Appendix (Proof of Lemma 3.4)

We first introduce a few definitions and facts about the matrices that are required for the proof. X is a density matrix if it is symmetric positive semi-definite and $\mathrm{Tr}(\mathrm{X}) = 1$. For a pair of density matrices A and B, the quantum relative entropy is defined as $\Delta(\mathrm{A},\mathrm{B}) = \mathrm{Tr}(\mathrm{A}(\log(\mathrm{A}) - \log(\mathrm{B})))$. For arbitrary symmetric matrices A and B, the Golden-Thompson inequality states that $\mathrm{Tr}\left(e^{\mathrm{A}+\mathrm{B}}\right) \le \mathrm{Tr}\left(e^{\mathrm{A}}e^{\mathrm{B}}\right)$. Also, for any symmetric matrix A, such that $0 \preceq \mathrm{A} \preceq \mathrm{I}$ and any $p_1, p_2 \in \mathbb{R}$, Jensen's inequality for matrix exponentials states that: $e^{\mathrm{A}p_1 + (\mathrm{I}-\mathrm{A})p_2} \preceq \mathrm{A}e^{p_1} + (\mathrm{I}-\mathrm{A})e^{p_2}$.

The following lemma is a straightforward consequence of Theorem 2 of (Warmuth & Kuzmin, 2006):

Lemma 5.1. *Let $\{\mathrm{C}_t\}_{t=1}^T$ be an arbitrary sequence of dilations of instantaneous covariance matrices after appropriate spectrum-shift, such that $0 \preceq \mathrm{C}_t \preceq r\mathrm{I}$. For arbitrary density matrix U and learning rate $\eta > 0$, the following bound holds on the MEG iterates (Algorithm 2):*

$$\frac{r\Delta(\mathrm{U},\mathrm{M}_0) - r\Delta(\mathrm{U},\mathrm{M}_T) + r\eta\sum_{t=1}^T \mathrm{Tr}(\mathrm{UC}_t)}{1-e^{-r\eta}} \ge \sum_{t=1}^T Tr(\mathrm{M}_{t-1}\mathrm{C}_t). \tag{17}$$

Proof. : We start by bounding the difference in divergences of two consecutive iterates of Algorithm 2 from the reference U.

$$\begin{aligned}
&\Delta(\mathrm{U},\mathrm{M}_{t-1}) - \Delta(\mathrm{U},\widehat{\mathrm{M}}_t)\\
=&\mathrm{Tr}\left(\mathrm{U}\log\left(\widehat{\mathrm{M}}_t\right) - \mathrm{U}\log(\mathrm{M}_{t-1})\right)\\
=&\mathrm{Tr}\left(\mathrm{U}\log\left(e^{\log(\mathrm{M}_{t-1})-\eta\mathrm{C}_t}\right)\right)\\
&- \mathrm{Tr}\left(\mathrm{U}\log\left(\mathrm{Tr}\left(e^{\log(\mathrm{M}_{t-1})-\eta\mathrm{C}_t}\right)\right)\right)\\
&- \mathrm{Tr}(\mathrm{U}\log(\mathrm{M}_{t-1}))\\
=&-\eta\mathrm{Tr}(\mathrm{UC}_t) - \log\left(\mathrm{Tr}\left(e^{\log(\mathrm{M}_{t-1})-\eta\mathrm{C}_t}\right)\right)
\end{aligned} \tag{18}$$

Using Golden-Thompson inequality with $\mathrm{B} = -\eta\mathrm{C}_t$ and $\mathrm{A} = \log(\mathrm{M}_{t-1})$ we get:

$$\mathrm{Tr}\left(e^{\log(\mathrm{M}_{t-1})-\eta\mathrm{C}_t}\right) \le \mathrm{Tr}\left(\mathrm{M}_{t-1}e^{-\eta\mathrm{C}_t}\right).$$

Now using Jensen's inequality for $0 \preceq \frac{\mathrm{C}_t}{r} \preceq \mathrm{I}$ with $p_1 = -\eta r, p_2 = 0$ we get: $e^{-\eta\mathrm{C}_t} \preceq \mathrm{I} - \frac{1-e^{-\eta r}}{r}\mathrm{C}_t$. Multiplying both sides by M_{t-1}, using the fact that $\mathrm{Tr}(\mathrm{AB}) \le \mathrm{Tr}(\mathrm{AC})$ if A is positive definite and $\mathrm{B} \preceq \mathrm{C}$, and taking logarithms of both sides we get:

$$\begin{aligned}
\log\left(\mathrm{Tr}\left(\mathrm{M}_{t-1}e^{-\eta\mathrm{C}_t}\right)\right) &\le \log\left(1 - \frac{1-e^{-\eta r}}{r}\mathrm{Tr}(\mathrm{M}_{t-1}\mathrm{C}_t)\right)\\
&\le -\frac{1-e^{-\eta r}}{r}\mathrm{Tr}(\mathrm{M}_{t-1}\mathrm{C}_t),
\end{aligned}$$

where we used the inequality $\log(1-x) \le -x$ to get the second inequality. Thus, we have:

$$\log\left(\mathrm{Tr}\left(e^{\log(\mathrm{M}_{t-1})-\eta\mathrm{C}_t}\right)\right) \le -\frac{1-e^{-\eta r}}{r}\mathrm{Tr}(\mathrm{M}_{t-1}\mathrm{C}_t).$$

Plugging the above back in equation (18) and rearranging we get:

$$\frac{r\Delta(\mathrm{U},\mathrm{M}_{t-1}) - r\Delta(\mathrm{U},\widehat{\mathrm{M}}_t) + r\eta\mathrm{Tr}(\mathrm{UC}_t)}{1-e^{-r\eta}} \ge \mathrm{Tr}(\mathrm{M}_{t-1}\mathrm{C}_t) \tag{19}$$

From the Generalized Pythagorean Theorem we have $\Delta(\mathrm{U},\widehat{\mathrm{M}}_t) \ge \Delta(\mathrm{U},\mathrm{M}_t)$. Hence, inequality (19) holds when we replace $\widehat{\mathrm{M}}_t$ by M_t. Now summing over T completes the proof. □

Note that lemma 5.1 holds for any density matrix U, and specifically for $\mathrm{U} = \mathrm{M}^*$, the optimum of Problem (12). We can now tune the learning rate using the following lemma, which we state without the proof:

Lemma 5.2. *(Lemma 4 in (Freund & Schapire, 1997)) Suppose $0 \le \mu \le \tilde{\mu}$ and $0 < \rho \le \tilde{\rho}$. Let $\beta = g(\frac{\tilde{\mu}}{\tilde{\rho}})$ where $g(z) = \frac{1}{1+\sqrt{2/z}}$. Then*

$$\frac{-\mu\log(\beta) + \rho}{1-\beta} \le \mu + \sqrt{2\tilde{\mu}\tilde{\rho}} + \rho \tag{20}$$

Letting $\rho := r\Delta(\mathrm{M}^*,\mathrm{M}_0) - r\Delta(\mathrm{M}^*,\mathrm{M}_T)$, it is easy to verify $\tilde{\rho} := r\log(d) \ge \rho$. Also, by assumptions of Theorem 3.3, we know $\mu := \sum_{t=1}^T \mathrm{Tr}(\mathrm{M}^*\mathrm{C}_t)$ is bounded above by $\tilde{\mu} := LT$. Setting $\beta = e^{-r\eta}$, the learning rate

$$\eta = \frac{1}{r}\log\left(\frac{1}{\beta}\right) = \frac{1}{r}\log\left(1 + \sqrt{\frac{2r\log(d)}{LT}}\right).$$

Substituting μ, $\tilde{\mu}$, ρ, and $\tilde{\rho}$ in (20), moving μ to the left hand side, and noting that replacing ρ with $\tilde{\rho}$ only makes the right hand side bigger completes the proof.