

APPENDIX

Multidimensional Projection Series Functional Estimation

Let $y_{ij} = p_i(u_{ij}) + \epsilon_{ij}$. For readability let $(\nu, \gamma, A) = (\nu_x, \gamma_x, A_x)$. Given P_i as in (4) our estimator for $p_i \in \mathcal{I}$ (31) will be:

$$\tilde{p}_i(x) = \sum_{\alpha : \kappa_\alpha(\nu, \gamma) \leq t} a_\alpha(P_i) \varphi_\alpha(x) \quad \text{where} \quad (40)$$

$$a_\alpha(P_i) = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij} \varphi_\alpha(u_{ij}). \quad (41)$$

That is, the set of indices for basis functions for input functions will be:

$$U \equiv M_t^x \equiv \{\alpha : \kappa_\alpha(\nu, \gamma) \leq t\}. \quad (42)$$

First, note that:

$$\begin{aligned} & \mathbb{E} [\|p_i - \tilde{p}_i\|_2^2] \\ &= \mathbb{E} \left[\left\| \sum_{\alpha \in \mathbb{Z}} a_\alpha(p_i) \varphi_\alpha - \sum_{\alpha \in M_t^x} a_\alpha(P_i) \varphi_\alpha \right\|_2^2 \right] \\ &= \mathbb{E} \left[\int_{[0,1]^l} \left(\sum_{\alpha \in M_t^x} (a_\alpha(p_i) - a_\alpha(P_i)) \varphi_\alpha(x) \right. \right. \\ & \quad \left. \left. + \sum_{\alpha \in (M_t^x)^c} a_\alpha(p_i) \varphi_\alpha(x) \right)^2 dx \right] \\ &= \mathbb{E} \left[\int_{[0,1]^l} \sum_{\alpha \in M_t^x} \sum_{\rho \in M_t^x} (a_\alpha(p_i) - a_\alpha(P_i)) \right. \\ & \quad \left. (a_\rho(p_i) - a_\rho(P_i)) \varphi_\alpha(x) \varphi_\rho(x) dx \right] \\ & \quad + 2 \mathbb{E} \left[\int_{[0,1]^l} \sum_{\alpha \in M_t^x} \sum_{\rho \in (M_t^x)^c} (a_\alpha(p_i) - a_\alpha(P_i)) \right. \\ & \quad \left. a_\rho(p_i) \varphi_\alpha(x) \varphi_\rho(x) dx \right] \\ & \quad + \mathbb{E} \left[\int_{[0,1]^l} \sum_{\alpha \in (M_t^x)^c} \sum_{\rho \in (M_t^x)^c} \right. \\ & \quad \left. a_\alpha(p_i) a_\rho(p_i) \varphi_\alpha(x) \varphi_\rho(x) dx \right] \\ &= \mathbb{E} \left[\sum_{\alpha \in M_t^x} (a_\alpha(p_i) - a_\alpha(P_i))^2 \right] + \mathbb{E} \left[\sum_{\alpha \in (M_t^x)^c} a_\alpha^2(p_i) \right], \end{aligned} \quad (44)$$

where the last line follows from the orthonormality of $\{\varphi\}_{\alpha \in \mathbb{Z}}$. Furthermore, note that $\forall p_i \in \mathcal{I}$:

$$\sum_{\alpha \in (M_t^x)^c} a_\alpha^2(p_i) = \frac{1}{t^2} \sum_{\alpha \in (M_t^x)^c} t^2 a_\alpha^2(p_i) \quad (45)$$

$$\leq \frac{1}{t^2} \sum_{\alpha \in \mathbb{Z}} \kappa_\alpha^2(\nu, \gamma) a_\alpha^2(p_i) \leq \frac{A}{t^2}. \quad (46)$$

Also,

$$\mathbb{E} [(a_\alpha(p_i) - a_\alpha(P_i))^2] = (\mathbb{E} [a_\alpha(P_i)] - a_\alpha(p_i))^2 + \text{Var} [a_\alpha(P_i)].$$

We may see that $a_\alpha(P_i)$ is unbiased. Let $u \sim \text{Unif}([0, 1]^l)$, then:

$$\begin{aligned} & a_\alpha(p_i) \\ &= \int_{[0,1]^l} \varphi_\alpha(z) p_i(z) (1) dz = \mathbb{E} [\varphi_\alpha(u) p_i(u)] \\ &= \mathbb{E} \left[\frac{1}{n_i} \sum_{j=1}^{n_i} p(u_{ij}) \varphi_\alpha(u_{ij}) \right] \\ &= \mathbb{E} \left[\frac{1}{n} \sum_{j=1}^n p(u_{ij}) \varphi_\alpha(u_{ij}) \right] + \mathbb{E} \left[\frac{1}{n} \sum_{j=1}^n \epsilon_{ij} \varphi_\alpha(u_{ij}) \right] \\ &= \mathbb{E} \left[\frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij} \varphi_\alpha(u_{ij}) \right] = \mathbb{E} [a_\alpha(P_i)] \end{aligned}$$

Also,

$$\begin{aligned} \text{Var} [a_\alpha(P_i)] &= \frac{1}{n_i^2} \sum_{j=1}^{n_i} \text{Var} [y_{ij} \varphi_\alpha(u_{ij})] \\ &\leq \frac{1}{n_i^2} \sum_{j=1}^{n_i} \mathbb{E} [(y_{ij} \varphi_\alpha(u_{ij}))^2] \\ &\leq \frac{\varphi_{\max}^2}{n_i^2} \sum_{j=1}^{n_i} \mathbb{E} [(p_i(u_{ij}) + \epsilon_{ij})^2] \\ &= \frac{\varphi_{\max}^2}{n_i^2} \sum_{j=1}^{n_i} \mathbb{E} [p_i(u_{ij})^2 + 2p_i(u_{ij})\epsilon_{ij} + \epsilon_{ij}^2] \\ &\leq \frac{\varphi_{\max}^2 (A^2 + \varsigma^2) n_i}{n_i^2} = O(n_i^{-1}), \end{aligned}$$

where $\varphi_{\max} \equiv \max_{\alpha \in \mathbb{Z}^l} \|\varphi_\alpha\|_\infty$. Thus,

$$\mathbb{E} [\|p_i - \tilde{p}_i\|_2^2] \leq \frac{C_1 |M_t^x|}{n_i} + \frac{C_2}{t^2}.$$

First note that if we have a bound $\forall \alpha \in M_t^x, |\alpha_i| \leq c_i$ then $|M_t^x| \leq \prod_{i=1}^l (2c_i + 1)$, by a simple counting argument. Let $\lambda = \arg \min_i \nu_i^{2\gamma_i}$. For $\alpha \in M_t^x$ we have:

$$\sum_{i=1}^l |\alpha_i|^{2\gamma_i} \leq \frac{1}{\nu_\lambda^{2\gamma_\lambda}} \sum_{i=1}^l (\nu_i |\alpha_i|)^{2\gamma_i} = \frac{\kappa_\alpha^2(\nu, \gamma)}{\nu_\lambda^{2\gamma_\lambda}} \leq \frac{t^2}{\nu_\lambda^{2\gamma_\lambda}}, \quad (47)$$

and

$$|\alpha_i|^{2\gamma_i} \leq \sum_{i=1}^l |\alpha_i|^{2\gamma_i} \leq t^2 \nu_\lambda^{-2\gamma_\lambda} \implies |\alpha_i| \leq \nu_\lambda^{-\frac{\gamma_\lambda}{\gamma_i}} t^{\frac{1}{\gamma_i}}. \quad (48)$$

Thus, $|M_t^x| \leq \prod_{i=1}^l (2\nu_\lambda^{-\frac{\gamma_\lambda}{\gamma_i}} t^{\frac{1}{\gamma_i}} + 1)$. Thus, $|M_t^x| = O(t^{\gamma^{-1}})$ where $\gamma^{-1} = \sum_{j=1}^l \gamma_j^{-1}$. Hence,

$$\begin{aligned} \frac{\partial}{\partial t} \left[\frac{C_1 t^{\gamma^{-1}}}{n_i} + \frac{C_2}{t^2} \right] &= \frac{C_1' t^{\gamma^{-1}-1}}{n_i} - C_2' t^{-3} = 0 \implies \\ t &= C n^{\frac{1}{2+\gamma^{-1}}} \implies \\ \mathbb{E} [\|p_i - \tilde{p}_i\|_2^2] &\leq \frac{C_1 |M_t^x|}{n_i} + \frac{C_2}{t^2} = O\left(n_i^{-\frac{2}{2+\gamma^{-1}}}\right). \end{aligned}$$

A similar result may be reached for \tilde{q}_i functions.

Theory

Assumptions

A.1 Sobolev Input/Output Functions. Suppose that (31) (32) hold.

A.2 FFR Mapping. We shall assume that $f \in \mathcal{F}_\sigma$ as in (14), (37).

A.3 Functional Observations Suppose that (4) holds and $n_i, m_i \asymp n$.

A.4 RKS Features Suppose that the number of RKS features D (16) is taken to be $D \asymp n \log(n)$.

Lemma 7.1. Let $V = M_u^\circ$ where V is defined as (17) and M_u° is analogous to (42). Also, let $\hat{q}_0(x) = \sum_{\alpha \in M_u^\circ} \hat{f}_\alpha(P_0) \varphi_\alpha(x)$. If $\forall \alpha \in \mathbb{Z}^k$, $\mathbb{E}[(a_\alpha(q_0) - \hat{f}_\alpha(P_0))^2] = O(\mathcal{R}(N, n))$, then $\mathbb{E}[\|q_0 - \hat{q}_0\|_2^2] = O(\mathcal{R}(N, n)^{2/2+\gamma_\circ^{-1}})$, where $\gamma_\circ^{-1} = \sum_{j=1}^k (\gamma_\circ)_j^{-1}$.

Proof. Note that:

$$\|q_0 - \hat{q}_0\|_2^2 = \sum_{\alpha \in M_u^\circ} (a_\alpha(q_0) - \hat{f}_\alpha(P_0))^2 \quad (49)$$

$$+ \sum_{\alpha \in (M_u^\circ)^c} a_\alpha^2(p_i), \quad (50)$$

by the orthonormality of $\{\varphi_\alpha\}_{\alpha \in \mathbb{Z}}$ (see above). Then,

$$\mathbb{E}[\|q_0 - \hat{q}_0\|_2^2] \leq \sum_{\alpha \in M_u^\circ} \mathbb{E}[(a_\alpha(q_0) - \hat{f}_\alpha(P_0))^2] \quad (51)$$

$$+ \mathbb{E} \left[\sum_{\alpha \in (M_u^\circ)^c} a_\alpha^2(q_0) \right]. \quad (52)$$

Furthermore, since $q_0 \in \Theta_k(\nu_\circ, \gamma_\circ, A_\circ)$,

$$\begin{aligned} \sum_{\alpha \in (M_u^\circ)^c} a_\alpha^2(q_0) &= \frac{1}{u^2} \sum_{\alpha \in (M_u^\circ)^c} u^2 a_\alpha^2(q_0) \quad (53) \\ &\leq \frac{1}{u^2} \sum_{\alpha \in \mathbb{Z}} \kappa_\alpha^2(\nu_\circ, \gamma_\circ) a_\alpha^2(q_0) \leq \frac{A_\circ^2}{u^2}. \end{aligned} \quad (54)$$

Thus, $\mathbb{E}[\|q_0 - \hat{q}_0\|_2^2] = O(\mathcal{R}(N, n) |M_u^\circ| + u^{-2})$.

For simplicity of notation let $(\nu, \gamma, A) = (\nu_\circ, \gamma_\circ, A_\circ)$. By an argument similar to (47) and (48) we have that $|M_u^\circ| = O(u^{\gamma^{-1}})$ where $\gamma^{-1} = \sum_{j=1}^l \gamma_j^{-1}$. Hence choosing $u \asymp \mathcal{R}(N, n)^{-1/(2+\gamma^{-1})}$ yields $\mathbb{E}[\|q_0 - \hat{q}_0\|_2^2] = O(\mathcal{R}(N, n)^{2/2+\gamma_\circ^{-1}})$. \square

For ease of notation below, let $T_B(x) \equiv \text{sign}(x) \min(|x|, B)$ and henceforth redefine $\hat{f}_\alpha(P_0)$ as $\hat{f}_\alpha(P_0) \equiv T_{B_\alpha}(\hat{f}_\alpha(P_0))$, where $\hat{f}_\alpha(P_0)$ on the RHS refers to (25) and B_α is as in (37).

Lemma 7.2. Let a small constant $\delta > 0$ be fixed. Then, asymptotically $\forall \alpha \in \mathbb{Z}^k$,

$$\begin{aligned} \mathbb{E}[(f_\alpha(p) - \hat{f}_\alpha(P_0))^2] &= \\ O\left(n^{-1/(2+\gamma_\circ^{-1})} + \max(1/n, B_\alpha) \frac{n \log(n) \log(N)}{N}\right) \end{aligned}$$

with probability at least $1 - \delta$.

Proof. Note that $\hat{f}_\alpha(P_0)$ is a function to real estimator attempting to estimate the mapping $p_0 \mapsto f_\alpha(p_0)$. Note that $a_\alpha(q_i) = a_\alpha(Q_i) + \epsilon_{\alpha i}$, where $\mathbb{E}[\epsilon_\alpha] = 0$ and $\text{Var}[\epsilon_\alpha] = O(1/n)$ (see above). Also, $\hat{f}_\alpha(P_0)$ is trained using a dataset $\mathcal{D}_\alpha = \{(P_i, a_\alpha(Q_i))\}_{i=1}^N = \{(P_i, a_\alpha(q_i) + \epsilon_{\alpha i})\}_{i=1}^N$. Thus, a straightforward analogue (using general functions rather than distributions) of the rate derived by Oliva et al. (2014a) yields the result. \square

Theorem 5.1. Let a small constant $\delta > 0$ be fixed. Suppose that $\hat{q}_0(x) = \sum_{\alpha \in M_u^\circ} \hat{f}_\alpha(P_0) \varphi_\alpha(x)$. Furthermore, suppose that (31) and (32) holds, and $f \in \mathcal{F}_\sigma$ as in (37). Moreover, assume that (4) holds and $n_i, m_i \asymp n$. Also, assume that the number of RKS features D (16) is taken to be $D \asymp n \log(n)$. Then,

$$\begin{aligned} \mathbb{E}[\|q_0 - \hat{q}_0\|_2^2] &\leq O\left(\left(n^{-1/(2+\gamma_\circ^{-1})} + \frac{n \log(n) \log(N)}{N}\right)^{2/(2+\gamma_\circ^{-1})}\right) \end{aligned}$$

with probability at least $1 - \delta$.

Proof. Follows from Lemmas 7.1 and 7.2. \square