Proofs

**Lemma 1.** The constrained optimization of (19) is equivalent to:

\[
\arg\max_{(f(\bar{u}_t||\bar{z}_t, \bar{x}_t))} H(\bar{U}_{1:T}||\bar{Z}_{1:T}, \bar{X}_{1:T})
\]

(34)

where: \( f(\bar{u}_t||\bar{z}_1:T, \bar{x}_1:T) = \prod_{t=1}^{T} f(\bar{u}_t|\bar{u}_{1:t-1}, \bar{z}_{1:t}, \bar{x}_{1:t}); \)

\( \forall t \in \{1 \cdots T\}, \bar{u}_{1:t} \in \bar{U}_{1:t}, \bar{z}_{1:t} \in \bar{Z}_{1:t}, \bar{x}_{1:t} \in \bar{X}_{1:t}, \bar{x}_t' \in \bar{X}_{1:t}. \)

such that \( f(\bar{u}_t|\bar{u}_{1:t-1}, \bar{z}_{1:t}, \bar{x}_t') \geq 0, \int_{\bar{u}_t \in \bar{U}_t} f(\bar{u}_t|\bar{u}_{1:t-1}, \bar{z}_{1:t}, \bar{x}_t') = 1, \)

\( f(\bar{u}_t|\bar{u}_{1:t-1}, \bar{z}_{1:t}, \bar{x}_t') = f(\bar{u}_t|\bar{u}_{1:t-1}, \bar{z}_{1:t}, \bar{x}_t). \)

(37)

**Proof of Lemma 1.** The previously developed theory of maximum causal entropy [28] shows the causally conditioned probability distribution defined according to affine constraint (15),(16) and (18) are equivalent to it defined by the decomposition into a product of conditional probabilities (35),(36). Then, we show partial observability constraint (17) implies (37).

\[
\forall \bar{u}_{1:T} \in \bar{U}_{1:T}, \bar{x}_{1:T}, \bar{x}_t' \in \bar{X}_{1:T}, \bar{z}_{1:T} \in \bar{Z}_{1:T},
\]

\[
\prod_{t=1}^{T} f(\bar{u}_t|\bar{u}_{1:t-1}, \bar{z}_{1:t}, \bar{x}_t') = \prod_{t=1}^{T} f(\bar{u}_t|\bar{u}_{1:t-1}, \bar{z}_{1:t}, \bar{x}_t')
\]

It is possible, \( f(\bar{u}_1|\bar{z}_1, \bar{x}_1) \cdots f(\bar{u}_t|\bar{u}_{1:t-1}, \bar{z}_{1:t}, \bar{x}_t') \cdots f(\bar{u}_T|\bar{u}_{1:T-1}, \bar{z}_{1:T}, \bar{x}_1) \)

\( = f(\bar{u}_1|\bar{z}_1, \bar{x}_1) \cdots f(\bar{u}_t|\bar{u}_{1:t-1}, \bar{z}_{1:t}, \bar{x}_t') \cdots f(\bar{u}_T|\bar{u}_{1:T-1}, \bar{z}_{1:T}, \bar{x}_1) \)

Thus, \( f(\bar{u}_t|\bar{u}_{1:t-1}, \bar{z}_{1:t}, \bar{x}_t') = f(\bar{u}_t|\bar{u}_{1:t-1}, \bar{z}_{1:t}, \bar{x}_t). \)

It is easy to show (37) implies (17). \( \square \)

**Lemma 2.** The constrained optimization defined in Lemma 1 is equivalent to:

\[
\arg\max_{(f(\bar{u}_t||\bar{z}_t))} H(\bar{U}_{1:T}||\bar{Z}_{1:T})
\]

\( \forall \bar{u}_{1:T} \in \bar{U}_{1:T}, \bar{z}_{1:T}, \bar{z}_t' \in \bar{Z}_{1:T}, \)

\( f(\bar{u}_{1:T}||\bar{z}_t) \geq 0, \int_{\bar{u}_{1:T} \in \bar{U}_{1:T}} f(\bar{u}_{1:T}||\bar{z}_t) d\bar{u}_{1:T} = 1, \)

\( \forall t \in \{1, \cdots, T\} \) such that \( \bar{z}_{1:t} = \bar{z}_{1:t}, \)

\( \int_{\bar{u}_{1:t+1} \in \bar{U}_{1:t+1}} f(\bar{u}_{1:t+1}||\bar{z}_{1:t}) d\bar{u}_{1:t+1} = \int_{\bar{u}_{1:t+1} \in \bar{U}_{1:t+1}} f(\bar{u}_{1:t+1}||\bar{z}'_{1:t}) d\bar{u}_{1:t+1}. \)

\( \square \)

**Proof of Lemma 2.**

\( \forall t \in \{1, \cdots, T\}, \bar{u}_{1:t} \in \bar{U}_{1:t}, \bar{z}_{1:t} \in \bar{Z}_{1:t}, \bar{x}_{1:t} \in \bar{X}_{1:t}, \bar{x}_t' \in \bar{X}_{1:t}. \)

\( f(\bar{u}_t|\bar{u}_{1:t-1}, \bar{z}_{1:t}, \bar{x}_t') = f(\bar{u}_t|\bar{u}_{1:t-1}, \bar{z}_{1:t}, \bar{x}_t') = f(\bar{u}_t|\bar{u}_{1:t-1}, \bar{z}_{1:t}) \)

Then, \( \prod_{t=1}^{T} f(\bar{u}_t|\bar{u}_{1:t-1}, \bar{z}_{1:t}, \bar{x}_t') = \prod_{t=1}^{T} f(\bar{u}_t|\bar{u}_{1:t-1}, \bar{z}_{1:t}) = f(\bar{u}_{1:T}||\bar{z}_{1:T}) \)

Similar to the proof of Lemma 1, the causally conditioned probability distribution defined by a product of conditional probabilities are equivalent to the affine constraint (39), (40).
Then the solution to this optimization problem has the form:

\[
\int_{\tilde{z}_{1:T} \in \tilde{X}_{1:T}} f(\tilde{z}_{1:T}, \tilde{x}_{1:T} | \tilde{u}_{1:T-1}) = \int_{\tilde{x}_{1:T} \in \tilde{X}_{1:T}} \prod_{t=1}^{T} f(\tilde{x}_{1:T} | \tilde{x}_{1:t}, \tilde{x}_{1:t-1}) \prod_{t=1}^{T} f(\tilde{u}_{1:T-1} | \tilde{u}_{1:t-1}, \tilde{z}_{1:t}) \prod_{t=1}^{T} f(\tilde{u}_{t} | \tilde{u}_{1:t-1}, \tilde{z}_{1:t}) = \int_{\tilde{x}_{1:T} \in \tilde{X}_{1:T}} f(\tilde{z}_{1:T} | \tilde{x}_{1:T}) = f(\tilde{z}_{1:T} | \tilde{x}_{1:T})
\]

Then, \( H(\tilde{U}_{1:T} | \tilde{Z}_{1:T}, \tilde{X}_{1:T}) \)

\[
= \int_{\tilde{u}_{1:T}} f(\tilde{u}_{1:T} | \tilde{Z}_{1:T}, \tilde{X}_{1:T}) f(\tilde{Z}_{1:T}, \tilde{X}_{1:T} | \tilde{u}_{1:T-1}) \log f(\tilde{u}_{1:T} | \tilde{Z}_{1:T}, \tilde{X}_{1:T})
\]

\[
= \int_{\tilde{u}_{1:T}} f(\tilde{u}_{1:T} | \tilde{Z}_{1:T}) \int_{\tilde{x}_{1:T}} f(\tilde{z}_{1:T}, \tilde{x}_{1:T} | \tilde{u}_{1:T}) \log f(\tilde{u}_{1:T} | \tilde{Z}_{1:T}, \tilde{X}_{1:T})
\]

\[
= \int_{\tilde{u}_{1:T}} f(\tilde{u}_{1:T} | \tilde{Z}_{1:T}) \log f(\tilde{u}_{1:T} | \tilde{Z}_{1:T}) f(\tilde{z}_{1:T} | \tilde{u}_{1:T-1})
\]

\[
= H(\tilde{U}_{1:T} | \tilde{Z}_{1:T})
\]

Lemma 3. Suppose the constrained optimization problem in Lemma 2 has the following additional constraint:

\[
(F : \tilde{U}_{1:T} \times \tilde{Z}_{1:T} \rightarrow R^N, \tilde{c} \in R^N)
\]

\[
E_{f(\tilde{u}_{1:T}, \tilde{z}_{1:T})} \left[ F(\tilde{U}_{1:T}, \tilde{Z}_{1:T}) \right] = \tilde{c}
\]

Then the solution to this optimization problem has the form:

\[
\hat{f}(\tilde{u}_{t} | \tilde{u}_{1:t-1}, \tilde{z}_{1:t}) = e^{Q(\tilde{u}_{1:t}, \tilde{z}_{1:t}) - V(\tilde{u}_{1:t-1}, \tilde{z}_{1:t})}
\]

where \( Q \) and \( V \) functions take the following recursive form:

\[
Q(\tilde{u}_{1:t}, \tilde{z}_{1:t}) = \begin{cases} 
\lambda^T F(\tilde{u}_{1:T}, \tilde{z}_{1:T}), & t = T; \\
\mathbb{E}[V(\tilde{U}_{1:t}, \tilde{Z}_{1:t+1}) | \tilde{u}_{1:t}, \tilde{z}_{1:t}], & t < T
\end{cases}
\]

\[
V(\tilde{u}_{1:t-1}, \tilde{z}_{1:t}) = \text{softmax} Q(\tilde{u}_{1:t}, \tilde{z}_{1:t}) \triangleq \log \int_{\tilde{u}_{t}} e^{Q(\tilde{u}_{1:t}, \tilde{z}_{1:t})} d\tilde{u}_{t}
\]

Proof of Lemma 3. We first show for any joint distribution \( g(\tilde{u}_{1:T}, \tilde{z}_{1:T}) \), the following equation holds:

\[
E_{g} \left[ - \log f(\tilde{U}_{1:T} | \tilde{Z}_{1:T}) \right] = \int_{\tilde{z}_{1}} f(\tilde{z}_{1}) V(\tilde{z}_{1}) - E_{g} \left[ \lambda^T F(\tilde{U}_{1:T}, \tilde{Z}_{1:T}) \right]
\]

\[
(42)
\]
\[ \mathbb{E}_g \left[ \sum_{t=1}^{T} - \log f(\tilde{U}_t|\tilde{Z}_{1:t-1}, \tilde{Z}_{1:t}) \right] \]
\[ = \mathbb{E}_g \left[ -\lambda^T F(\tilde{U}_{1:T}, \tilde{Z}_{1:T}) - \sum_{t=1}^{T-1} Q(\tilde{U}_{1:t}, \tilde{Z}_{1:t}) + \sum_{t=1}^{T} V(\tilde{U}_{1:t-1}, \tilde{Z}_{1:t}) \right] \]
\[ = \mathbb{E}_g \left[ -\lambda^T F(\tilde{U}_{1:T}, \tilde{Z}_{1:T}) - \int_{\tilde{u}_{1:T}, \tilde{x}_{1:T}} g(\tilde{u}_{1:T}, \tilde{z}_{1:T}) \sum_{t=1}^{T-1} \int_{\tilde{z}_{t+1}} f(\tilde{z}_{t+1} | \tilde{u}_{1:t}, \tilde{z}_{1:t}) V(\tilde{u}_{1:t}, \tilde{z}_{1:t+1}) \right. \]
\[ + \int_{\tilde{u}_{1:T}, \tilde{x}_{1:T}} g(\tilde{u}_{1:T}, \tilde{z}_{1:T}) \sum_{t=1}^{T} V(\tilde{u}_{1:t-1}, \tilde{Z}_{1:t}) \]
\[ = \mathbb{E}_g \left[ -\lambda^T F(\tilde{U}_{1:T}, \tilde{Z}_{1:T}) - \sum_{t=1}^{T-1} \int_{\tilde{u}_{1:T}, \tilde{x}_{1:t+1}} g(\tilde{u}_{1:t}, \tilde{z}_{1:t+1}) V(\tilde{u}_{1:t}, \tilde{Z}_{1:t+1}) \right. \]
\[ + \int_{\tilde{u}_{1:T}, \tilde{x}_{1:T}} g(\tilde{u}_{1:T}, \tilde{z}_{1:T}) \sum_{t=1}^{T} V(\tilde{u}_{1:t-1}, \tilde{Z}_{1:t}) \]
which implies equation (42).

For any arbitrary causally conditional probability distribution \( g(\tilde{u}_{1:T} | \tilde{Z}_{1:T}) \) satisfies with expectation constraint (41), we show:
\[ H_g(\tilde{U}_{1:T} | \tilde{Z}_{1:T}) \leq H_f(\tilde{U}_{1:T} | \tilde{Z}_{1:T}) \]
\[ \mathbb{E}_g \left[ -\log g(\tilde{U}_{1:T} | \tilde{Z}_{1:T}) \right] \]
\[ = -\int_{\tilde{u}_{1:T}, \tilde{x}_{1:T}} g(\tilde{u}_{1:T}, \tilde{z}_{1:T}) \log \left( \frac{g(\tilde{u}_{1:T}, \tilde{z}_{1:T}) f(\tilde{z}_{1:T} | \tilde{u}_{1:T-1})}{f(\tilde{u}_{1:T} | \tilde{z}_{1:T}) f(\tilde{z}_{1:T} | \tilde{u}_{1:T-1})} \right) f(\tilde{u}_{1:T} | \tilde{z}_{1:T}) \]
\[ = -D_{KL} \left( g(\tilde{u}_{1:T}, \tilde{z}_{1:T}) | f(\tilde{u}_{1:T}, \tilde{z}_{1:T}) \right) - \int_{\tilde{u}_{1:T}, \tilde{x}_{1:T}} g(\tilde{u}_{1:T}, \tilde{z}_{1:T}) \log f(\tilde{u}_{1:T} | \tilde{z}_{1:T}) \]
\[ \leq -\int_{\tilde{u}_{1:T}, \tilde{x}_{1:T}} g(\tilde{u}_{1:T}, \tilde{z}_{1:T}) \log \hat{f}(\tilde{u}_{1:T} | \tilde{z}_{1:T}) \]
\[ = \int_{\tilde{z}_{1:T}} f(\tilde{z}_{1:T}) V(\tilde{z}_{1:T}) - \mathbb{E}_g \left[ \lambda^T F(\tilde{U}_{1:T}, \tilde{Z}_{1:T}) \right] \]
\[ = \int_{\tilde{z}_{1:T}} f(\tilde{z}_{1:T}) V(\tilde{z}_{1:T}) - \mathbb{E}_f \left[ \lambda^T F(\tilde{U}_{1:T}, \tilde{Z}_{1:T}) \right] \]
\[ = H_f(\tilde{U}_{1:T} | \tilde{Z}_{1:T}) \]

\( D_{KL} \) is the Kullback-Leibler divergence which is non-negative[8]. Thus, \( \hat{f}(\tilde{u}_{1:t-1}, \tilde{z}_{1:t}) \) is the solution to the optimization problem in Lemma 2 incorporates with expectation constraint (41).

**Proof of Theorem 1.** We first incorporate the expectation constraint (20) into the constrained optimization problem defined in Lemma 2
\[ \mathbb{E}_f(\tilde{u}_{1:T}, \tilde{x}_{1:T}, \tilde{x}_{1:T+1}) \left[ \sum_{t=1}^{T+1} \tilde{x}_t \tilde{x}_t^T \right] \]
\[ = \int_{\tilde{u}_{1:T}, \tilde{x}_{1:T+1}, \tilde{x}_{1:T+1}} f(\tilde{u}_{1:T}, \tilde{x}_{1:T}, \tilde{x}_{1:T+1}) f(\tilde{z}_{1:T} | \tilde{u}_{1:T+1}, \tilde{x}_{1:T+1}) \sum_{t=1}^{T+1} \tilde{x}_t \tilde{x}_t^T \]
\[ = \int_{\tilde{u}_{1:T}, \tilde{x}_{1:T}, \tilde{z}_{1:T}} f(\tilde{u}_{1:T}, \tilde{x}_{1:T} | \tilde{u}_{1:T-1}) \int_{\tilde{x}_{1:T+1}, \tilde{x}_{T+1}} f(\tilde{z}_{1:T+1}, \tilde{x}_{1:T+1} | \tilde{u}_{1:T}) \sum_{t=1}^{T+1} \tilde{x}_t \tilde{x}_t^T \]
\[ = \mathbb{E}_f(\tilde{u}_{1:T}, \tilde{x}_{1:T}) \left[ \int_{\tilde{x}_{1:T+1}, \tilde{x}_{T+1}} f(\tilde{z}_{1:T+1}, \tilde{x}_{1:T+1} | \tilde{U}_{1:T}) \sum_{t=1}^{T+1} \tilde{x}_t \tilde{x}_t^T \right] / f(\tilde{Z}_{1:T} | \tilde{U}_{1:T-1}) \]
According to Lemma 3, the solution to the constrained problem defined in Lemma 2 incorporates with the expected constraint (41) takes the following recursive form:

\[
Q(\tilde{u}_{1:t}, \tilde{z}_{1:t}) = \begin{cases} 
\int_{\tilde{x}_{1:T+1}} f(\tilde{z}_{1:T+1} | \tilde{x}_{1:T+1} \mid \tilde{u}_{1:T}) \sum_{t=1}^{T+1} \tilde{z}_t^T \tilde{M}_t, & t = T; \\
\frac{\int_{\tilde{x}_{1:T+1}} f(\tilde{z}_{1:T} \mid \tilde{x}_{1:T} \mid \tilde{u}_{1:T}) \sum_{t=1}^{T} \tilde{z}_t^T \tilde{M}_t,}{E[V(\tilde{U}_{1:T}, \tilde{Z}_{1:T+1}) \mid \tilde{u}_{1:T}, \tilde{z}_{1:T}]} & t < T
\end{cases}
\]

\[
V(\tilde{u}_{1:t-1}, \tilde{z}_{1:t}) = \text{softmax} Q(\tilde{u}_{1:t}, \tilde{z}_{1:t}) \triangleq \log \int_{\tilde{u}_t} c^{Q(\tilde{u}_{1:t}, \tilde{x}_{1:t})} \tilde{u}_t
\]

\[
Q(\tilde{u}_{1:T}, \tilde{z}_{1:T}) = \int_{\tilde{x}_{1:T}} f(\tilde{z}_{1:T} \mid \tilde{x}_{1:T} \mid \tilde{u}_{1:T}) \frac{f(\tilde{x}_{1:T} \mid \tilde{u}_{1:T-1}) E \left[ \tilde{X}_{T+1}^T \tilde{M} \tilde{X}_{T+1} \mid \tilde{X}_{T}, \tilde{u}_T \right]}{f(\tilde{z}_{1:T} \mid \tilde{u}_{1:T-1})}
\]

We define it \(Q'(\tilde{u}_{1:T}, \tilde{z}_{1:T})\)

\[
Q'(\tilde{u}_{1:T}, \tilde{z}_{1:T}) = \int_{\tilde{x}_{1:T}} f(\tilde{x}_{1:T} \mid \tilde{u}_{1:T}) f(\tilde{z}_{1:T} \mid \tilde{x}_{1:T}) E \left[ \tilde{X}_{T+1}^T \tilde{M} \tilde{X}_{T+1} \mid \tilde{X}_{T}, \tilde{u}_T \right]
\]

This is a constant term with respect to \(\tilde{u}_T\), we define it \(W_T\)

\[
Q'(\tilde{u}_{1:T-1}, \tilde{z}_{1:T-1}) = \int_{\tilde{x}_{1:T-1}} f(\tilde{x}_{1:T-1} \mid \tilde{u}_{1:T-1}) f(\tilde{x}_{1:T-1} \mid \tilde{z}_{1:T-1}) E \left[ \tilde{X}_{T}^T \tilde{M} \tilde{X}_{T} \mid \tilde{X}_{T-1}, \tilde{u}_T \right] + W_{T-1}
\]

We define it \(Q'(\tilde{u}_{1:T-1}, \tilde{x}_{1:T-1})\)

And let \(V'(\tilde{u}_{1:T-2}, \tilde{z}_{1:T-1}) = \log \int_{\tilde{u}_{T-1}} Q'(\tilde{u}_{1:T-1}, \tilde{z}_{1:T-1})\)

For \(t < T - 1\), the argument to \(Q'(\tilde{u}_{1:t}, \tilde{z}_{1:t}), V'(\tilde{u}_{1:t-1}, \tilde{z}_{1:t})\) is similar. We redefined \(Q(\tilde{u}_{1:t}, \tilde{z}_{1:t}) = Q'(\tilde{u}_{1:t}, \tilde{z}_{1:t})\) and \(V(\tilde{u}_{1:t-1}, \tilde{z}_{1:t}) = V'(\tilde{u}_{1:t-1}, \tilde{z}_{1:t})\) which gives the recursive form in Theorem 1.

\[\square\]

**Lemma 4.** The distribution of belief state \(\tilde{X}_t \mid b_t \sim N(\mu_{b_t}, \Sigma_{b_t})\) is recursively defined as following and \(\Sigma_{b_t}\) is independent of \(b_t\).

\[
\bar{\mu}_{b_t} = \mu + \Sigma_{b_t} C^T (\Sigma_o + C \Sigma_{b_t} C^T)^{-1} (\tilde{Z}_1 - C \bar{\mu})
\]

\[
\Sigma_{b_t} = \Sigma_d - \Sigma_{d_t} C^T (\Sigma_o + C \Sigma_{d_t} C^T)^{-1} C \Sigma_{d_t}
\]

(43)

(44)

(45)

(46)
Proof of Lemma 4. Since \( \tilde{Z}_1|\tilde{x}_1 \sim N(C\tilde{x}_1, \Sigma_o) \) and \( \tilde{X}_1 \sim N(\tilde{\mu}, \Sigma_d) \), applying Gaussian transformation techniques, it is easy to show that the distribution of initial belief state \( \bar{X}_1|b_1 \) (that is \( X_1|z_1 \)) is a Gaussian distribution with mean (43) and variance (44).

Note that \( f(\tilde{x}_{t+1}|\tilde{x}_t, \tilde{u}_t, b_1) = f(\tilde{x}_{t+1}|\tilde{x}_t, \tilde{u}_t) \quad \bar{X}_{t+1}|\tilde{x}_t, \tilde{u}_t \sim N(A\tilde{x}_t + B\tilde{u}_t, \Sigma_d) \)

\[
f(\tilde{x}_t|\tilde{u}_t, b_1) = f(\tilde{x}_t|b_1) \quad \bar{X}_t|b_1 \sim N(\tilde{\mu}_b, \Sigma_b)
\]

Thus the distribution of \( \tilde{X}_{t+1}|\tilde{u}_t, b_1 \) is:

\[
N \left( \begin{bmatrix} B\tilde{u}_t + A\tilde{\mu}_b \cr C(B\tilde{u}_t + A\tilde{\mu}_b) \end{bmatrix}, \begin{bmatrix} \Sigma_d + A\Sigma_d^T A^T & (\Sigma_d + A\Sigma_d^T A^T)C^T \\ C(C(B\tilde{u}_t + A\tilde{\mu}_b) & \Sigma_o + C(\Sigma_d + A\Sigma_d^T A^T)C^T \\ \end{bmatrix} \right)
\]

Finally, \( f(\tilde{x}_{t+1}|b_{t+1}) = f(\tilde{x}_{t+1}|\tilde{x}_{t+1}, \tilde{u}_t, b_1) = f(\tilde{x}_{t+1}, \tilde{z}_{t+1}|\tilde{u}_t, b_1)/f(\tilde{z}_{t+1}|\tilde{u}_t, b_1) \) which gives the distribution of \( \bar{X}_{t+1}|b_{t+1} \) with mean (45) and variance (46).

Proof of Theorem 2.

\[
E[X_{t+1}^T M \bar{X}_{t+1}|\bar{u}_{1:t}, \bar{z}_{1:t}] = E[\bar{X}_{t+1}^T M \bar{X}_{t+1}|\bar{u}_t, b_1]
\]

\[
= (B\tilde{u}_t + A\tilde{\mu}_b)^T M(B\tilde{u}_t + A\tilde{\mu}_b) + tr(M(\Sigma_d + A\Sigma_d^T A^T))
\]

\[
= \begin{bmatrix} \tilde{u}_t \\ \tilde{\mu}_b \end{bmatrix} \begin{bmatrix} B & A \\ A & \tilde{\mu}_b \end{bmatrix} M \begin{bmatrix} \tilde{u}_t \\ \tilde{\mu}_b \end{bmatrix} + \text{constant}
\]

Thus \( Q(\tilde{u}_{1:T}, \tilde{z}_{1:T}) = Q(\tilde{u}_T, \tilde{\mu}_b) = E[X_{t+1}^T M \bar{X}_{t+1}|\bar{u}_t, b_1] \) gives \( \mathbf{W}_T \).

\[
V(\tilde{u}_{1:T-1}, \tilde{z}_{1:T}) = V(\tilde{\mu}_b) = V(\tilde{z}_T, \tilde{\mu}_{T-1}) = \log \int \tilde{u}_T e^{Q(\tilde{\mu}_{T-1}, \tilde{\mu}_T)}
\]

\[
= \tilde{\mu}_b^T (\mathbf{W}_T(\tilde{\mu}_b) - \mathbf{W}_T(\tilde{\mu}_b) \mathbf{W}_T^{-1}(\tilde{\mu}_b)) \tilde{\mu}_b + \text{constant}
\]

Thus \( \mathbf{D}_T \) gives \( D_{t+1} \).

\[
E[V(\tilde{u}_{1:t+1}, \tilde{z}_{1:t+1})|\tilde{u}_{1:t}, \tilde{z}_{1:t}] = E[V(\tilde{Z}_{t+1}, \tilde{U}_{t+1}, \tilde{\mu}_b)|\tilde{u}_t, \tilde{\mu}_b]
\]

\[
= \mathbf{E}\left[ \begin{bmatrix} \tilde{u}_t \\ \tilde{\mu}_b \end{bmatrix}^T D_{t+1(u, \mu, z)} \tilde{Z}_{t+1} + D_{t+1(z, \mu)} \begin{bmatrix} \tilde{u}_t \\ \tilde{\mu}_b \end{bmatrix} \right]
\]

\[
+ \mathbf{E}\left[ \begin{bmatrix} \tilde{Z}_{t+1} D_{t+1(z, \mu)} \tilde{Z}_{t+1} + D_{t+1(z, \mu)} \begin{bmatrix} \tilde{u}_t \\ \tilde{\mu}_b \end{bmatrix} \right] \begin{bmatrix} \tilde{u}_t \\ \tilde{\mu}_b \end{bmatrix} + \text{constant}
\]

\[
= \begin{bmatrix} \tilde{u}_t \\ \tilde{\mu}_b \end{bmatrix}^T D_{t+1(u, \mu, z)} C_{BA} + C_{BA}^T D_{t+1(z, \mu)}
\]

\[
\quad + C_{BA}^T D_{t+1(z, \mu)} C_{BA} + D_{t+1(z, \mu)} \begin{bmatrix} \tilde{u}_t \\ \tilde{\mu}_b \end{bmatrix}
\]

\( Q(\tilde{u}_t, \tilde{\mu}_b) = E[\bar{X}_{t+1}^T M \bar{X}_{t+1} + V(\tilde{Z}_{t+1}, \tilde{\mu}_b)|\tilde{u}_{1:t}, \tilde{z}_{1:t}] \) which gives \( \mathbf{W}_T(\tilde{\mu}_b) \).

The quadratic form of \( V(\tilde{z}_T, \tilde{\mu}_{T-1}) \) is similar to \( V(\tilde{z}_T, \tilde{\mu}_{T-1}, \tilde{\mu}_{T-1}) \) which gives \( \mathbf{D}_T(\tilde{\mu}_{T-1}) \).
Proof of Theorem 3. It’s easy to check the initial setting $W_T = [B ~ A]^T M [B ~ A]$ matches (5). For general case, we plug $D_{t+1}$ (27) into $W_t$ (26) and check with $W_{t(U,U)}$ first. To simplify proof, let’s define

$$\phi_t = W_{t(\mu,\mu)} - W_{t(U,\mu)}^T W_{t(U,U)}^{-1} W_{t(U,\mu)}.$$ 

Then from (26),(27)

$$W_t = [B ~ A]^T M [B ~ A] + [B - E_{t+1}CB ~ A - E_{t+1}CA]^T \phi_{t+1} E_{t+1} [CB ~ CA] +$$

$$[CB ~ CA]^T E_{t+1}^T \phi_{t+1} [B - E_{t+1}CB ~ A - E_{t+1}CA] +$$

$$[CB ~ CA]^T E_{t+1}^T \phi_{t+1} E_{t+1} [CB ~ CA] +$$

$$[B - E_{t+1}CB ~ A - E_{t+1}CA]^T \phi_{t+1} [B - E_{t+1}CB ~ A - E_{t+1}CA]$$

$$W_{t(U,U)} = B^T MB + (B - E_{t+1}CB)^T \phi_{t+1} E_{t+1} CB + (E_{t+1}CB)^T \phi_{t+1} (B - E_{t+1}CB) +$$

$$(E_{t+1}CB)^T \phi_{t+1} E_{t+1} CB + (B - E_{t+1}CB)^T \phi_{t+1} (B - E_{t+1}CB)$$

$$= B^T MB + B^T \phi_{t+1} B$$

That is $B^T F_{t+1} B = B^T MB + B^T \phi_{t+1} B$. (47)

By plugging out $\phi_{t+1}$, the equation(47) matches equation(5). $W_{t(U,\mu)}, W_{t(U,U)}, W_{t(\mu,\mu)}$ follow similar argument. \qed