## A. Missing Proofs

**Proposition 3.1.** *Let $\{\ell_t\}_{t=1}^T$ be a sequence of loss functions, and $\{\tilde{\ell}_t\}_{t=1}^T$ a sequence of corresponding approximation functions for which it holds that*

$$\mathbb{E}[\nabla\tilde{\ell}_t(a) \mid \mathcal{F}_{t-1}] = \nabla\ell_t(a) + b_t(a),$$

*for all $t$ and $a \in \mathcal{K}$. Denote by $\{a_t\}_{t=1}^T$ the sequence of actions that a first-order algorithm $\mathcal{A}$ outputs for $\{h_t\}_{t=1}^T$, where $h_t(a) = \ell_t(a) + a^\top(\nabla\tilde{\ell}_t(a_t) - \nabla\ell_t(a_t))$. Then,*

$$\mathbb{E}[\mathcal{R}_T(\ell_1,\ldots,\ell_T)] = \sum_{t=1}^T \mathbb{E}\left[\ell_t(a_t)\right] - \sum_{t=1}^T \ell_t(a^*)$$
$$\leq \mathbb{E}\left[\mathcal{B}_T^{\mathcal{A}}(h_1,\ldots,h_T)\right] - \sum_{t=1}^T \mathbb{E}\left[(a_t - a^*)^\top b_t(a_t)\right],$$

*where $a^* = \arg\min_{a\in\mathcal{K}} \sum_{t=1}^T \ell_t(a)$, and $\mathcal{B}_T^{\mathcal{A}}(h_1,\ldots,h_T)$ is the regret bound of algorithm $\mathcal{A}$ applied to $\{h_t\}_{t=1}^T$.*

*Proof.* Notice that $\nabla h_t(a_t) = \nabla\tilde{\ell}_t(a_t)$. Thus, applying algorithm $\mathcal{A}$ to $\{h_t\}_{t=1}^T$ yields the following guarantee:

$$\mathcal{R}_T^{\mathcal{A}}(h_1,\ldots,h_T) = \sum_{t=1}^T h_t(a_t) - \sum_{t=1}^T h_t(a)$$
$$\leq \mathcal{B}_T^{\mathcal{A}}(h_1,\ldots,h_T), \tag{8}$$

for any fixed action $a \in \mathcal{K}$. Next, note that

$$\begin{aligned}\mathbb{E}[h_t(a_t)] &= \mathbb{E}[\ell_t(a_t)] + \mathbb{E}[a_t^\top(\nabla\tilde{\ell}_t(a_t) - \nabla\ell_t(a_t))]\\ &= \mathbb{E}[\ell_t(a_t)] + \mathbb{E}[\mathbb{E}[a_t^\top(\nabla\tilde{\ell}_t(a_t) - \nabla\ell_t(a_t)) \mid \mathcal{F}_{t-1}]]\\ &= \mathbb{E}[\ell_t(a_t)] + \mathbb{E}[a_t^\top\mathbb{E}[(\nabla\tilde{\ell}_t(a_t) - \nabla\ell_t(a_t)) \mid \mathcal{F}_{t-1}]]\\ &= \mathbb{E}[\ell_t(a_t)] + \mathbb{E}[a_t^\top b_t(a_t)],\end{aligned}$$

and also that

$$\begin{aligned}\mathbb{E}[h_t(a)] &= \mathbb{E}[\ell_t(a)] + \mathbb{E}[a^\top(\nabla\tilde{\ell}_t(a_t) - \nabla\ell_t(a_t))]\\ &= \mathbb{E}[\ell_t(a)] + \mathbb{E}[a^\top\mathbb{E}[(\nabla\tilde{\ell}_t(a_t) - \nabla\ell_t(a_t)) \mid \mathcal{F}_{t-1}]]\\ &= \mathbb{E}[\ell_t(a)] + \mathbb{E}[a^\top b_t(a_t)],\end{aligned}$$

for any fixed action $a \in \mathcal{K}$. Finally, taking expectation on Equation (8) and substituting $\mathbb{E}[h_t(a_t)]$, $\mathbb{E}[h_t(a)]$ as computed above yields

$$\mathbb{E}[\mathcal{R}_T(\ell_1,\ldots,\ell_T)] = \sum_{t=1}^T \mathbb{E}\left[\ell_t(a_t)\right] - \sum_{t=1}^T \ell_t(a)$$
$$\leq \mathbb{E}\left[\mathcal{B}_T^{\mathcal{A}}(h_1,\ldots,h_T)\right] - \sum_{t=1}^T \mathbb{E}\left[(a_t - a)^\top b_t(a_t)\right],$$

for any fixed action $a \in \mathcal{K}$, and in particular for $a^*$. □

**Lemma A.1.** *Algorithm 2 generates online predictions for which it holds that:*

$$\sum_{t=1}^T \mathbb{E}\left[\left(u_t^\top\phi(x_t) - u_0^\top\phi(x_t)\right)^2\right]$$
$$- \min_{u\in\mathcal{K}} \sum_{t=1}^T \left(u^\top\phi(x_t) - u_0^\top\phi(x_t)\right)^2 \leq 8T^{1/2}.$$

*Proof.* Define an auxiliary function

$$\tilde{\ell}_t^{\text{Sig}}(u) = \left(u^\top\phi(x_t) - u_0^\top\phi(x_t)\right)^2,$$

and notice that $\mathbb{E}[\nabla\ell_t^{\text{Sig}}(u_t) \mid \mathcal{F}_{t-1}] = \nabla\tilde{\ell}_t^{\text{Sig}}(u_t)$, where $\mathcal{F}_{t-1}$ denotes the sigma-algebra that consists of the actions $u_1,\ldots,u_t$, and the losses $\ell_1^{\text{Sig}},\ldots,\ell_{t-1}^{\text{Sig}}$. Next, define $h_t(u)$ as in Claim 3.1:

$$h_t(u) = \tilde{\ell}_t^{\text{Sig}}(u) + u^\top(\nabla\ell_t^{\text{Sig}}(u_t) - \nabla\tilde{\ell}_t^{\text{Sig}}(u_t)),$$

and notice that $\nabla h_t(u_t) = \nabla\ell_t^{\text{Sig}}(u_t)$. Thus, by Claim 3.1 we can obtain

$$\begin{aligned}\mathbb{E}[\mathcal{R}_T(\tilde{\ell}_1^{\text{Sig}},\ldots,\tilde{\ell}_T^{\text{Sig}})] &\leq \mathbb{E}[\mathcal{B}_T^{\mathcal{A}}(h_1,\ldots,h_T)]\\ &= \mathbb{E}[\mathcal{B}_T^{\mathcal{A}}(\ell_1^{\text{Sig}},\ldots,\ell_T^{\text{Sig}})]\\ &\leq 8T^{1/2},\end{aligned}$$

as stated in the lemma. □

**Proposition 4.2.** *Let $\ell_t^{Sig}$, $\ell_t^{Var}$ and $\tilde{\ell}_t^{Var}$ be as defined above and let $\alpha \in (0,1)$. Then, Algorithm 2 generates online sequences $\{u_t\}_{t=1}^T$ and $\{v_t\}_{t=1}^T$ for which it holds that:*

$$\frac{1}{T}\sum_{t=1}^T P\left(|u_t^\top\phi(x_t) - y_t| \geq c_t\right) \leq \alpha,$$

*for $c_t = \sqrt{\frac{2\max\{\beta, v_t^\top\psi(x_t)\}}{\alpha}}$ and $\beta = 16T^{-1/4}$.*

*Proof.* Using the techniques of Section 3.2, we have that

$$\sum_{t=1}^T \ell_t^{\text{Sig}}(u_t) - \min_{\|u\|\leq 1}\sum_{t=1}^T \ell_t^{\text{Sig}}(u) \leq \frac{4T^{1/2}\alpha}{\beta},$$

and also that

$$\sum_{t=1}^T \mathbb{R}\left[\ell_t^{\text{Var}}(v_t)\right] - \min_{\|v\|\leq 1}\sum_{t=1}^T \ell_t^{\text{Var}}(v) \leq \frac{16T^{1/2}\alpha^2}{\beta^2}.$$

where we used the fact that $\frac{1}{c_t^2} \leq \frac{\alpha}{2\beta}$. Equipped with the above, we can bound the quantity of interest:

$$
\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T} P\left(|u_t^\top \phi(x_t) - y_t| \geq c_t\right) \\
&= \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}\left[\mathbb{E}\left[1_{\left\{(u_t^\top \phi(x_t)-y_t)^2 \geq c_t^2\right\}} \mid \mathcal{F}_{t-1}\right]\right] \\
&\overset{(1)}{\leq} \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}\left[\frac{\mathbb{E}\left[(u_t^\top \phi(x_t) - y_t)^2 \mid \mathcal{F}_{t-1}\right]}{c_t^2}\right] \\
&= \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}\left[\frac{1}{c_t^2}(u_t^\top \phi(x_t) - y_t)^2\right] \\
&\overset{(2)}{\leq} \frac{1}{T}\left(\mathbb{E}\left[\min_{\|u\|\leq 1}\sum_{t=1}^{T}\frac{1}{c_t^2}\left(y_t - u^\top \phi(x_t)\right)^2\right] + \frac{4T^{1/2}\alpha}{\beta}\right) \\
&\leq \frac{1}{T}\left(\sum_{t=1}^{T}\mathbb{E}\left[\frac{1}{c_t^2}\left(y_t - u_0^\top \phi(x_t)\right)^2\right] + \frac{4T^{1/2}\alpha}{\beta}\right) \\
&= \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\mathbb{E}\left[\frac{1}{c_t^2}\left(y_t - u_0^\top \phi(x_t)\right)^2 \mid \mathcal{F}_{t-1}\right]\right] + \frac{4T^{-1/2}\alpha}{\beta} \\
&= \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\frac{v_0^\top \psi(x_t)}{c_t^2}\right] + \frac{4T^{-1/2}\alpha}{\beta} \\
&\leq \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\frac{|v_0^\top \psi(x_t) - v_t^\top \psi(x_t)|}{c_t^2}\right] \\
&\quad + \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\frac{v_t^\top \psi(x_t)}{c_t^2}\right] + \frac{4T^{-1/2}\alpha}{\beta} \\
&\overset{(3)}{\leq} \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\frac{v_t^\top \psi(x_t)}{c_t^2}\right] + \frac{4T^{-1/2}\alpha}{\beta} + \frac{4T^{-1/4}\alpha}{\beta} \leq \alpha,
\end{aligned}
$$

where (1) follows by Markov's inequality; (2) follows by the regret bound for $\{\ell_t^{\text{Sig}}\}_{t=1}^T$; and (3) follows by the regret bound for $\{\ell_t^{\text{Var}}\}_{t=1}^T$ and the relationship between the $\ell_1$ and the $\ell_2$ norms. □