Inferring Block Structure of Graphical Models in Exponential Families (Supplemantary Material)

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1 Detailed calculation of Proposition 2

It is not difficult to infer the form variational distribution for Θ based on its complete conditionals, where $q(\beta|\lambda)$ is Beta distribution, $q(\pi|\gamma)$ is Dirichlet distribution, and $q(z|\phi)$ is categorical distribution. More specifically,

$$
\begin{aligned}
E_q[\log p(\beta_k|\eta_k)] &= E_q[(\eta-1)\log\beta_k] + E_q[(\eta-1)\log(1-\beta_k)] - C \\
&= (\eta-1)(\psi(\lambda_{k1}) + \psi(\lambda_{k2}) - \psi(\lambda_{k1}+\lambda_{k2})) \\
E_q[\log q(\beta_k|\lambda_k)] &= (\lambda_{k1}-1)E_q[\log\beta_k] + (\lambda_{k2}-1)E_q[\log(1-\beta_k)] - C \\
&= (\lambda_{k1}-1)\psi(\lambda_{k1}) + (\lambda_{k2}-1)\psi(\lambda_{k2}) - (\lambda_{k1}+\lambda_{k2}-2)\psi(\lambda_{k1}+\lambda_{k2})) \\
E_q[\log p(\pi_a|\alpha)] &= \alpha E_q[\sum_k \log\pi_{ak}] = \alpha\sum_k \psi(\gamma_{ak}) - \psi(\gamma_a) \\
E_q[\log q(\pi_a|\gamma_a)] &= E_q[\sum_k \gamma_{ak}\log\pi_{ak}] = \sum_k \gamma_{ak}(\psi(\gamma_{ak}) - \psi(\gamma_a)) \\
E_q[\log p(z_{a\to b}|\pi_a)] &= E_q[\sum_k z^k_{a\to b}\log\pi_{ak}] = \sum_k E_q[z^k_{a\to b}]E_q[\log\pi_{ak}] = \sum_k \phi^k_{a\to b}(\psi(\gamma_{ak}) - \psi(\gamma_a)) \\
E_q[\log p(z_{a\leftarrow b}|\pi_b)] &= E_q[\sum_k z^k_{a\leftarrow b}\log\pi_{bk}] = \sum_k E_q[z^k_{a\leftarrow b}]E_q[\log\pi_{bk}] = \sum_k \phi^k_{a\leftarrow b}(\psi(\gamma_{bk}) - \psi(\gamma_b)) \\
E_q[\log q(z_{a\to b}|\phi_{a\to b})] &= E_q[\sum_k z^k_{a\to b}\log\phi^k_{a\to b}] = \sum_k \phi^k_{a\to b}\log\phi^k_{a\to b} \\
E_q[\log q(z_{a\leftarrow b}|\phi_{a\leftarrow b})] &= E_q[\sum_k z^k_{a\leftarrow b}\log\phi^k_{a\leftarrow b}] = \sum_k \phi^k_{a\leftarrow b}\log\phi^k_{a\leftarrow b} \\
E_q[\log p(\bar{y}_{ab}|z_{a\to b}, z_{a\leftarrow b}, \beta)] &= E_q[\bar{w}_{ab}\log r_{ab} + (1-\bar{w}_{ab})\log(1-r_{ab})] \\
&= \bar{w}_{ab}\sum_k \phi^k_{a\to b}\phi^k_{a\leftarrow b}(\psi(\lambda_{k1}) - \psi(\lambda_k)) + (1 - \sum_k \phi^k_{a\to b}\phi^k_{a\leftarrow b})\log\epsilon \\
&+ (1-\bar{w}_{ab})(\sum_k \phi^k_{a\to b}\phi^k_{a\leftarrow b}(\psi(\lambda_{k2}) - \psi(\lambda_k)) + (1 - \sum_k \phi^k_{a\to b}\phi^k_{a\leftarrow b})\log(1-\epsilon))
\end{aligned}
$$

Therefore the ELBO can be calculated accordingly.

2 Proposition 1

$$
\begin{aligned}
P(X,w,\Theta) =& P(X|w)P(w|\Theta)P(\Theta) \\
=& \Big(\prod_a P(X_a|X_{-a},w)\Big)\Big(\prod_{a,b} P(w_{ab}|\Theta)\Big)P(\beta|\eta)P(z|\pi)P(\pi|\alpha) \quad (1) \\
& \quad (2)
\end{aligned}
$$

And it is eay to see that

$$
\begin{aligned}
P(X|w) =& \prod_{i=1}^{n} \exp(\sum_{a=1}^{p}\sum_{b=1}^{p} w_{ab}X_a^i X_b^i + C(X_a) - D(X_{-a})) \\
P(w|\Theta) =& \prod_{a,b} \frac{1}{2\rho_{ab}(r_{ab})} \exp(-\frac{|w_{ab}|}{\rho_{ab}(r_{ab})}) \\
P(\Theta) =& \prod_k \frac{1}{B(\eta_{k1},\eta_{k2})} \beta_{k1}^{\eta_{k1}-1}(1-\beta_{k1})^{\eta_{k2}-1} \\
& \prod_{a\leq b}\prod_k \pi_{a,k}^{z_{a\rightarrow b}^k}\pi_{b,k}^{z_{a\leftarrow b}^k} \prod_a \frac{1}{B(\alpha)} \prod_k \pi_{a,k}^{\alpha_k - 1}, \quad (3)
\end{aligned}
$$