X(H)$ and $X'(H)$ respectively. For any measurable function $h : \mathcal{X}^{a_n\mu_n} \to \mathbb{R}$ bounded by B, we have*

$$|\mathbb{E}_Q[h(X(H)] - \mathbb{E}_{Q'}[h(X'(H)]| \leq B\mu_n\beta_{a_n}.$$

By applying Lemma 6, Equation (29) leads to:

$$\mathbb{P}\left(\left\|\frac{1}{n-m}\sum_{i=m}^{n-1} U_i^m - \mathbb{E}[U_i^m]\right\|_2 \geq \epsilon\right) \leq 2\mathbb{P}\left(\left\|\sum_{l=1}^{\mu_{n-m}} U'(H_l) - \mathbb{E}[U'(H_l)]\right\|_2 \geq \frac{(n-m)\epsilon}{4}\right)$$
$$+ 2\mu_{n-m}\beta_{a_{n-m}}. \tag{30}$$

The variables $U'(H_l)$ are independent. Furthermore, it can be seen that $(\sum_{l=1}^{\mu_{n-m}} U'(H_l) - \mathbb{E}[U'(H_l)])_{\mu_{n-m}}$ is a $\sigma(U'(H_1),\dots,U'(H_{\mu_{n-m}}))$ martingale:

$$\mathbb{E}\left[\sum_{l=1}^{\mu_{n-m}} U'(H_l) - \mathbb{E}[U'(H_l)] \,\middle|\, U'(H_1),\dots,U'(H_{\mu_{n-m}-1})\right]$$
$$= \sum_{l=1}^{\mu_{n-m}-1} U'(H_l) - \mathbb{E}[U'(H_l)] + \mathbb{E}[U'_{H_{\mu_{n-m}}} - \mathbb{E}[U'_{H_{\mu_{n-m}}}]]$$
$$= \sum_{l=1}^{\mu_{n-m}-1} U'(H_l) - \mathbb{E}[U'(H_l)].$$

We can now use the following concentration result for martingales.

Lemma 7 ((Hayes, 2005))**.** *Let $X = (X_0,\dots,X_n)$ be a discrete time martingale taking values in an Euclidean space such that $X_0 = 0$ and for all i, $\|X_i - X_{i-1}\|_2 \leq B_2$ almost surely. Then for all ϵ,*

$$P\{\|X_n\|_2 \geq \epsilon\} < 2e^2 e^{-\frac{\epsilon^2}{2n(B_2)^2}}.$$

Indeed, taking $X_{\mu_{n-m}} = \sum_{l=1}^{\mu_{n-m}} U'(H_l) - \mathbb{E}[U'(H_l)]$, and observing that $\|X_i - X_{i-1}\| = \|U'(H_l) - \mathbb{E}[U'(H_l)]\|_2 \leq a_{n-m}C$ with $C = \frac{2\sqrt{dk}LL'}{1-\lambda\gamma}$, the lemma leads to

$$\mathbb{P}\left(\left\|\sum_{l=1}^{\mu_{n-m}} U'(H_l) - \mathbb{E}[U'(H_l)]\right\|_2 \geq \frac{(n-m)\epsilon}{4}\right) \leq 2e^2 e^{-\frac{(n-m)^2\epsilon^2}{32\mu_{n-m}(a_{n-m}C)^2}}$$
$$= 2e^2 e^{-\frac{(n-m)\epsilon^2}{16a_{n-m}C^2}}.$$

where the second line is obtained by using the fact that $2a_{n-m}\mu_{n-m} = n-m$. With Equations (29) and (30), we finally obtain

$$\mathbb{P}\left(\left\|\frac{1}{n-m}\sum_{i=m}^{n-1} U_i^m - \mathbb{E}[U_i^m]\right\|_2 \geq \epsilon\right) \leq 4e^2 e^{-\frac{(n-m)\epsilon^2}{16a_{n-m}C^2}} + 2(n-m)\beta^U_{a_{n-m}}.$$

The vector U_i^m is a function of $Z_i = (X_{i-m+1}, \ldots, X_{i+1})$, and Lemma 3 tells us that for all $j > m$,

$$\beta_j^U \leq \beta_j^Z \leq \beta_{j-m}^X \leq \overline{\beta} e^{-b(j-m)^\kappa}.$$

So the equation above may be re-written as

$$\mathbb{P}\left(\left\|\frac{1}{n-m}\sum_{i=m}^{n-1} U_i^m - \mathbb{E}[U_i^m]\right\|_2 \geq \epsilon\right) \leq 4e^2 e^{-\frac{(n-m)\epsilon^2}{16a_{n-m}C^2}} + 2(n-m)\overline{\beta} e^{-b(a_{n-m}-m)^\kappa} = \delta'. \quad (31)$$

We now follow a reasoning similar to that of (Lazaric et al., 2012) in order to get the same exponent in both of the above exponentials. Taking $a_{n-m} - m = \left\lceil \frac{C_2(n-m)\epsilon^2}{b} \right\rceil^{\frac{1}{\kappa+1}}$ with $C_2 = (16C^2\zeta)^{-1}$, and $\zeta = \frac{a_{n-m}}{a_{n-m}-m}$, we have

$$\delta' \leq (4e^2 + (n-m)\overline{\beta}) \exp\left(-\min\left\{\left(\frac{b}{(n-m)\epsilon^2 C_2}\right), 1\right\}^{\frac{1}{k+1}} \frac{1}{2}(n-m)C_2\epsilon^2\right). \quad (32)$$

Define

$$\Lambda(n,\delta) = \log\left(\frac{2}{\delta}\right) + \log(\max\{4e^2, n\overline{\beta}\}),$$

and

$$\epsilon(\delta) = \sqrt{2\frac{\Lambda(n-m,\delta)}{C_2(n-m)} \max\left\{\frac{\Lambda(n-m,\delta)}{b}, 1\right\}^{\frac{1}{\kappa}}}.$$

It can be shown that

$$\exp\left(-\min\left\{\left(\frac{b}{(n-m)(\epsilon(\delta))^2 C_2}\right), 1\right\}^{\frac{1}{k+1}} \frac{1}{2}(n-m)C_2(\epsilon(\delta))^2\right) \leq \exp\left(-\Lambda(n-m,\delta)\right). \quad (33)$$

Indeed[6], there are two cases:

1. Suppose that $\min\left\{\left(\frac{b}{(n-m)(\epsilon(\delta))^2 C_2}\right), 1\right\} = 1$. Then

$$\begin{aligned}
&\exp\left(-\min\left\{\left(\frac{b}{(n-m)(\epsilon(\delta))^2 C_2}\right), 1\right\}^{\frac{1}{k+1}} \frac{1}{2}(n-m)C_2(\epsilon(\delta))^2\right) \\
=& \exp\left(-\Lambda(n-m,\delta)\max\left\{\frac{\Lambda(n-m,\delta)}{b}, 1\right\}^{\frac{1}{k}}\right) \\
\leq& \exp\left(-\Lambda(n-m,\delta)\right).
\end{aligned}$$

[6]This inequality exists in (Lazaric et al., 2012), and is developped here for completeness.

2. Suppose now that $\min\left\{\left(\frac{b}{(n-m)(\epsilon(\delta))^2C_2}\right),1\right\} = \left(\frac{b}{(n-m)(\epsilon(\delta))^2C_2}\right)$. Then

$$\begin{aligned}
&\exp\left(-\min\left\{\left(\frac{b}{(n-m)(\epsilon(\delta))^2C_2}\right),1\right\}^{\frac{1}{k+1}}\frac{1}{2}(n-m)C_2(\epsilon(\delta))^2\right)\\
=&\exp\left(-\frac{1}{2}b^{\frac{1}{k+1}}((n-m)C_2(\epsilon(\delta))^2)^{\frac{k}{k+1}}\right)\\
=&\exp\left(-\frac{1}{2}b^{\frac{1}{k+1}}(\Lambda(n-m,\delta)^{\frac{k}{k+1}}\max\left\{\frac{\Lambda(n-m,\delta)}{b},1\right\}^{\frac{1}{k+1}}\right)\\
=&\exp\left(-\frac{1}{2}\Lambda(n-m,\delta)^{\frac{k}{k+1}}\max\left\{\Lambda(n-m,\delta),b\right\}^{\frac{1}{k+1}}\right)\\
\leq&\exp\left(-\Lambda(n-m,\delta)\right).
\end{aligned}$$

By combining Equations (32) and (33), we get

$$\delta' \leq (4e^2+(n-m)\overline{\beta})\exp\left(-\Lambda(n-m,\delta)\right).$$

If we replace $\Lambda(n-m,\delta)$ with its expression, we obtain

$$\exp\left(-\Lambda(n-m,\delta)\right) = \frac{\delta}{2}\max\{4e^2,(n-m)\overline{\beta}\}^{-1}.$$

Since $4e^2\max\{4e^2,(n-m)\overline{\beta}\}^{-1}\leq 1$ and $(n-m)\overline{\beta}\max\{4e^2,(n-m)\overline{\beta}\}^{-1}\leq 1$, we consequently have

$$\delta' \leq 2\frac{\delta}{2}\leq \delta.$$

Now, note that since $a_{n-m}-m\geq 1$, we have

$$\zeta = \frac{a_{n-m}}{a_{n-m}-m} = \frac{a_{n-m}-m+m}{a_{n-m}-m}\leq 1+m.$$

Let $J(n,\delta) = 32\Lambda(n,\delta)\max\left\{\frac{\Lambda(n,\delta)}{b},1\right\}^{\frac{1}{\kappa}}$. Then Equation (31) is reduced to

$$\mathbb{P}\left(\left\|\frac{1}{n-m}\sum_{i=m}^{n-1}(U_i^m-\mathbb{E}[U_i^m])\right\|_2 \geq \frac{C}{\sqrt{n-m}}\left(\zeta J(n-m,\delta)\right)^{\frac{1}{2}}\right)\leq\delta. \tag{34}$$

Since $J(n,\delta)$ is an increasing function on n, and $\frac{n-1}{\sqrt{n-1}(n-m)} = \frac{1}{\sqrt{n-m}}\sqrt{\frac{n-1}{n-m}}\geq\frac{1}{\sqrt{n-m}}$, we have

$$\begin{aligned}
&\mathbb{P}\left(\left\|\frac{1}{n-1}\sum_{i=m}^{n-1}(G_i^m-\mathbb{E}[G_i^m])\right\|_2 \geq \frac{C}{\sqrt{n-1}}\left(\zeta J(n-1,\delta)\right)^{\frac{1}{2}}\right)\\
\leq\;&\mathbb{P}\left(\left\|\frac{1}{n-m}\sum_{i=m}^{n-1}(G_i^m-\mathbb{E}[G_i^m])\right\|_2 \geq \frac{C}{\sqrt{n-1}}\frac{n-1}{n-m}\left((m+1)J(n-1,\delta)\right)^{\frac{1}{2}}\right)\\
\leq\;&\mathbb{P}\left(\left\|\frac{1}{n-m}\sum_{i=m}^{n-1}(G_i^m-\mathbb{E}[G_i^m])\right\|_2 \geq \frac{C}{\sqrt{n-m}}\left((m+1)J(n-m,\delta)\right)^{\frac{1}{2}}\right).
\end{aligned}$$

By using Equations (25) and (34), we deduce that

$$\mathbb{P}\left(\left\|\frac{1}{n-1}\sum_{i=m}^{n-1}(G_i^m-\mathbb{E}[G_i^m])\right\|_2 \geq \frac{C}{\sqrt{n-1}}\left((m+1)J(n-1,\delta)\right)^{\frac{1}{2}}\right)\leq\delta. \tag{35}$$

By combining Equations (23), (24) and (35), plugging the value of $C=\frac{2\sqrt{dk}LL'}{1-\lambda\gamma}$, and taking $m=\left\lceil\frac{\log(n-1)}{\log\frac{1}{\lambda\gamma}}\right\rceil$—so that $\|\epsilon_1+\epsilon_2\|_2\leq\epsilon(n)$—, we get the announced result.

B. Proof of Theorem 3

We prove here the following result: for any $\delta \in (0,1)$, for all $n \geq 1$, consider $\hat{v}^{\rho}_{LSTD(\lambda)} = \Phi\hat{\theta}_\rho$ with penalization parameter $\rho = 2\Xi^2(n,\delta)$. Then, with at least probability $1-\delta$, for all n,

$$\|\hat{v}^{\rho}_{LSTD(\lambda)} - v_{LSTD(\lambda)}\|_\mu \leq \frac{4V_{\max}\sqrt{d}L(3+\sqrt{d}L)}{\sqrt{n-1}(1-\gamma)\sqrt{\nu}}\sqrt{(m_n^\lambda+1)\, I(n-1,\delta)} + g(n,\delta),$$

where $g(n,\delta)$ and $I(n,\delta)$ are defined as in Theorem 1.

Proof. Let $\hat{\theta}_\rho$ be the vector that satisfies

$$\hat{\theta}_\rho = \arg\min_{\theta\in\mathbb{R}^d}\left\{\|\hat{A}\theta_\rho - \hat{b}\|_2^2 + \rho\|\theta_\rho\|_2^2\right\}. \tag{36}$$

We have

$$\|A\hat{\theta}_\rho - b\|_2 \leq \|\epsilon_A\|_2\|\hat{\theta}_\rho\|_2 + \|\epsilon_b\|_2 + \|\hat{A}\hat{\theta}_\rho - \hat{b}\|_2.$$

Then by using the inequality $(a+b)^2 \leq 2(a^2+b^2)$ twice on $\underbrace{\|\epsilon_A\|_2\|\hat{\theta}\|_2 + \|\epsilon_b\|_2}_{a} + \underbrace{\|\hat{A}\hat{\theta} - \hat{b}\|_2}_{b}$ and then on $\underbrace{\|\epsilon_A\|_2\|\hat{\theta}\|_2}_{a} + \underbrace{\|\epsilon_b\|_2}_{b}$ we have

$$\|A\hat{\theta}_\rho - b\|_2^2 \leq 4\|\epsilon_A\|_2^2\|\hat{\theta}_\rho\|_2^2 + 4\|\epsilon_b\|_2^2 + 2\|\hat{A}\hat{\theta}_\rho - \hat{b}\|_2^2.$$

From Equation (36) we can write that

$$\left\{\|\hat{A}\hat{\theta}_\rho - \hat{b}\|_2^2 + \rho\|\hat{\theta}_\rho\|_2^2\right\} = \min_{\theta\in\mathbb{R}^d}\left\{\|\hat{A}\theta_\rho - \hat{b}\|_2^2 + \rho\|\theta\|_2^2\right\}$$
$$\|\hat{\theta}_\rho\|_2^2 = \frac{1}{\rho}\min_{\theta\in\mathbb{R}^d}\left\{\|\hat{A}\theta - \hat{b}\|_2^2 + \rho\|\theta\|_2^2 - \|\hat{A}\hat{\theta}_\rho - \hat{b}\|_2^2\right\},$$

and

$$\begin{aligned}\|\hat{A}\hat{\theta}_\rho - \hat{b}\|_2^2 &= \min_{\theta\in\mathbb{R}^d}\left\{\|\hat{A}\theta - \hat{b}\|_2^2 + \rho(\|\theta\|_2^2 - \|\hat{\theta}_\rho\|_2^2)\right\}\\ &\leq \min_{\theta\in\mathbb{R}^d}\left\{\|\hat{A}\theta - \hat{b}\|_2^2 + \rho\|\theta\|_2^2\right\}.\end{aligned}$$

So that

$$\begin{aligned}\|A\hat{\theta}_\rho - b\|_2^2 &\leq 4\frac{\|\epsilon_A\|_2^2}{\rho}\min_{\theta\in\mathbb{R}^d}\left\{\|\hat{A}\theta - \hat{b}\|_2^2 + \rho\|\theta\|_2^2 - \|\hat{A}\hat{\theta} - \hat{b}\|_2^2\right\} + 4\|\epsilon_b\|_2^2 + 2\|\hat{A}\hat{\theta} - \hat{b}\|_M^2\\ &\leq 4\frac{\|\epsilon_A\|_2^2}{\rho}\min_{\theta\in\mathbb{R}^d}\left\{\|\hat{A}\theta - \hat{b}\|_2^2 + \rho\|\theta\|_2^2\right\} + \max\left(0, 2 - 4\frac{\|\epsilon_A\|_2^2}{\rho}\right)\|\hat{A}\hat{\theta} - \hat{b}\|_2^2 + 4\|\epsilon_b\|_2^2\\ &\leq \max\left(4\frac{\|\epsilon_A\|_2^2}{\rho}, 2\right)\min_{\theta\in\mathbb{R}^d}\left\{\|\hat{A}\theta - \hat{b}\|_2^2 + \rho\|\theta\|_2^2\right\} + 4\|\epsilon_b\|_2^2.\end{aligned}$$

In Section 4.3, we derived high-probability bounds on $\|\epsilon_A\|_2$ and $\|\hat{A}\theta^* - \hat{b}\|_2 = \|\epsilon_A\theta^* - \epsilon_b\|_2$ with $\theta^* = A^{-1}b$. It is easy to also derive a high-probability bound on $\|\epsilon_b\|_2^2$. More precisely, with the definitions of ϵ_1 and ϵ_2 given in Equations (17) and (21), and with $\epsilon_3(n,\delta_n) = \frac{2\sqrt{d}L^2}{(1-\lambda\gamma)\sqrt{n-1}}\sqrt{(m_n^\lambda+1)\, J(n-1,\delta_n)} + \tilde{O}(\frac{1}{n})$, we know that with probability at least $1-\delta$,

$$\|\epsilon_A\|_2 \leq \epsilon_1(n,\delta_n), \quad \|\epsilon_A\theta^* - \epsilon_b\|_2 \leq \epsilon_2(n,\delta_n) \quad \text{and} \quad \|\epsilon_b\|_2 \leq \epsilon_3(n,\delta_n).$$

As a consequence,

$$\|A\hat{\theta}_\rho - b\|_2^2 \leq \max\left(4\frac{\|\epsilon_A\|_2^2}{\rho}, 2\right)\left\{(\epsilon_2(n,\delta_n)^2 + \rho)\|\theta^*\|_2^2\right\} + 4\|\epsilon_b\|_2^2.$$

With $\rho = 2(\epsilon_1(n,\delta_n))^2$, we obtain with probability at $1-\delta$,

$$\|A\hat{\theta}_\rho - b\|_2^2 \leq 2(2(\epsilon_1(n,\delta_n))^2 + (\epsilon_2(n,\delta_n))^2)\|\theta^*\|_2^2 + 4(\epsilon_3(n,\delta_n))^2$$

By using the fact that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, this implies

$$\|A\hat{\theta}_\rho - b\|_2 \leq \sqrt{2(2\epsilon_1(n,\delta_n) + \epsilon_2(n,\delta_n))}\|\theta^*\|_2 + 2(\epsilon_3(n,\delta_n))$$

We conclude by using Equation (8) in which we take the norm, by bounding $\|\Phi A^{-1}\|_\mu$ in the same way as we did in the proof of Lemma 1, and finish in the way similar to the unregularized proof with $\delta_n = \frac{\delta}{6n^2}$. □