Stochastic Block Transition Models for Dynamic Networks: Supplementary Material

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A Proofs

Proof of Proposition 1. Begin with property 1. Let $i \in a$ and $j \in b$ at both times $t-1$ and t. From (15) and (16),

$$\begin{aligned}\xi_{ij}^{t|0} &= \alpha(a,b) + \frac{\beta(a,b) - \alpha(a,b)}{\gamma(a,b)} \\ &= \alpha(a,b) + (1 - \alpha(a,b)) \\ &= 1. \end{aligned} \tag{A.1}$$

Substituting (A.1) and (7) into (10),

$$\xi_{ij}^{t|1} = \frac{\pi_{ab}^{t|0}(\theta_{ab}^{t-1} - 1)}{\pi_{ab}^{t|1}\theta_{ab}^{t-1}} + \frac{\pi_{ab}^{t|0}(1 - \theta_{ab}^{t-1}) + \pi_{ab}^{t|1}\theta_{ab}^{t-1}}{\pi_{ab}^{t|1}\theta_{ab}^{t-1}} = 1.$$

Thus property 1 is satisfied.

From the derivation of the scaling factor assignment, it was shown that properties 2 and 3 are satisfied provided $\gamma(a', b') \geq 1$ for all (a', b'). From (16), this is true if and only if $\beta(a,b) \geq 1$ for all (a,b). From (14), $\beta(a,b) \geq 1/\pi_{ab}^{t|0} \geq 1$ because $\pi_{ab}^{t|0}$ is a probability and hence must be between 0 and 1, and

$$\beta(a,b) \geq \frac{\theta_{ab}^{t}}{\pi_{ab}^{t|0}(1 - \theta_{ab}^{t-1})} = 1 + \frac{\pi_{ab}^{t|1}\theta_{ab}^{t-1}}{\pi_{ab}^{t|0}(1 - \theta_{ab}^{t-1})} \geq 1,$$

where the equality follows from (7), and the final inequality results from $\pi_{ab}^{t|0}$, $\pi_{ab}^{t|1}$, and θ_{ab}^{t-1} all being probabilities and hence between 0 and 1. Thus properties 2 and 3 are also satisfied. □

Proof of Theorem 1. The scaled adjacencies w_{ij}^t/ξ_{ij}^t are independent, but not identically distributed, so the classical Central Limit Theorem (CLT) no longer applies. However, the Lyapunov CLT can be applied provided Lyapunov's condition is satisfied (Billingsley, 1995). Let $\mathrm{Var}_u(\cdot)$ denote the conditional variance $\mathrm{Var}(\cdot|w_{ij}^{t-1} = u)$. The conditional variance of the scaled adjacencies is given by

$$\mathrm{Var}_u\left(\frac{w_{ij}^t}{\xi_{ij}^t}\right) = \left(\frac{1}{\xi_{ij}^t}\right)^2 \left(\xi_{ij}^t \pi_{ab}^{t|u}\right)\left(1 - \xi_{ij}^t \pi_{ab}^{t|u}\right) = \frac{\pi_{ab}^{t|u}}{\xi_{ij}^t} - \left(\pi_{ab}^{t|u}\right)^2.$$

Thus

$$\sum_{(i,j)\in\mathcal{B}_{ab}^{t|u}} \mathrm{Var}_u\left(\frac{w_{ij}^t}{\xi_{ij}^t}\right) = \pi_{ab}^{t|u} \sum_{(i,j)\in\mathcal{B}_{ab}^{t|u}} \frac{1}{\xi_{ij}^t} - n_{ab}^{t|u}\left(\pi_{ab}^{t|u}\right)^2 = \left(s_{ab}^{t|u}\right)^2,$$

where $s_{ab}^{t|u}$ was defined in (19). In this setting, Lyapunov's condition specifies that for some $\delta > 0$,

$$\lim_{n_{ab}^{t|u}\to\infty} \frac{1}{\left(s_{ab}^{t|u}\right)^{2+\delta}} \sum_{(i,j)\in\mathcal{B}_{ab}^{t|u}} E_u\left[\left|\frac{w_{ij}^t}{\xi_{ij}^t} - \pi_{ab}^{t|u}\right|^{2+\delta}\right] = 0,$$

where $E_u[\cdot]$ denotes the conditional expectation $E[\cdot|w_{ij}^{t-1} = u]$.

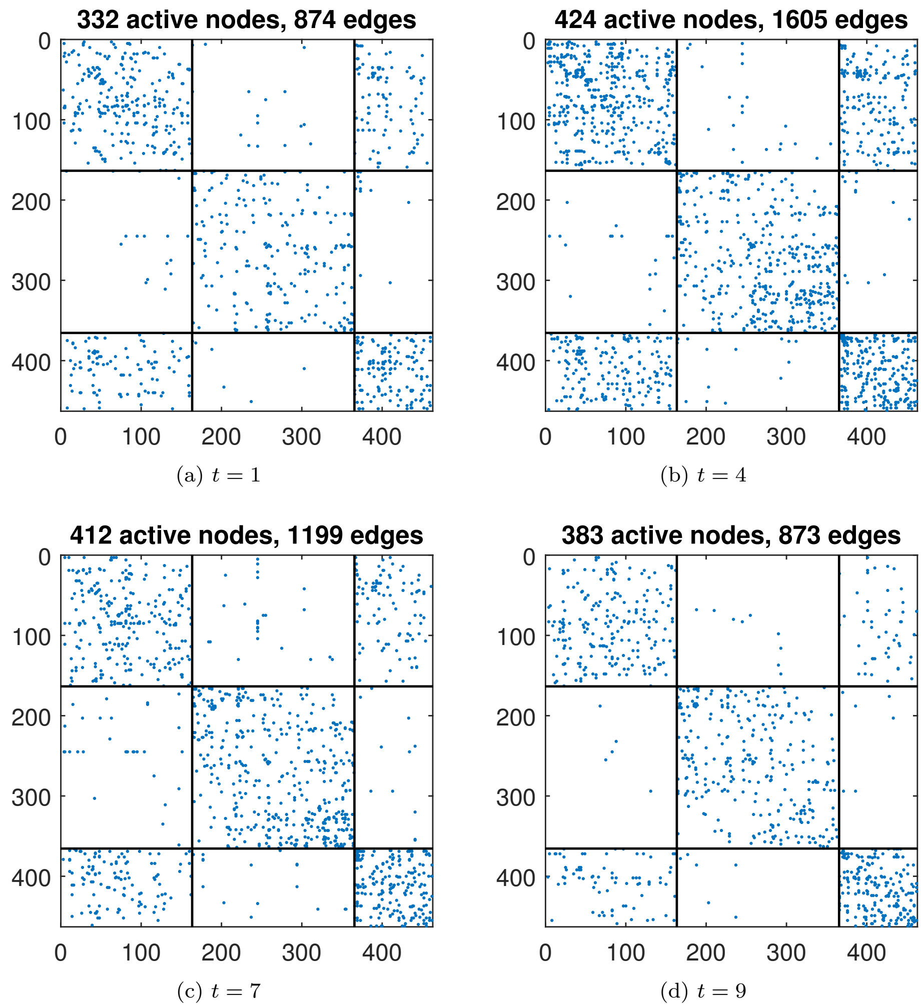
I demonstrate that Lyapunov's condition is satisfied for $\delta = 2$. First note that, although there are an infinite number of terms in the summation (in the limit), there are a finite number of unique terms. Specifically $w_{ij}^t \in \{0,1\}$, and ξ_{ij}^t depends only on i, j through their current and previous class memberships a, b, a', and b', which are all in $\{0, 1, \ldots, k\}$. Hence

$$\begin{aligned}
&\frac{1}{\left(s_{ab}^{t|u}\right)^4} \sum_{(i,j)\in\mathcal{B}_{ab}^{t|u}} E_u\left[\left(\frac{w_{ij}^t}{\xi_{ij}^t} - \pi_{ab}^{t|u}\right)^4\right] \qquad \text{(A.2)}\\
&\le \frac{n_{ab}^{t|u}}{\left(s_{ab}^{t|u}\right)^4} \max_{(i,j)\in\mathcal{B}_{ab}^{t|u}} E_u\left[\left(\frac{w_{ij}^t}{\xi_{ij}^t} - \pi_{ab}^{t|u}\right)^4\right]\\
&= \frac{1}{O\left(n_{ab}^{t|u}\right)},
\end{aligned}$$

where the last equality follows from (19). Thus (A.2) approaches 0 as $n_{ab}^{t|u} \to \infty$, and Lyapunov's condition is satisfied. The Lyapunov CLT states that

$$\frac{1}{s_{ab}^{t|u}} \sum_{(i,j)\in\mathcal{B}_{ab}^{t|u}} \left(\frac{w_{ij}^t}{\xi_{ij}^t} - \pi_{ab}^{t|u}\right) \xrightarrow{d} \mathcal{N}(0,1)$$

where $\xrightarrow{d}$ denotes convergence in distribution. By rearranging terms one obtains the desired result. □



(a) $t = 1$ (b) $t = 4$ (c) $t = 7$ (d) $t = 9$

Figure B.1: Adjacency matrices at four different time steps constructed from Facebook wall posts. The estimated classes at the final time $t = 9$ are overlaid onto the adjacency matrices.

B Visualizations of Facebook Wall Posts Data

Visualizations of the class structure overlaid onto the adjacency matrices at several time steps constructed from the Facebook wall posts data (Viswanath et al., 2009) are shown in Figure B.1.

References

P. Billingsley. *Probability and measure*. Wiley-Interscience, 3rd edition, 1995.

B. Viswanath, A. Mislove, M. Cha, and K. P. Gummadi. On the evolution of user interaction in Facebook. In *Proceedings of the 2nd ACM Workshop on Online Social Networks*, pages 37–42, 2009.