A Proofs of Main Theorems

A.1 Proof of Lemma 1

Let \( R_t = R(A_t, w_t) \) be the stochastic regret of CombUCB1 at time \( t \), where \( A_t \) and \( w_t \) are the solution and the weights of the items at time \( t \), respectively. Furthermore, let \( \mathcal{E}_t = \{ 3e \in E : \tilde{w}(e) - \tilde{w}_{T_{t-1}}(e) \geq c_{t-1}, T_{t-1}(e) \} \) be the event that \( \tilde{w}(e) \) is outside of the high-probability confidence interval around \( \tilde{w}_{T_{t-1}}(e) \) for some item \( e \) at time \( t \); and let \( \mathcal{E}_t^c \) be the complement of \( \mathcal{E}_t \), \( \tilde{w}(e) \) is in the high-probability confidence interval around \( \tilde{w}_{T_{t-1}}(e) \) for all \( e \) at time \( t \). Then we can decompose the regret of CombUCB1 as:

\[
R(n) = \mathbb{E} \left[ \sum_{t=1}^{n-1} R_t \right] + \mathbb{E} \left[ \sum_{t=0}^{n} 1 \{ \mathcal{E}_t \} R_t \right] + \mathbb{E} \left[ \sum_{t=0}^{n} 1 \{ \mathcal{E}_t^c \} R_t \right].
\]

Now we bound each term in our regret decomposition.

The regret of the initialization, \( \mathbb{E} \left[ \sum_{t=1}^{n-1} R_t \right] \), is bounded by \( KL \) because Algorithm 2 terminates in at most \( L \) steps, and \( R_t \leq K \) for any \( A_t \) and \( w_t \).

The second term in our regret decomposition, \( \mathbb{E} \left[ \sum_{t=0}^{n} 1 \{ \mathcal{E}_t \} R_t \right] \), is small because all of our confidence intervals hold with high probability. In particular, for any \( e, s, t \) and \( t \):

\[
P \left( |\tilde{w}(e) - \tilde{w}_s(e)| \geq c_{t,s} \right) \leq 2 \exp\left[-3 \log t \right],
\]

and therefore:

\[
\mathbb{E} \left[ \sum_{t=0}^{n} 1 \{ \mathcal{E}_t \} \right] \leq \sum_{e \in E} \sum_{t=1}^{n} \sum_{s=1}^{t} P \left( |\tilde{w}(e) - \tilde{w}_s(e)| \geq c_{t,s} \right) \leq 2 \sum_{e \in E} \sum_{t=1}^{n} \exp\left[-3 \log t \right] \leq 2 \sum_{t=1}^{\infty} t^{-2} \leq \frac{\pi^2}{3} K L.
\]

Since \( R_t \leq K \) for any \( A_t \) and \( w_t \), \( \mathbb{E} \left[ \sum_{t=0}^{n} 1 \{ \mathcal{E}_t \} R_t \right] \leq \frac{\pi^2}{3} KL \).

Finally, we rewrite the last term in our regret decomposition as:

\[
\mathbb{E} \left[ \sum_{t=0}^{n} 1 \{ \mathcal{E}_t^c \} R_t \right] \overset{(a)}{=} \sum_{t=0}^{n} \mathbb{E} \left[ 1 \{ \mathcal{E}_t^c \} \mathbb{E} [R_t | A_t] \right] \overset{(b)}{=} \sum_{t=0}^{n} \mathbb{E} \left[ \Delta_{A_t} 1 \{ \mathcal{E}_t, \Delta_{A_t} > 0 \} \right].
\]

In equality (a), the outer expectation is over the history of the agent up to time \( t \), which in turn determines \( A_t \) and \( \mathcal{E}_t \); and \( \mathbb{E} [R_t | A_t] \) is the expected regret at time \( t \) conditioned on solution \( A_t \). Equality (b) follows from \( \Delta_{A_t} = \mathbb{E} [R_t | A_t] \). Now we bound \( \Delta_{A_t} \{ \mathcal{E}_t, \Delta_{A_t} > 0 \} \) for any suboptimal \( A_t \). The bound is derived based on two facts. First, when CombUCB1 chooses \( A_t, f(A_t, U_t) \geq f(A^*, U_t) \). This further implies that \( \sum_{e \in A_t A^*} U_t(e) \geq \sum_{e \in A A^*} U_t(e) \). Second, when event \( \mathcal{E}_t \) happens, \( |\tilde{w}(e) - \tilde{w}_{T_{t-1}}(e)| < c_{t-1}, T_{t-1}(e) \) for all items \( e \). Therefore:

\[
\sum_{e \in A_t A^*} \tilde{w}(e) + 2 \sum_{e \in A \setminus A^*} c_{t-1}, T_{t-1}(e) \geq \sum_{e \in A \setminus A^*} U_t(e) \geq \sum_{e \in A A^*} U_t(e) \geq \sum_{e \in A A^*} \tilde{w}(e),
\]

and \( 2 \sum_{e \in A \setminus A^*} c_{t-1}, T_{t-1}(e) \geq \Delta_{A_t} \) follows from the observation that \( \Delta_{A_t} = \sum_{e \in A \setminus A^*} \tilde{w}(e) - \sum_{e \in A A^*} \tilde{w}(e) \). Now note that \( c_{t, T_{t-1}(e)} \geq c_{t-1}, T_{t-1}(e) \) for any time \( t \leq n \). Therefore, the event \( \mathcal{F}_t \) in (3) must happen and:

\[
\mathbb{E} \left[ \sum_{t=0}^{n} \Delta_{A_t} 1 \{ \mathcal{E}_t, \Delta_{A_t} > 0 \} \right] \leq \mathbb{E} \left[ \sum_{t=0}^{n} \Delta_{A_t} 1 \{ \mathcal{F}_t \} \right].
\]

This concludes our proof.

A.2 Proof of Theorem 2

By Lemma 1, it remains to bound \( \hat{R}(n) = \sum_{t=0}^{n} \Delta_{A_t} 1 \{ \mathcal{F}_t \} \), where the event \( \mathcal{F}_t \) is defined in (3). By Lemma 2 and from the assumption that \( \Delta_{A_t} = \Delta \) for all suboptimal \( A_t \), it follows that:

\[
\hat{R}(n) = \Delta \sum_{t=0}^{n} 1 \{ \mathcal{F}_t \} = \Delta \sum_{t=0}^{n} 1 \{ G_{1,t}, \Delta_{A_t} > 0 \} + \Delta \sum_{t=0}^{n} 1 \{ G_{2,t}, \Delta_{A_t} > 0 \}.
\]
To bound the above quantity, it is sufficient to bound the number of times that events $G_{1,t}$ and $G_{2,t}$ happen. Then we set the tunable parameters $d$ and $\alpha$ such that the two counts are of the same magnitude.

**Claim 1.** Event $G_{1,t}$ happens at most $\frac{\alpha}{d} K^2 L \frac{\Delta}{\Delta^2} \log n$ times.

**Proof.** Recall that event $G_{1,t}$ can happen only if at least $d$ chosen suboptimal items are not observed “sufficiently often” up to time $t$, $T_{t-1}(e) \leq \alpha K^2 \frac{\Delta}{\Delta^2} \log n$ for at least $d$ items in $A_t$. After the event happens, the observation counters of these items increase by one. Therefore, after the event happens $\frac{\alpha}{d} K^2 L \frac{\Delta}{\Delta^2} \log n$ times, all suboptimal items are guaranteed to be observed at least $\alpha K^2 \frac{\Delta}{\Delta^2} \log n$ times and $G_{1,t}$ cannot happen anymore. ■

**Claim 2.** Event $G_{2,t}$ happens at most $\frac{\alpha d^2}{(\sqrt{\alpha} - 1)^2} L \frac{\Delta}{\Delta^2} \log n$ times.

**Proof.** Event $G_{2,t}$ can happen only if there exists $e \in \hat{A}_t$ such that $T_{t-1}(e) \leq \frac{\alpha d^2}{(\sqrt{\alpha} - 1)^2} \frac{\Delta}{\Delta^2} \log n$. After the event happens, the observation counter of item $e$ increases by one. Therefore, the number of times that event $G_{2,t}$ can happen is bounded trivially by $\frac{\alpha d^2}{(\sqrt{\alpha} - 1)^2} L \frac{\Delta}{\Delta^2} \log n$. ■

Based on Claims 1 and 2, $\hat{R}(n)$ is bounded as:

$$\hat{R}(n) \leq \left( \frac{\alpha}{d} K^2 + \frac{\alpha d^2}{(\sqrt{\alpha} - 1)^2} \right) L \frac{\Delta}{\Delta^2} \log n.$$

Finally, we choose $\alpha = 4$ and $d = K^\frac{2}{3}$; and it follows that the regret is bounded as:

$$R(n) \leq \mathbb{E} \left[ \hat{R}(n) \right] + \left( \frac{\pi^2}{3} + 1 \right) KL \leq K^\frac{2}{3} L \frac{48}{\Delta} \log n + \left( \frac{\pi^2}{3} + 1 \right) KL.$$

### A.3 Proof of Theorem 3

Let $\mathcal{F}_t$ be the event in (3). By Lemmas 1 and 2, it remains to bound:

$$\hat{R}(n) = \sum_{t=t_0}^{n} \Delta_{A_t} \mathbb{I} \{ \mathcal{F}_t \} = \sum_{t=t_0}^{n} \Delta_{A_t} \mathbb{I} \{ G_{1,t}, \Delta_{A_t} > 0 \} + \sum_{t=t_0}^{n} \Delta_{A_t} \mathbb{I} \{ G_{2,t}, \Delta_{A_t} > 0 \}.$$

In the next step, we introduce item-specific variants of events $G_{1,t}$ (6) and $G_{2,t}$ (7), and then associate the regret at time $t$ with these events. In particular, let:

$$G_{e,1,t} = G_{1,t} \cap \left\{ e \in \hat{A}_t, T_{t-1}(e) \leq \alpha K^2 \frac{\Delta}{\Delta^2} \log n \right\}$$

(14)

$$G_{e,2,t} = G_{2,t} \cap \left\{ e \in \hat{A}_t, T_{t-1}(e) \leq \frac{\alpha d^2}{(\sqrt{\alpha} - 1)^2} \frac{\Delta}{\Delta^2} \log n \right\}$$

(15)

be the events that item $e$ is not observed “sufficiently often” under events $G_{1,t}$ and $G_{2,t}$, respectively. Then by the definitions of the above events, it follows that:

$$\mathbb{I} \{ G_{1,t}, \Delta_{A_t} > 0 \} \leq \frac{1}{d} \sum_{e \in \hat{E}} \mathbb{I} \{ G_{e,1,t}, \Delta_{A_t} > 0 \}$$

$$\mathbb{I} \{ G_{2,t}, \Delta_{A_t} > 0 \} \leq \sum_{e \in \hat{E}} \mathbb{I} \{ G_{e,2,t}, \Delta_{A_t} > 0 \},$$

where $\hat{E} = E \setminus A^*$ is the set of suboptimal items; and we bound $\hat{R}(n)$ as:

$$\hat{R}(n) \leq \sum_{e \in \hat{E}} \sum_{t=t_0}^{n} \mathbb{I} \{ G_{e,1,t}, \Delta_{A_t} > 0 \} \frac{\Delta_{A_t}}{d} + \sum_{e \in \hat{E}} \sum_{t=t_0}^{n} \mathbb{I} \{ G_{e,2,t}, \Delta_{A_t} > 0 \} \Delta_{A_t}.$$
Let each item $e$ be contained in $N_e$ suboptimal solutions and $\Delta_{e,1} \geq \ldots \geq \Delta_{e,N_e}$ be the gaps of these solutions, ordered from the largest gap to the smallest one. Then $\hat{R}(n)$ can be further bounded as:

$$\hat{R}(n) \leq \sum_{e \in \hat{E}} \sum_{t=t_0}^{n} \sum_{k=1}^{N_e} \mathbb{1}\{G_{e,1,t}, \Delta_{A_t} = \Delta_{e,k}\} \frac{\Delta_{e,k}}{d} + \sum_{e \in \hat{E}} \sum_{t=t_0}^{n} \sum_{k=1}^{N_e} \mathbb{1}\{G_{e,2,t}, \Delta_{A_t} = \Delta_{e,k}\} \Delta_{e,k}$$

(a) $\leq \sum_{e \in \hat{E}} \sum_{t=t_0}^{n} \sum_{k=1}^{N_e} \mathbb{1}\{e \in \hat{A}_t, T_{t-1}(e) \leq \alpha K^2 6 \frac{1}{\Delta_{e,k}} \log n, \Delta_{A_t} = \Delta_{e,k}\} \Delta_{e,k}$

(b) $\leq \sum_{e \in \hat{E}} \frac{6\alpha K^2 \log n}{d} \left[ \Delta_{e,1} \frac{1}{\Delta_{e,1}} + \sum_{k=2}^{N_e} \Delta_{e,k} \left( \frac{1}{\Delta_{e,k}} - \frac{1}{\Delta_{e,k-1}} \right) \right] + \sum_{e \in \hat{E}} \frac{6\alpha d^2 \log n}{(\sqrt{\alpha} - 1)^2} \left[ \Delta_{e,1} \frac{1}{\Delta_{e,1}} + \sum_{k=2}^{N_e} \Delta_{e,k} \left( \frac{1}{\Delta_{e,k}} - \frac{1}{\Delta_{e,k-1}} \right) \right]$

(c) $\leq \sum_{e \in \hat{E}} \left( \frac{\alpha}{d} K^2 + \frac{6\alpha d^2}{(\sqrt{\alpha} - 1)^2} \right) \frac{12}{\Delta_{e,\min}} \log n$

where inequality (a) is by the definitions of events $G_{e,1,t}$ and $G_{e,2,t}$, inequality (b) is from the solution to:

$$\max \sum_{t=t_0}^{n} \sum_{k=1}^{N_e} \mathbb{1}\{e \in \hat{A}_t, T_{t-1}(e) \leq \frac{C}{\Delta_{e,k}} \log n, \Delta_{A_t} = \Delta_{e,k}\} \Delta_{e,k}$$

for appropriate $C$, and inequality (c) follows from Lemma 3 of Kveton et al. [12]:

$$\left[ \Delta_{e,1} \frac{1}{\Delta_{e,1}} + \sum_{k=2}^{N_e} \Delta_{e,k} \left( \frac{1}{\Delta_{e,k}} - \frac{1}{\Delta_{e,k-1}} \right) \right] \leq \frac{2}{\Delta_{e,N_e}} = \frac{2}{\Delta_{e,\min}}.$$  \hspace{1cm} (16)

Finally, we choose $\alpha = 4$ and $d = K^{\frac{2}{3}}$; and it follows that the regret is bounded as:

$$R(n) \leq \mathbb{E} \left[ \hat{R}(n) \right] + \left( \frac{\pi^2}{3} + 1 \right) KL \leq \sum_{e \in \hat{E}} K^{\frac{2}{3}} \frac{96}{\Delta_{e,\min}} \log n + \left( \frac{\pi^2}{3} + 1 \right) KL.$$

### A.4 Proof of Theorem 4

The first step of the proof is identical to that of Theorem 2. By Lemma 1, it remains to bound $\hat{R}(n) = \sum_{t=t_0}^{n} \Delta_{A_t} \mathbb{1}\{F_t\}$, where the event $F_t$ is defined in (3). By Lemma 3 and from the assumption that $\Delta_{A_t} = \Delta$ for all suboptimal $A_t$, it follows that:

$$\hat{R}(n) = \Delta \sum_{t=t_0}^{n} \mathbb{1}\{F_t\} = \Delta \sum_{t=t_0}^{n} \sum_{i=1}^{n} \mathbb{1}\{G_{i,t}, \Delta_{A_t} > 0\}.$$ 

Note that $\Delta_{A_t} > 0$ implies $\Delta_{A_t} = \Delta$. Therefore, $m_{i,t}$ does not depend on $t$ and we denote it by $m_i = \alpha_i K^2 \frac{\beta_i}{n} \log n$. Based on the same argument as in Claim 1, event $G_{i,t}$ cannot happen more than $\frac{Lm_i}{\beta_i K}$ times, because at least $\beta_i K$ items that are observed at most $m_i$ times have their observation counters incremented in each event $G_{i,t}$. Therefore:

$$\hat{R}(n) \leq \Delta \sum_{i=1}^{\infty} \frac{Lm_i}{\beta_i K} = KL \frac{1}{\Delta} \left( \sum_{i=1}^{\infty} \frac{\alpha_i}{\beta_i} \right) \log n.$$  \hspace{1cm} (17)

It remains to choose $(\alpha_i)$ and $(\beta_i)$ such that:
• \( \lim_{i \to \infty} \alpha_i = \lim_{i \to \infty} \beta_i = 0; \)
• Monotonicity conditions in (9) and (10) hold;
• Inequality (12) holds, \( \sqrt{6} \sum_{i=1}^{\infty} \frac{\beta_i - \beta_{i-1}}{\sqrt{\alpha_i}} \leq 1; \)
• \( \sum_{i=1}^{\infty} \frac{\alpha_i}{\beta_i} \) is minimized.

We choose \((\alpha_i)\) and \((\beta_i)\) to be geometric sequences, \(\beta_i = \beta^i\) and \(\alpha_i = d\alpha^i\) for \(0 < \alpha, \beta < 1\) and \(d > 0\). For this setting, \(\alpha_i \to 0\) and \(\beta_i \to 0\), and the monotonicity conditions are also satisfied. Moreover, if \(\beta < \sqrt{\alpha}\), we have:

\[
\sqrt{6} \sum_{i=1}^{\infty} \frac{\beta_i - \beta_{i-1}}{\sqrt{\alpha_i}} \leq 1
\]

provided that \(d \geq 6 \left( \frac{1 - \beta}{\sqrt{\alpha - \beta}} \right)^2\). Furthermore, if \(\alpha < \beta\), we have:

\[
\sum_{i=1}^{\infty} \frac{\alpha_i}{\beta_i} = \sum_{i=1}^{\infty} \frac{d\alpha^i}{\beta^i} = \frac{d\alpha}{\beta - \alpha}.
\]

Given the above, the best choice of \(d\) is \(6 \left( \frac{1 - \beta}{\sqrt{\alpha - \beta}} \right)^2\) and the problem of minimizing the constant in our regret bound can be written as:

\[
\inf_{\alpha, \beta} \left[ 6 \left( \frac{1 - \beta}{\sqrt{\alpha - \beta}} \right)^2 \frac{\alpha}{\beta - \alpha} \right] \quad \text{s.t.} \quad 0 < \alpha < \beta < \sqrt{\alpha} < 1.
\]

We find the solution to the above problem numerically, and determine it to be \(\alpha = 0.1459\) and \(\beta = 0.2360\). For these \(\alpha\) and \(\beta\), \(6 \left( \frac{1 - \beta}{\sqrt{\alpha - \beta}} \right)^2 \frac{\alpha}{\beta - \alpha} < 267\). We apply this upper bound to (17) and it follows that the regret is bounded as:

\[
R(n) \leq \mathbb{E} \left[ \hat{R}(n) \right] + \left( \frac{\pi^2}{3} + 1 \right) KL \leq KL \frac{267}{\Delta} \log n + \left( \frac{\pi^2}{3} + 1 \right) KL.
\]

### A.5 Proof of Theorem 5

Let \(F_i\) be the event in (3). By Lemmas 1 and 3, it remains to bound:

\[
\hat{R}(n) = \sum_{t=0}^{n} \Delta_{A_t} 1\{F_t\} = \sum_{i=1}^{\infty} \sum_{t=0}^{n} \Delta_{A_t} 1\{G_{i,t}, \Delta_{A_t} > 0\}.
\]

In the next step, we define item-specific variants of events \(G_{i,t}\) (11) and associate the regret at time \(t\) with these events. In particular, let:

\[
G_{e,i,t} = G_{i,t} \cap \left\{ e \in \tilde{A}_t, T_{t-1}(e) \leq m_{i,t} \right\}
\]

be the event that item \(e\) is not observed “sufficiently often” under event \(G_{i,t}\). Then it follows that:

\[
1\{G_{i,t}, \Delta_{A_t} > 0\} \leq \frac{1}{\beta_i K} \sum_{e \in E} 1\{G_{e,i,t}, \Delta_{A_t} > 0\},
\]

because at least \(\beta_i K\) items are not observed “sufficiently often” under event \(G_{i,t}\). Therefore, we can bound \(\hat{R}(n)\) as:

\[
\hat{R}(n) \leq \sum_{e \in E} \sum_{i=1}^{\infty} \sum_{t=0}^{n} 1\{G_{e,i,t}, \Delta_{A_t} > 0\} \frac{\Delta_{A_t}}{\beta_i K}.
\]
Let each item $e$ be contained in $N_e$ suboptimal solutions and $\Delta_{e,1} \geq \ldots \geq \Delta_{e,N_e}$ be the gaps of these solutions, ordered from the largest gap to the smallest one. Then $\hat{R}(n)$ can be further bounded as:

$$
\hat{R}(n) \leq \sum_{e \in E} \sum_{t=t_0}^{\infty} \sum_{i=1}^{n} \sum_{k=1}^{N_e} \mathbb{I}\{G_{e,i,t}, \Delta_{A_e} = \Delta_{e,k}\} \Delta_{e,k} \beta_i \frac{K}{l_i} 
$$

where inequality (a) is by the definition of event $G_{e,i,t}$, inequality (b) follows from the solution to:

$$
\max_{A_1, \ldots, A_e} \sum_{t=t_0}^{n} \sum_{i=1}^{N_e} \mathbb{I}\{e \in A_t, T_{t-1}(e) \leq \alpha_i K^2 \Delta_{e,k}^2 \log n, \Delta_{A_e} = \Delta_{e,k}\} \Delta_{e,k} \beta_i \frac{K}{l_i},
$$

and inequality (c) follows from (16). For the same $(\alpha_i)$ and $(\beta_i)$ as in Theorem 4, we have $\sum_{i=1}^{\infty} \frac{\alpha_i}{\beta_i} < 267$ and it follows that the regret is bounded as:

$$
R(n) \leq \mathbb{E}\left[ \hat{R}(n) \right] + \left( \frac{\pi^2}{3} + 1 \right) KL \leq K \sum_{e \in E} \frac{534}{\Delta_{e,\min}} \log n + \left( \frac{\pi^2}{3} + 1 \right) KL.
$$

### A.6 Proof of Theorem 6

The key idea is to decompose the regret of CombUCB1 into two parts, where the gaps are larger than $\epsilon$ and at most $\epsilon$. We analyze each part separately and then set $\epsilon$ to get the desired result.

By Lemma 1, it remains to bound $\hat{R}(n) = \sum_{t=t_0}^{n} \Delta_{A_e} \mathbb{I}\{F_t\}$, where the event $F_t$ is defined in (3). We partition $\hat{R}(n)$ as:

$$
\hat{R}(n) = \sum_{t=t_0}^{n} \Delta_{A_e} \mathbb{I}\{F_t, \Delta_{A_e} < \epsilon\} + \sum_{t=t_0}^{n} \Delta_{A_e} \mathbb{I}\{F_t, \Delta_{A_e} \geq \epsilon\}
$$

and bound the first term trivially. The second term is bounded in the same way as $\hat{R}(n)$ in the proof of Theorem 5, except that we only consider the gaps $\Delta_{e,k} \geq \epsilon$. Therefore, $\Delta_{e,\min} \geq \epsilon$ and we get:

$$
\sum_{t=t_0}^{n} \Delta_{A_e} \mathbb{I}\{F_t, \Delta_{A_e} \geq \epsilon\} \leq \sum_{e \in E} K \frac{534}{\epsilon} \log n \leq KL \frac{534}{\epsilon} \log n.
$$

Based on the above inequalities:

$$
R(n) \leq \frac{534KL}{\epsilon} \log n + \epsilon n + \left( \frac{\pi^2}{3} + 1 \right) KL.
$$

Finally, we choose $\epsilon = \sqrt{\frac{534KL \log n}{n}}$ and get:

$$
R(n) \leq 2 \sqrt{534KLn \log n} + \left( \frac{\pi^2}{3} + 1 \right) KL < 47 \sqrt{KLn \log n} + \left( \frac{\pi^2}{3} + 1 \right) KL,
$$

which concludes our proof.
B Technical Lemmas

Lemma 4. Let $S_i$, $\bar{S}_i$, and $m_i$ be defined as in Lemma 3; and $|S_i| < \beta_i K$ for all $i > 0$. Then:

$$\sum_{i=1}^{\infty} \frac{|\bar{S}_i \setminus S_{i-1}|}{\sqrt{m_i}} < \sum_{i=1}^{\infty} \frac{(\beta_{i-1} - \beta_i) K}{\sqrt{m_i}}.$$ 

Proof. The lemma is proved as:

$$\sum_{i=1}^{\infty} |\bar{S}_i \setminus S_{i-1}| \frac{1}{\sqrt{m_i}} = \sum_{i=1}^{\infty} (|S_{i-1} \setminus S_i|) \frac{1}{\sqrt{m_i}}$$

$$= \sum_{i=1}^{\infty} (|S_{i-1}| - |S_i|) \frac{1}{\sqrt{m_i}}$$

$$= \frac{|S_0|}{\sqrt{m_1}} + \sum_{i=1}^{\infty} |S_i| \left( \frac{1}{\sqrt{m_{i+1}}} - \frac{1}{\sqrt{m_i}} \right)$$

$$< \frac{\beta_0 K}{\sqrt{m_1}} + \sum_{i=1}^{\infty} \beta_i K \left( \frac{1}{\sqrt{m_{i+1}}} - \frac{1}{\sqrt{m_i}} \right)$$

$$= \sum_{i=1}^{\infty} (\beta_{i-1} - \beta_i) K \frac{1}{\sqrt{m_i}}.$$ 

The first two equalities follow from the definitions of $\bar{S}_i$ and $S_i$. The inequality follows from the facts that $|S_i| < \beta_i K$ for all $i > 0$ and $|S_0| \leq \beta_0 K$. ⌣