A. Proof for Theorem 1

To prove Theorem 1, we first prove the following theorem:

**Theorem 3.** If (1) \( w = \Phi^* \), (2) the prior on \( \theta^* \) is \( N(0, \lambda^2 I) \), and (3) the noises are i.i.d. sampled from \( N(0, \sigma^2) \), then under CombLinTS algorithm with parameter \( (\Phi, \lambda, \sigma) \), then we have

\[
R_{Bayes}(n) \leq 1 + K\lambda \min \left\{ \sqrt{\ln \left( \frac{\lambda L \ln n}{2\pi} \right)}, \sqrt{d \ln \left( \frac{2dK \lambda}{\sqrt{2\pi}} \right)} \right\} \sqrt{2dn \ln \left( 1 + \frac{2K \lambda^2}{d} \right)}.
\]  

Notice that Theorem 1 follows immediately from Theorem 3. Specifically, if \( \lambda \geq \sigma \), then we have

\[
B_{Bayes}(n) \leq 1 + K\lambda \min \left\{ \sqrt{\ln \left( \frac{\lambda L \ln n}{2\pi} \right)}, \sqrt{d \ln \left( \frac{2dK \lambda}{\sqrt{2\pi}} \right)} \right\} \sqrt{2dn \log_2 \left( 1 + \frac{nK \lambda^2}{d} \right)} = \tilde{O} \left( K\lambda \sqrt{d \min \{ \ln(L), d \}} \right).
\]

We now outline the proof of Theorem 3, which is based on (Russo & Van Roy, 2013; Dani et al., 2008). Let \( H_t \) denote the “history” (i.e. all the available information) by the start of episode \( t \). Note that from the Bayesian perspective, conditioning on \( H_t \), \( \theta^* \) and \( \theta_t \) are i.i.d. drawn from \( N(\bar{\theta}_t, \Sigma_t) \) (see (Russo & Van Roy, 2013)). This is because that conditioning on \( H_t \), the posterior belief in \( \theta^* \) is \( N(\bar{\theta}_t, \Sigma_t) \) and based on Algorithm 2, \( \theta_t \) is independently sampled from \( N(\bar{\theta}_t, \Sigma_t) \). Since \( \theta^* \) is a fixed combinatorial optimization algorithm (even though it can be independently randomized), and \( E, A, \Phi \) are all fixed, then conditioning on \( H_t \), \( A^* \) and \( A^t \) are i.i.d., furthermore, \( A^* \) is conditionally independent of \( \theta_t \), and \( A^t \) is conditionally independent of \( \theta^* \).

To simplify the exposition, \( \forall \theta \in \mathbb{R}^d \) and \( \forall A \subseteq E \), we define

\[
g(A, \theta) = \sum_{e \in A} \langle \phi_e, \theta \rangle,
\]

then we have \( \mathbb{E}[f(A^*, w_t) | H_t, \theta^*, \theta_t, A^*, A^t] = g(A^*, \theta^*) \) and \( \mathbb{E}[f(A^t, w_t) | H_t, \theta^*, \theta_t, A^*, A^t] = g(A^t, \theta^*) \), hence we have \( \mathbb{E}[R_t | H_t] = \mathbb{E}[g(A^*, \theta^*) - g(A^t, \theta^*) | H_t] \). We also define the upper confidence bound (UCB) function \( U_t : 2^E \rightarrow \mathbb{R} \) as

\[
U_t(A) = \sum_{e \in A} \left[ \langle \phi_e, \bar{\theta}_t \rangle + c \sqrt{\phi_e^T \Sigma_t \phi_e} \right],
\]

where \( c > 0 \) is a constant to be specified. Notice that conditioning on \( H_t \), \( U_t \) is a deterministic function and \( A^*, A^t \) are i.i.d., then \( \mathbb{E}[U_t(A^t) - U_t(A^*) | H_t] = 0 \) and

\[
\mathbb{E}[R_t | H_t] = \mathbb{E}[g(A^*, \theta^*) - U_t(A^*) | H_t] + \mathbb{E}[U_t(A^t) - g(A^t, \theta^*) | H_t].
\]

One key observation is that

\[
\mathbb{E}[U_t(A^t) - g(A^t, \theta^*) | H_t] \overset{(a)}{=} \sum_{e \in E} \mathbb{E} \left[ \mathbb{I} \{ e \in A^t \} \left( \langle \phi_e, \bar{\theta}_t - \theta^* \rangle + c \sqrt{\phi_e^T \Sigma_t \phi_e} \right) \right] | H_t
\]

\[
\overset{(b)}{=} \sum_{e \in E} \mathbb{E} \left[ \mathbb{I} \{ e \in A^t \} | H_t \right] \mathbb{E} \left[ \langle \phi_e, \bar{\theta}_t - \theta^* \rangle | H_t \right] + c \mathbb{E} \left[ \sum_{e \in A^t} \sqrt{\phi_e^T \Sigma_t \phi_e} \right] | H_t
\]

\[
\overset{(c)}{=} c \mathbb{E} \left[ \sum_{e \in A^t} \sqrt{\phi_e^T \Sigma_t \phi_e} \right] | H_t,
\]

where (b) follows from the fact that \( A^t \) and \( \theta^* \) are conditionally independent, and (c) follows from \( \mathbb{E}[\theta^* | H_t] = \bar{\theta}_t \). Hence \( B_{Bayes}(n) = \sum_{t=1}^{n} \mathbb{E}[g(A^*, \theta^*) - U_t(A^*)] + c \sum_{t=1}^{n} \mathbb{E} \left[ \sum_{e \in A^t} \sqrt{\phi_e^T \Sigma_t \phi_e} \right]. \) We can show that (1)
We have the following naive bound:

\[ c \geq \min \left\{ \sqrt{\ln \left( \frac{\lambda L n}{\sqrt{2\pi}} \right)}, \sqrt{\frac{d K n \lambda}{\sqrt{2\pi}}} \right\}, \quad (16) \]

and (2) \[ \sum_{i=1}^{n} \mathbb{E} \left[ \sum_{e \in A^*} \sqrt{\phi_e^T \Sigma_t \phi_e} \right] \leq K \lambda \sqrt{2dn \ln \left( 1 + \frac{n K \lambda^2}{d} \right)} / \ln \left( 1 + \frac{\lambda^2}{d} \right). \]

Thus, the bound in Theorem 3 holds. Please refer to the remainder of this section for the full proof.

### A.1. Bound on \[ \sum_{i=1}^{n} \mathbb{E}[g(A^*, \theta^*) - U_i(A^*)] \]

We first prove that if we choose

\[ c \geq \min \left\{ \sqrt{\ln \left( \frac{\lambda L n}{\sqrt{2\pi}} \right)}, \sqrt{\frac{d K n \lambda}{\sqrt{2\pi}}} \right\}, \quad (17) \]

then \[ \sum_{i=1}^{n} \mathbb{E}[g(A^*, \theta^*) - U_i(A^*)] \leq 1. \] To prove this result, we use the following inequality for truncated Gaussian distribution.

**Lemma 1.** If \( X \sim N(\mu, s^2) \), then we have

\[ \mathbb{E}[X \mathbb{1}\{X \geq 0\}] = \mu \left[ 1 - \Phi \left( \frac{\mu}{s} \right) \right] + \frac{s}{\sqrt{2\pi}} \exp \left( -\frac{\mu^2}{2s^2} \right), \]

where \( \Phi \) is the cumulative distribution function (CDF) of the standard Gaussian distribution \( N(0, 1) \). Furthermore, if \( \mu \leq 0 \), we have \( \mathbb{E}[X \mathbb{1}\{X \geq 0\}] \leq \frac{s}{\sqrt{2\pi}} \exp \left( -\frac{\mu^2}{2s^2} \right) \).

Based on Lemma 1, we can prove the following lemmas:

**Lemma 2.** If \( c \geq \sqrt{\ln \left( \frac{\lambda L n}{\sqrt{2\pi}} \right)} \), then we have \[ \sum_{i=1}^{n} \mathbb{E}[g(A^*, \theta^*) - U_i(A^*)] \leq 1. \]

**Proof.** We have the following naive bound:

\[
g(A^*, \theta^*) - U_i(A^*) = \sum_{e \in A^*} \left[ \langle \phi_e, \theta^* - \bar{\theta}_i \rangle - c \sqrt{\phi_e^T \Sigma_t \phi_e} \right] \\
\leq \sum_{e \in A^*} \left[ \langle \phi_e, \theta^* - \bar{\theta}_i \rangle - c \sqrt{\phi_e^T \Sigma_t \phi_e} \right] \mathbb{1}\left\{ \langle \phi_e, \theta^* - \bar{\theta}_i \rangle - c \sqrt{\phi_e^T \Sigma_t \phi_e} \geq 0 \right\} \\
\leq \sum_{e \in E} \left[ \langle \phi_e, \theta^* - \bar{\theta}_i \rangle - c \sqrt{\phi_e^T \Sigma_t \phi_e} \right] \mathbb{1}\left\{ \langle \phi_e, \theta^* - \bar{\theta}_i \rangle - c \sqrt{\phi_e^T \Sigma_t \phi_e} \geq 0 \right\}.
\]

Notice that conditioning on \( \mathcal{H}_t, \bigl\langle \phi_e, \theta^* - \bar{\theta}_i \bigr\rangle - c \sqrt{\phi_e^T \Sigma_t \phi_e} \) is a Gaussian random variable with mean \(-c \sqrt{\phi_e^T \Sigma_t \phi_e}\) and variance \(\phi_e^T \Sigma_t \phi_e\). Thus, from Lemma 1, we have

\[
\mathbb{E} \left[ g(A^*, \theta^*) - U_i(A^*) \big| \mathcal{H}_t \right]
\leq (a) \sum_{e \in E} \mathbb{E}_{\theta^*} \left[ \left\langle \phi_e, \theta^* - \bar{\theta}_i \right\rangle - c \sqrt{\phi_e^T \Sigma_t \phi_e} \right] \mathbb{1}\left\{ \left\langle \phi_e, \theta^* - \bar{\theta}_i \right\rangle - c \sqrt{\phi_e^T \Sigma_t \phi_e} \geq 0 \right\} \mathcal{H}_t \\
\leq (b) \sum_{e \in E} \sqrt{\phi_e^T \Sigma_t \phi_e} \frac{\lambda}{\sqrt{2\pi}} \exp \left( -\frac{c^2}{2} \right) \\
\leq (c) \exp \left( -\frac{c^2}{2} \right) \sum_{e \in E} \frac{\lambda \|\phi_e\|}{\sqrt{2\pi}} \leq \exp \left( -\frac{c^2}{2} \right) \frac{\lambda L}{\sqrt{2\pi}}, \quad (18)
\]
where the last two inequalities follow from the fact that $\phi_e^T \Sigma_t \phi_e \leq \alpha_e^2 \Sigma_t \phi_e \leq \lambda^2 \|\phi_e\|^2 \leq \lambda^2$, since $\|\phi_e\| \leq 1$ by assumption\(^8\). Thus we have

$$
\mathbb{E} \left[ \sum_{t=1}^{n} [g(A^*, \theta^*) - U_t(A^*)] \right] \leq \exp \left( -\frac{c^2}{2} \right) \frac{n\lambda L}{\sqrt{2\pi}}
$$

(19)

If we choose $c \geq \sqrt{2 \ln \left( \frac{2L_n \lambda}{\sqrt{2\pi}} \right)}$, then we have $\mathbb{E} \left[ \sum_{t=1}^{n} [g(A^*, \theta^*) - U_t(A^*)] \right] \leq 1$.

**Lemma 3.** If $c \geq \sqrt{d \ln \left( \frac{2dK_n \lambda}{\sqrt{2\pi}} \right)}$, then we also have $\sum_{t=1}^{n} \mathbb{E} [g(A^*, \theta^*) - U_t(A^*)] \leq 1$.

**Proof.** We use $v_1, \ldots, v_d$ to denote a fixed set of $d$ orthonormal eigenvectors of $\Sigma_i$, and $\Lambda_1^2, \ldots, \Lambda_d^2$ to denote the associated eigenvalues. Notice that for $i \neq j$, we have $v_i^T \Sigma_i v_j = \Lambda_i^2 v_i^T v_j = 0, \forall i = 1, \ldots, d$, we define $v_{i+d} = -v_i$ and $\Lambda_{i+d} = \Lambda_i$, which allows us to define the following conic decomposition:

$$
\phi_e = \sum_{i=1}^{2d} \alpha_{ei} v_i, \quad \forall e \in E,
$$

subject to the constraints that $\alpha_{ei} \geq 0, \forall (e, i)$. Notice that $\alpha_{ei}$’s are uniquely determined. Furthermore, for $i$ and $j$ s.t. $|i-j| = d$, by definition of conic decomposition, we have $\alpha_{ei} \alpha_{ej} = 0$. In other words, $\alpha_e$ is a $d$-sparse vector.

Since we assume that $\|\phi_e\| \leq 1$, we have that $\sum_{i=1}^{2d} \alpha_{ei}^2 \leq 1, \forall e \in E$. Thus, for any $e$, we have that $\langle \phi_e, e^* - \bar{\theta}_t \rangle = \sum_{i=1}^{2d} \alpha_{ei} \langle v_i, \theta^* - \bar{\theta}_t \rangle$ and

$$
\phi_e^T \Sigma_t \phi_e = \left( \sum_{i=1}^{2d} \alpha_{ei} v_i^T \right) \Sigma_t \left( \sum_{j=1}^{2d} \alpha_{ej} v_j \right) = \sum_{i=1}^{2d} \sum_{j=1}^{2d} \alpha_{ei} \alpha_{ej} v_i^T \Sigma_i v_j.
$$

Notice that for $i \neq j$, if $|i-j| \neq d$, then $v_i^T \Sigma_i v_j = 0$; on the other hand, if $|i-j| = d$, $\alpha_{ei} \alpha_{ej} = 0$. Thus, if $i \neq j$, we have $\alpha_{ei} \alpha_{ej} v_i^T \Sigma_i v_j = 0$. Consequently,

$$
\phi_e^T \Sigma_i \phi_e = \sum_{i=1}^{2d} \alpha_{ei}^2 v_i^T \Sigma_i v_i = \sum_{i=1}^{2d} \alpha_{ei}^2 \Lambda_i^2.
$$

Thus we have

$$
\sqrt{\phi_e^T \Sigma_t \phi_e} = \sqrt{\sum_{i=1}^{2d} \alpha_{ei}^2 \Lambda_i^2} \geq \frac{1}{\sqrt{d}} \sum_{i=1}^{2d} \alpha_{ei} \Lambda_i,
$$

(21)

where the inequality follows from Cauchy-Schwartz inequality, specifically, define $s_i = 1$ if $\alpha_{ei} \Lambda_i \neq 0$, and $s_i = 0$ if $\alpha_{ei} \Lambda_i = 0$, then we have

$$
\sum_{i=1}^{2d} \alpha_{ei} \Lambda_i = \sum_{i=1}^{2d} \alpha_{ei} \Lambda_i s_i \leq \sqrt{\sum_{i=1}^{2d} s_i^2} \sqrt{\sum_{i=1}^{2d} \alpha_{ei}^2 \Lambda_i^2} \leq \sqrt{d} \sum_{i=1}^{2d} \alpha_{ei}^2 \Lambda_i^2,
$$

\(^8\)Notice that in the derivation of Inequality (18), we implicitly assume that $\phi_e^T \Sigma_t \phi_e > 0, \forall e \in E$. It is worth pointing out that the case with $\phi_e^T \Sigma_t \phi_e = 0$ is a trivial case and this inequality still holds in this case.
Consequently, we have

\[ \langle \phi_e, \theta^* - \tilde{\theta}_t \rangle - c \sqrt{\phi_e^T \Sigma_t \phi_e} \leq \sum_{i=1}^{2d} \alpha_{ei} \langle v_i, \theta^* - \tilde{\theta}_t \rangle - c \frac{\sqrt{d}}{\sqrt{d}} \sum_{i=1}^{2d} \alpha_{ei} \Lambda_i. \]  

(22)

Consequently, we have

\[
\sum_{e \in A^*} \left[ \langle \phi_e, \theta^* - \tilde{\theta}_t \rangle - c \sqrt{\phi_e^T \Sigma_t \phi_e} \right] \leq \sum_{i=1}^{2d} \left( \langle v_i, \theta^* - \tilde{\theta}_t \rangle - c \frac{\Lambda_i}{\sqrt{d}} \right) \left( \sum_{e \in A^*} \alpha_{ei} \right).
\]

(23)

Define \( X_i = \langle v_i, \theta^* - \tilde{\theta}_t \rangle - c \frac{\Lambda_i}{\sqrt{d}} \), notice that conditioning on \( \mathcal{H}_t \), we have \( X_i | \mathcal{H}_t \sim N \left( -c \frac{\Lambda_i}{\sqrt{d}}, \Lambda_i^2 \right) \). Hence we have

\[
\sum_{e \in A^*} \langle \phi_e, \theta^* - \tilde{\theta}_t \rangle - c \sqrt{\phi_e^T \Sigma_t \phi_e} \leq \sum_{i=1}^{2d} X_i \left[ \sum_{e \in A^*} \alpha_{ei} \right] \leq \sum_{i=1}^{2d} X_i \mathbb{I} \{ X_i \geq 0 \} \left[ \sum_{e \in A^*} \alpha_{ei} \right],
\]

where the inequality (b) follows from the fact that \( X_i \leq \sum_{e \in A^*} \alpha_{ei} \geq 0 \) and \( \left[ \sum_{e \in A^*} \alpha_{ei} \right] \geq 0 \). On the other hand, notice that \( |A^*| \leq K \)

\[
\sum_{e \in A^*} \alpha_{ei} \leq \sqrt{|A^*|} \sqrt{\sum_{e \in A^*} \alpha_{ei}^2} \leq \sqrt{|A^*|} \sqrt{\sum_{e \in A^*} \sum_{j=1}^{d} \alpha_{ej}^2} \leq \sqrt{|A^*|} \sqrt{\sum_{e \in A^*} 1} = |A^*| \leq K.
\]

Since \( \sum_{i=1}^{2d} X_i \mathbb{I} \{ X_i \geq 0 \} \geq 0 \), we have

\[
\sum_{e \in A^*} \langle \phi_e, \theta^* - \tilde{\theta}_t \rangle - c \sqrt{\phi_e^T \Sigma_t \phi_e} \leq K \sum_{i=1}^{2d} X_i \mathbb{I} \{ X_i \geq 0 \},
\]

notice that the RHS does not include \( A^* \). Hence we have

\[
\mathbb{E}_{\theta^{*}} [g(A^*, \theta^*) - U_i(A^*) | \mathcal{H}_t] = \mathbb{E}_{\theta^{*}} \left[ \sum_{e \in A^*} \langle \phi_e, \theta^* - \tilde{\theta}_t \rangle - c \sqrt{\phi_e^T \Sigma_t \phi_e} | \mathcal{H}_t \right] \leq K \sum_{i=1}^{2d} \mathbb{E}_{\theta^{*}} [X_i \mathbb{I} \{ X_i \geq 0 \} | \mathcal{H}_t] \leq K \sum_{i=1}^{2d} \frac{\Lambda_i}{\sqrt{2\pi}} \exp \left( -\frac{c^2}{2d} \right) \leq \frac{2dK \lambda}{\sqrt{2\pi}} \exp \left( -\frac{c^2}{2d} \right),
\]

where the last inequality follows from the fact that \( \Lambda_i \leq \lambda \). Hence we have

\[
\sum_{i=1}^{n} \mathbb{E} [g(A^*, \theta^*) - U_i(A^*)] \leq \frac{2dK n \lambda}{\sqrt{2\pi}} \exp \left( -\frac{c^2}{2d} \right),
\]

if we choose \( c \geq \sqrt{2d \ln \left( \frac{2dK n \lambda}{\sqrt{2\pi}} \right)} \), then we have \( \sum_{i=1}^{n} \mathbb{E} [f(A^*, \theta^*) - U_i(A^*)] \leq 1. \)
Combining the results from Lemma 2 and 3, we have proved that if

\[ c \geq \min \left\{ \sqrt{\ln \left( \frac{\lambda L n}{\sqrt{2 \pi}} \right)}, \sqrt{d \ln \left( \frac{2dK n \lambda}{\sqrt{2 \pi}} \right)} \right\}, \]

then \( \sum_{t=1}^{n} \mathbb{E}[g(A^t, \theta^*) - U_t(A^t)] \leq 1. \)

### A.2. Bound on \( \sum_{t=1}^{n} \mathbb{E} \left[ \sum_{e \in A^t} \sqrt{\phi^T e \Sigma_t \phi_e} \right] \)

In this subsection, we derive a bound on \( \sum_{t=1}^{n} \mathbb{E} \left[ \sum_{e \in A^t} \sqrt{\phi^T e \Sigma_t \phi_e} \right] \). Our analysis is motivated by the analysis in (Dani et al., 2008). Specifically, we provide a worst-case bound on \( \sum_{t=1}^{n} \sum_{e \in A^t} \sqrt{\phi^T e \Sigma_t \phi_e} \), for any realization of random variable \( w_t \)'s, \( \theta_t \)'s, \( A_t \)'s, \( A^* \), and \( \theta^* \).

**Lemma 4.** \( \sum_{t=1}^{n} \sum_{e \in A^t} \sqrt{\phi^T e \Sigma_t \phi_e} \leq K \sqrt{\frac{d \log \left( 1 + \frac{K \lambda^2 }{d \sigma^2} \right)}{\log(1 + \frac{d}{\lambda^2})}} \).

**Proof.** To simplify the exposition, we define

\[ z_{t,k} = \sqrt{\phi^T a_k \Sigma_t \phi_{a_k}}. \] (24)

First, notice that \( \Sigma_t^{-1} \) is the Gramian matrix and satisfies

\[ \Sigma_{t+1} = \Sigma_t^{-1} + \frac{1}{\sigma^2} \sum_{k=1}^{A_t} \phi_{a_k} \phi_{a_k}^T. \] (25)

Hence for any \( t, k \), we have that

\[
\det \left[ \Sigma_{t+1}^{-1} \right] \geq \det \left[ \Sigma_t^{-1} + \frac{1}{\sigma^2} \phi_{a_k} \phi_{a_k}^T \right] = \det \left[ \Sigma_t^{-\frac{1}{2}} \left( I + \frac{1}{\sigma^2} \Sigma_t^\frac{1}{2} \phi_{a_k} \phi_{a_k}^T \Sigma_t^\frac{1}{2} \right) \Sigma_t^{-\frac{1}{2}} \right] \\
= \det \left[ \Sigma_t^{-1} \right] \det \left[ I + \frac{1}{\sigma^2} \Sigma_t^\frac{1}{2} \phi_{a_k} \phi_{a_k}^T \Sigma_t^\frac{1}{2} \right] = \det \left[ \Sigma_t^{-1} \right] \left( 1 + \frac{z_{t,k}^2}{\sigma^2} \right).
\] (26)

Hence we have that

\[
\left( \det \left[ \Sigma_{t+1}^{-1} \right] \right)^{|A_t|} \geq \left( \det \left[ \Sigma_t^{-1} \right] \right)^{|A_t|} \prod_{k=1}^{|A_t|} \left( 1 + \frac{z_{t,k}^2}{\sigma^2} \right).
\] (27)

**Remark 1.** This is where the extra \( O(\sqrt{K}) \) factor arises. Notice that this extra factor is purely due to linear generalization. Specifically, if \( \Phi = I \), then \( \Sigma_t \)'s and \( \Sigma_t^{-1} \)'s will be diagonal, and we have

\[
\det \left[ \Sigma_{t+1}^{-1} \right] = \det \left[ \Sigma_t^{-1} \right] \prod_{k=1}^{A_t} \left( 1 + \frac{z_{t,k}^2}{\sigma^2} \right).
\] (28)

Notice that Equation 27 further implies that

\[
\left( \det \left[ \Sigma_{t+1}^{-1} \right] \right)^K \geq \left( \det \left[ \Sigma_t^{-1} \right] \right)^K \prod_{k=1}^{A_t} \left( 1 + \frac{z_{t,k}^2}{\sigma^2} \right),
\] (29)
since \( \det \left[ \Sigma_{t+1}^{-1} \right] \geq \det \left[ \Sigma_t^{-1} \right] \) and \( |A^t| \leq K \). Recall that \( \det \left[ \Sigma_1^{-1} \right] = \left( \frac{1}{\sigma^2} \right)^d \), we have that

\[
\left( \det \left[ \Sigma_{n+1}^{-1} \right] \right)^K \geq \left( \det \left[ \Sigma_1^{-1} \right] \right)^K \prod_{t=1}^{n} \prod_{k=1}^{|A^t|} \left( 1 + \frac{z_{t,k}^2}{\sigma^2} \right) = \frac{1}{\lambda^{2dK}} \prod_{t=1}^{n} \prod_{k=1}^{|A^t|} \left( 1 + \frac{z_{t,k}^2}{\sigma^2} \right). \tag{30}
\]

On the other hand, we have

\[
\text{trace} \left[ \Sigma_{n+1}^{-1} \right] = \text{trace} \left[ \frac{1}{\lambda^2} I + \frac{1}{\sigma^2} \sum_{t=1}^{n} \sum_{k=1}^{|A^t|} \phi_k^T \Sigma \phi_k^T \right] = \frac{d}{\lambda^2} + \frac{1}{\sigma^2} \sum_{t=1}^{n} \sum_{k=1}^{|A^t|} ||\phi_k^T||^2 \leq \frac{d}{\lambda^2} + nK \sigma^2. \tag{31}
\]

where the last inequality follows from the assumption that \( ||\phi_e|| \leq 1, \forall e \in E \) and \( |A^t| \leq K \). From the trace-determinant inequality, we have

\[
\frac{1}{d} \text{trace} \left[ \Sigma_{n+1}^{-1} \right] \geq \left( \det \left[ \Sigma_{n+1}^{-1} \right] \right)^{\frac{1}{d}},
\]

which implies that

\[
\left( \frac{1}{\lambda^2} + \frac{nK}{d \sigma^2} \right)^{dK} \geq \left( \frac{1}{d} \text{trace} \left[ \Sigma_{n+1}^{-1} \right] \right)^{dK} \geq \left( \det \left[ \Sigma_{n+1}^{-1} \right] \right)^K \geq \frac{1}{\lambda^{2dK}} \prod_{t=1}^{n} \prod_{k=1}^{|A^t|} \left( 1 + \frac{z_{t,k}^2}{\sigma^2} \right). \tag{32}
\]

Taking the logarithm, we have

\[
dK \log \left( 1 + \frac{nK \lambda^2}{d \sigma^2} \right) \geq \sum_{t=1}^{n} \sum_{k=1}^{|A^t|} \log \left( 1 + \frac{z_{t,k}^2}{\sigma^2} \right). \tag{33}
\]

Notice that \( z_{t,k}^2 = \phi_k^T \Sigma \phi_k \), hence we have that \( 0 \leq z_{t,k}^2 \leq \phi_k^T \Sigma \phi_k \leq \lambda^2 ||\phi_k||^2 \leq \lambda^2 \). We have the following technical lemma:

**Lemma 5.** For any real number \( x \in [0, \lambda^2] \), we have \( x \leq \frac{\lambda^2}{\log(1 + \frac{\lambda^2}{\sigma^2})} \log \left( 1 + \frac{\sigma^2}{\sigma^2} \right). \)

**Proof.** Define \( h(x) = \frac{\lambda^2}{\log(1 + \frac{\lambda^2}{\sigma^2})} \log \left( 1 + \frac{\sigma^2}{\sigma^2} \right) - x \), thus we only need to prove \( h(x) \geq 0 \) for \( x \in [0, \lambda^2] \). Notice that \( h(x) \) is a strictly concave function for \( x \geq 0 \), and \( h(0) = 0, h(\lambda^2) = 0 \). From Jensen’s inequality, for any \( x \in (0, \lambda^2) \), we have \( h(x) > 0 \).

Hence we have that

\[
\sum_{t=1}^{n} \sum_{k=1}^{|A^t|} z_{t,k}^2 \leq \frac{\lambda^2}{\log(1 + \frac{\lambda^2}{\sigma^2})} \sum_{t=1}^{n} \sum_{k=1}^{|A^t|} \log \left( 1 + \frac{z_{t,k}^2}{\sigma^2} \right) \leq \frac{dK \lambda^2}{\log(1 + \frac{\lambda^2}{\sigma^2})} \log \left( 1 + \frac{\lambda^2}{\sigma^2} \right). \tag{33}
\]

Finally, we have that

\[
\sum_{t=1}^{n} \sum_{k=1}^{|A^t|} z_{t,k} \leq \sqrt{nK} \sqrt{\sum_{t=1}^{n} \sum_{k=1}^{|A^t|} z_{t,k}^2} \leq K \lambda \sqrt{dn \log \left( 1 + \frac{nK \lambda^2}{d \sigma^2} \right)} \log \left( 1 + \frac{\lambda^2}{\sigma^2} \right). \tag{34}
\]
Recall that the above bound holds for any realization of random variables, thus, we have
\[
\mathbb{E} \left[ \sum_{t=1}^{n} \left[ U_t(A^t) - g(A^t, \theta^*) \right] \right] = c \mathbb{E} \left[ \sum_{t=1}^{n} \sum_{k=1}^{|A^t|} z_{t,k} \right] \leq cK \sqrt{\frac{dn \log \left( 1 + \frac{nK^2}{d} \right)}{\log \left( 1 + \frac{\lambda^2}{\sigma^2} \right)}}.
\]

With
\[
c = \min \left\{ \sqrt{\ln \left( \frac{\lambda L n}{\sqrt{2 \pi}} \right)}, \sqrt{d \ln \left( \frac{2dK n \lambda}{\sqrt{2 \pi}} \right)} \right\},
\]
and combining the results in the previous subsection, we have proved Theorem 3.

### B. Proof for Theorem 2

We start by writing an alternative formula for \( \Sigma_t \) and \( \bar{\theta}_t \). Notice that based on Algorithm 1, we have:
\[
\Sigma_t^{-1} = \frac{1}{\lambda^2} I + \frac{1}{\sigma^2} \sum_{\tau=1}^{t-1} \sum_{k=1}^{|A^\tau|} \phi_{a^\tau_k} \phi_{a^\tau_k}^T
\]
\[
\Sigma_t^{-1} \bar{\theta}_t = \frac{1}{\sigma^2} \sum_{\tau=1}^{t-1} \sum_{k=1}^{|A^\tau|} \phi_{a^\tau_k} w_{\tau} (a^\tau_k)
\]
(36)

Interested readers might refer to Appendix C for the derivation of Equation (36). The proof proceeds as follows. We first construct a confidence set of \( \theta^* \) based on the “self normalized bound” developed in (Abbasi-Yadkori et al., 2011). Then we derive a regret bound based on Lemma 4 derived above.

#### B.1. Confidence Set

Our construction of confidence set is motivated by the analysis in (Agrawal & Goyal, 2013). We start by defining some useful notation. Specifically, for any \( t = 1, 2, \ldots, n \), any \( k = 1, 2, \ldots, |A^t| \), we define
\[
\eta_{t,k} = w_t (a^t_k) - \bar{w} (a^t_k).
\]
One key observation is that \( \eta_{t,k} \)'s form a Martingale difference sequence (MDS)\(^9\) since \( w(e) \)'s are statistically independent under \( P \). Moreover, since \( w_t (a^t_k) \) is bounded in interval \([0, 1]\), \( \eta_{t,k} \)'s are sub-Gaussian with constant \( R = 1 \). We further define
\[
V_t = \frac{\sigma^2}{\lambda^2} I + \sum_{\tau=1}^{t-1} \sum_{k=1}^{|A^\tau|} \phi_{a^\tau_k} \phi_{a^\tau_k}^T \\
\xi_t = \sum_{\tau=1}^{t-1} \sum_{k=1}^{|A^\tau|} \phi_{a^\tau_k} \eta_{\tau,k}
\]
As we will see later, we define \( V_t \) and \( \xi_t \) to use the “self normalized bound” developed in (Abbasi-Yadkori et al., 2011) (see Theorem 1 of (Abbasi-Yadkori et al., 2011)). Notice that based on the above definition, we have \( \Sigma_t^{-1} = \frac{1}{\sigma^2} V_t \), and
\[
\bar{\theta}_t - \theta^* = \Sigma_t \left( \frac{1}{\sigma^2} \xi_t - \frac{1}{\lambda^2} \theta^* \right).
\]
\(^9\)Note that the notion of “time” is indexed by a pair \((t, k)\), and follows the lexicographical order.
To see why the second equality holds, notice that
\[
\Sigma_t^{-1} \bar{\theta}_t = \frac{1}{\sigma^2} \sum_{\tau=1}^{t-1} \sum_{k=1}^{|A^\tau|} \phi_{a_k^\tau}^T (\phi_{a_k^\tau}^T \theta^* + \eta_{\tau,k}) \\
= \left( \Sigma_t^{-1} - \frac{1}{\lambda^2} I \right) \theta^* + \frac{1}{\sigma^2} \xi_t.
\]
Hence, for any \( e \in E \), we have
\[
|\langle \phi_e, \bar{\theta}_t - \theta^* \rangle| = \left| \phi_e^T \Sigma_t \left( \frac{1}{\sigma^2} \xi_t - \frac{1}{\lambda^2} \theta^* \right) \right| \\
\leq \| \phi_e \|_{\Sigma_t} \left| \phi_e^T \xi_t - \frac{1}{\lambda^2} \theta^* \right|_{\Sigma_t} \\
\leq \| \phi_e \|_{\Sigma_t} \left[ \frac{1}{\sigma^2} \| \xi_t \|_{\Sigma_t} + \frac{1}{\lambda^2} \| \theta^* \|_{\Sigma_t} \right],
\]
where the first inequality follows from the Cauchy-Schwarz inequality, and the second inequality follows from the triangular inequality. Notice that
\[
\| \theta^* \|_{\Sigma_t} \leq \| \theta^* \|_{\Sigma_t} = \lambda \| \theta^* \|_2,
\]
hence we have
\[
|\langle \phi_e, \bar{\theta}_t - \theta^* \rangle| \leq \| \phi_e \|_{\Sigma_t} \left[ \frac{1}{\sigma^2} \| \xi_t \|_{\Sigma_t} + \frac{1}{\lambda^2} \| \theta^* \|_2 \right].
\]
Moreover, we have
\[
\frac{1}{\sigma^2} \| \xi_t \|_{\Sigma_t} = \frac{1}{\sigma^2} \| \xi_t \|_{\sigma^2 V^{-1}_t} = \frac{1}{\sigma} \| \xi_t \|_{V^{-1}_t}.
\]
So we have
\[
|\langle \phi_e, \bar{\theta}_t - \theta^* \rangle| \leq \| \phi_e \|_{\Sigma_t} \left[ \frac{1}{\sigma} \| \xi_t \|_{V^{-1}_t} + \frac{1}{\lambda} \| \theta^* \|_2 \right]. \tag{37}
\]
The above inequality always holds. We now provide a high probability bound on \( \| \xi_t \|_{V^{-1}_t} \), based on the “self normalized bound” proposed in (Abbasi-Yadkori et al., 2011). From Theorem 1 of (Abbasi-Yadkori et al., 2011), we know for any \( \delta \in (0, 1) \), with probability at least \( 1 - \delta \),
\[
\| \xi_t \|_{V^{-1}_t} \leq \sqrt{2 \log \left( \frac{\text{det}(V_t)}{\delta} \right)} \quad \forall t = 1, 2, \ldots.
\]
Obviously, \( \text{det}(V_1) = \left[ \frac{\sigma^2}{\lambda^2} \right]^d \). On the other hand, we have
\[
|\text{det}(V_t)|^{1/d} \leq \frac{\text{trace}(V_t)}{d} = \frac{\sigma^2}{\lambda^2} + \frac{1}{d} \sum_{\tau=1}^{t-1} \sum_{k=1}^{|A^\tau|} \| \phi_{a_k^\tau} \|^2 \leq \frac{\sigma^2}{\lambda^2} + \frac{(t-1)K}{d},
\]
where the last inequality follows from the assumption that \( \| \phi_e \| \leq 1 \). Hence, for \( t \leq n \), we have
\[
|\text{det}(V_t)|^{1/d} \leq \frac{\sigma^2}{\lambda^2} + \frac{nK}{d}.
\]
Thus, with probability at least \( 1 - \delta \), we have
\[
\| \xi_t \|_{V^{-1}_t} \leq \sqrt{d \log \left( 1 + \frac{nK\lambda^2}{d\sigma^2} \right)} + 2 \log \left( \frac{1}{\delta} \right) \quad \forall t = 1, 2, \ldots, n.
\]
We define event $G$. Moreover, by definition, under event $G$, we have $\bar{w}(e) \leq \hat{w}_t(e)$, for all $t = 1, \ldots, n$ and any $e \in E$.

### B.2. Regret Analysis

We define event $G$ as

$$
G = \left\{ \langle \phi_e, \bar{\theta}_t \rangle - c \sqrt{\phi_e^T \Sigma_t \phi_e} \leq \langle \phi_e, \theta^* \rangle \leq \langle \phi_e, \bar{\theta}_t \rangle + c \sqrt{\phi_e^T \Sigma_t \phi_e} \quad \forall e \in E, \forall t = 1, \ldots, n \right\},
$$

and use $\bar{G}$ to denote the complement of event $G$. Recall that Lemma 7 states that if

$$
c \geq \frac{1}{\sigma} \sqrt{d \log \left( 1 + \frac{nK\lambda^2}{\sigma^2} \right) + 2 \log \left( \frac{1}{\delta} \right)} + \frac{\|\theta^*\|_2}{\lambda},
$$

then $\mathbb{P}(G) \geq 1 - \delta$. Moreover, by definition, under event $G$, we have $\bar{w}(e) \leq \hat{w}_t(e)$, for all $t = 1, \ldots, n$ and any $e \in E$.

Notice that

$$
R(n) = \sum_{t=1}^{n} \mathbb{E} \left[ \sum_{e \in A^t} \hat{w}_t(e) - \sum_{e \in A^t} \bar{w}(e) \right]
$$

for all $t = 1, \ldots, n$, and for all $e \in E$.

Notice that $\|\phi_e\|_{\Sigma_t} = \sqrt{\phi_e^T \Sigma_t \phi_e}$, thus, the above lemma immediately implies the following lemma:

### Lemma 7. For any $\lambda, \sigma > 0$, any $\delta \in (0, 1)$, and any

$$
c \geq \frac{1}{\sigma} \sqrt{d \log \left( 1 + \frac{nK\lambda^2}{\sigma^2} \right) + 2 \log \left( \frac{1}{\delta} \right)} + \frac{\|\theta^*\|_2}{\lambda},
$$

with probability at least $1 - \delta$, we have

$$
\langle \phi_e, \bar{\theta}_t \rangle - c \sqrt{\phi_e^T \Sigma_t \phi_e} \leq \langle \phi_e, \theta^* \rangle \leq \langle \phi_e, \bar{\theta}_t \rangle + c \sqrt{\phi_e^T \Sigma_t \phi_e},
$$

for all $e \in E$ and $t = 1, 2, \ldots n$.

Notice that

$$
\langle \phi_e, \theta^* \rangle \leq \langle \phi_e, \bar{\theta}_t \rangle + c \sqrt{\phi_e^T \Sigma_t \phi_e}
$$

is exactly $\bar{w}(e) \leq \hat{w}_t(e)$.
where the last inequality follows from the naive bound on the realized regret. If $c$ satisfies inequality (40), we have $\mathbb{P}(\bar{G}) \leq \delta$, hence we have

$$ R(n) \leq \sum_{t=1}^{n} \mathbb{E} \left[ \sum_{e \in A} \bar{w}(e) - \sum_{e \in A'} \bar{w}(e) \right] G + nK\delta. $$

Finally, we bound $\sum_{t=1}^{n} \mathbb{E} \left[ \sum_{e \in A} \bar{w}(e) - \sum_{e \in A'} \bar{w}(e) \right] G$ using a worst-case bound conditioning on $G$ (worst-case over all possible random realizations), notice that conditioning on $G$, we have

$$ \sum_{e \in A} \bar{w}(e) \leq \sum_{e \in A} \bar{w}(e) \leq \sum_{e \in A'} \bar{w}(e), $$

where the first inequality follows from the definition of event $G$, and the second inequality follows from that $A'$ is the exact solution of the combinatorial optimization problem $(E, A, w_t)$. Thus we have

$$ \sum_{e \in A} \bar{w}(e) - \sum_{e \in A'} \bar{w}(e) \leq \sum_{e \in A} \bar{w}(e) - \sum_{e \in A'} \bar{w}(e) $$

$$ = \sum_{e \in A'} \left[ \langle \phi_e, \bar{\theta}_t - \theta^* \rangle + c \sqrt{\phi_e^T \Sigma_t \phi_e} \right] $$

$$ \leq 2c \sum_{e \in A'} \sqrt{\phi_e^T \Sigma_t \phi_e}, $$

where the last inequality follows from the definition of $G$. Recall that from Lemma 4, we have

$$ \sum_{t=1}^{n} \sum_{e \in A'} \sqrt{\phi_e^T \Sigma_t \phi_e} \leq K\lambda \sqrt{\frac{dn \log \left( 1 + \frac{nK\lambda^2}{d\sigma^2} \right)}{\log \left( 1 + \frac{\lambda^2}{\sigma^2} \right)}}. $$

Thus we have

$$ \sum_{t=1}^{n} \mathbb{E} \left[ \sum_{e \in A} \bar{w}(e) - \sum_{e \in A'} \bar{w}(e) \right] G \leq 2c \mathbb{E} \left[ \sum_{t=1}^{n} \sum_{e \in A'} \sqrt{\phi_e^T \Sigma_t \phi_e} \right] \leq 2cK\lambda \sqrt{\frac{dn \log \left( 1 + \frac{nK\lambda^2}{d\sigma^2} \right)}{\log \left( 1 + \frac{\lambda^2}{\sigma^2} \right)}}, $$

which implies

$$ R(n) \leq 2cK\lambda \sqrt{\frac{dn \log \left( 1 + \frac{nK\lambda^2}{d\sigma^2} \right)}{\log \left( 1 + \frac{\lambda^2}{\sigma^2} \right)}} + nK\delta. $$

C. Technical Lemma

In this section, we derive Equation (36). We first prove the following technical lemma:

**Lemma 8.** For any $\phi, \bar{\theta} \in \mathbb{R}^d$, any positive definite $\Sigma \in \mathbb{R}^{d \times d}$, any $\sigma > 0$, and any $w \in \mathbb{R}$, if we define

$$ \Sigma_{\text{new}} = \Sigma - \frac{\Sigma \phi \phi^T \Sigma}{\phi^T \Sigma \phi + \sigma^2} $$

$$ \bar{\theta}_{\text{new}} = \left[ I - \frac{\Sigma \phi \phi^T}{\phi^T \Sigma \phi + \sigma^2} \right] \bar{\theta} + \left[ \frac{\Sigma \phi}{\phi^T \Sigma \phi + \sigma^2} \right] w, $$

then we have

$$ \Sigma_{\text{new}}^{-1} = \Sigma^{-1} + \frac{1}{\sigma^2} \phi \phi^T $$

(41)

$$ \Sigma_{\text{new}}^{-1} \bar{\theta}_{\text{new}} = \Sigma^{-1} \bar{\theta} + \frac{1}{\sigma^2} \phi w. $$

(42)
Proof. Notice that Equation (41) follows directly from the Woodbury matrix identity (matrix inversion lemma). We now prove Equation (42). Notice that we have

\[ \tilde{\theta}_{\text{new}} = \left[ I - \frac{\Sigma \phi T \Sigma}{\phi T \Sigma + \sigma^2} \right] \hat{\theta} + \left[ \frac{\Sigma \phi}{\phi T \Sigma + \sigma^2} \right] w \]

\[ = \Sigma_{\text{new}} \Sigma^{-1} \hat{\theta} + \left[ \frac{\Sigma \phi}{\phi T \Sigma + \sigma^2} \right] w, \]

that is,

\[ \Sigma_{\text{new}}^{-1} \tilde{\theta}_{\text{new}} = \Sigma^{-1} \hat{\theta} + \left[ \frac{\Sigma_{\text{new}} \Sigma \phi}{\phi T \Sigma + \sigma^2} \right] w. \] (43)

Notice that

\[ \Sigma_{\text{new}}^{-1} \Sigma \phi = \left[ \Sigma^{-1} + \frac{1}{\sigma^2} \phi \phi^T \right] \Sigma \phi = \phi + \frac{\phi^T \Sigma \phi}{\sigma^2} \phi = \frac{\sigma^2 + \phi^T \Sigma \phi}{\sigma^2} \phi. \] (44)

Plug Equation (44) into Equation (43), we have Equation (42).

Based on Lemma 8, by mathematical induction, we have

\[ \Sigma_{t}^{-1} = \Sigma_{1}^{-1} + \frac{1}{\sigma^2} \sum_{r=1}^{t-1} \sum_{k=1}^{|A^r|} \phi_{a_k^r} \phi_{a_k^r}^T \]

\[ \Sigma_{t}^{-1} \tilde{\theta}_t = \Sigma_{1}^{-1} \tilde{\theta}_1 - \frac{1}{\sigma^2} \sum_{r=1}^{t-1} \sum_{k=1}^{|A^r|} \phi_{a_k^r} w_r (a_k^r), \]

further noting that \( \Sigma_1 = \lambda^2 I \) and \( \tilde{\theta}_1 = 0 \), we can derive Equation (36).

D. A Variant of Theorem 2 for Approximation Algorithms

By suitably redefining the realized regret, we can prove a variant of Theorem 2 in which ORACLE can be an approximation algorithm. Specifically, for a (possibly approximation) algorithm ORACLE, let \( A^* (w) \) be the solution of ORACLE to the optimization problem \( (E, A, w) \), we say \( \gamma \in [0, 1) \) is a sub-optimality gap of ORACLE if

\[ f(A^*(w), w) \geq (1 - \gamma) \max_{A \in A} f(A, w), \quad \forall w. \] (45)

Then we define the (scaled) realized regret \( R_{l t}^\gamma \) as

\[ R_{l t}^\gamma = f(A_{l t}^{\text{opt}}, w_t) - f(A^l, w_t) \frac{1 - \gamma}{1 - \gamma}, \] (46)

where \( A_{l t}^{\text{opt}} \) is the exact solution to the optimization problem \( (E, A, w) \). The (scaled) cumulative regret \( R^\gamma (n) \) is defined as

\[ R^\gamma (n) = \sum_{t=1}^{n} E [R_{l t}^\gamma | \bar{w}]. \]

Under the assumptions that (1) the support of \( P \) is a subset of \( [0, 1]^L \) (i.e. \( w_t(e) \in [0, 1] \forall t \) and \( \forall e \in E \)), (2) the item weight \( w(e)'s \) are statistically independent under \( P \), and (3) the oracle ORACLE has sub-optimality gap \( \gamma \in [0, 1) \), we have the following variant of Theorem 2 when CombLinUCB is applied to coherent learning cases:
Theorem 4. For any $\lambda, \sigma > 0$, any $\delta \in (0,1)$, and any $c$ satisfying
\[
c \geq \frac{1}{\sigma} \sqrt{d \ln \left( 1 + \frac{nK\lambda^2}{d\sigma^2} \right) + 2 \ln \left( \frac{1}{\delta} \right) + \frac{\|\theta^*\|_2}{\lambda}}, \tag{47}
\]
if $\bar{w} = \Phi \theta^*$ and the above three assumptions hold, then under CombLinUCB algorithm with parameter $(\Phi, \lambda, \sigma, c)$, we have
\[
R^\gamma(n) \leq \frac{2cK\lambda}{1 - \gamma} \sqrt{dn \ln \left( 1 + \frac{nK\lambda^2}{d\sigma^2} \right)} + nK\delta.
\]

Proof. Notice that Lemma 7 in Section B.1 still holds. With $G$ defined in Equation (39), we have
\[
R^\gamma(n) = \sum_{t=1}^n \mathbb{E} \left[ \sum_{e \in A^{opt}} \tilde{w}_t(e) - \frac{1}{1 - \gamma} \sum_{e \in A^t} \tilde{w}_t(e) \right]
\]
\[
= \sum_{t=1}^n \mathbb{E} \left[ \sum_{e \in A^{opt}} \tilde{w}(e) - \frac{1}{1 - \gamma} \sum_{e \in A^t} \tilde{w}(e) \right]
\]
\[
= \mathbb{P}(\bar{G}) \sum_{t=1}^n \mathbb{E} \left[ \sum_{e \in A^{opt}} \tilde{w}_t(e) - \frac{1}{1 - \gamma} \sum_{e \in A^t} \tilde{w}_t(e) \right] + \mathbb{P}(\bar{\bar{G}}) \sum_{t=1}^n \mathbb{E} \left[ \sum_{e \in A^{opt}} \tilde{w}_t(e) - \frac{1}{1 - \gamma} \sum_{e \in A^t} \tilde{w}_t(e) \right]
\]
\[
\leq \sum_{t=1}^n \mathbb{E} \left[ \sum_{e \in A^{opt}} \tilde{w}(e) - \frac{1}{1 - \gamma} \sum_{e \in A^t} \tilde{w}(e) \right] + \mathbb{P}(\bar{\bar{G}}) nK,
\]
where the last inequality follows from the naive bound on $R^\gamma$. If $c$ satisfies inequality (40), we have $\mathbb{P}(\bar{\bar{G}}) \leq \delta$, hence we have
\[
R^\gamma(n) \leq \sum_{t=1}^n \mathbb{E} \left[ \sum_{e \in A^{opt}} \tilde{w}_t(e) - \frac{1}{1 - \gamma} \sum_{e \in A^t} \tilde{w}_t(e) \right] + nK\delta.
\]
Finally, we bound $\sum_{t=1}^n \mathbb{E} \left[ \sum_{e \in A^{opt}} \tilde{w}_t(e) - \frac{1}{1 - \gamma} \sum_{e \in A^t} \tilde{w}_t(e) \right]$ using a worst-case bound conditioning on $G$ (worst-case over all the possible random realizations), notice that conditioning on $G$, we have
\[
\sum_{e \in A^{opt}} \tilde{w}_t(e) \leq \sum_{e \in A^{opt}} \bar{w}_t(e) \leq \max_{A \in A} \sum_{e \in A} \bar{w}_t(e) \leq \frac{1}{1 - \gamma} \sum_{e \in A^t} \bar{w}_t(e),
\]
where
\begin{itemize}
  \item The first inequality follows from the definition of event $G$. Specifically, under event $G$, $\bar{w}(e) \leq \bar{w}_t(e)$ for all $i = 1, \ldots, n$ and all $e \in E$.
  \item The second inequality follows from $A^{opt} \in A$.
  \item The last inequality follows from $A^t \leftarrow ORACLE(E, A, \bar{w}_t)$ and ORACLE has sub-optimality gap $\gamma$ (see Equation (45)).
\end{itemize}

Thus we have
\[
\sum_{e \in A^{opt}} \tilde{w}(e) - \frac{1}{1 - \gamma} \sum_{e \in A^t} \tilde{w}(e) \leq \frac{1}{1 - \gamma} \left[ \sum_{e \in A^t} \bar{w}_t(e) - \sum_{e \in A^t} \bar{w}(e) \right]
\]
\[
= \frac{1}{1 - \gamma} \sum_{e \in A^t} \left[ \langle \phi_e, \tilde{\theta}_t \rangle - \theta^* \right] + c \sqrt{\phi_e^T \Sigma_h \phi_e}
\]
\[
\leq \frac{2c}{1 - \gamma} \sum_{e \in A^t} \sqrt{\phi_e^T \Sigma_h \phi_e},
\]
where the last inequality follows from the definition of $G$. Recall that from Lemma 4, we also have

$$
\sum_{t=1}^{n} \sum_{e \in \mathcal{A}^t} \sqrt{\phi_t^T \Sigma_t \phi_e} \leq K \lambda \sqrt{\frac{dn \log \left(1 + \frac{n K \lambda^2}{d \sigma^2}\right)}{\log \left(1 + \frac{\lambda^2}{\sigma^2}\right)}}.
$$

Putting the above inequalities together, we have proved the theorem. \qed