Appendix (Proof of Lemma 3.4)

We first introduce a few definitions and facts about the matrices that are required for the proof. X is a density matrix if it is symmetric positive semi-definite and $\text{Tr}(X) = 1$. For a pair of density matrices $A$ and $B$, the quantum relative entropy is defined as $\Delta(A, B) = \text{Tr}(A \log(A) - \log(B))$. For arbitrary symmetric matrices $A$ and $B$, the Golden-Thompson inequality states that $\text{Tr}(e^{A+B}) \leq \text{Tr}(e^{A} e^{B})$. Also, for any symmetric matrix $A$, such that $0 \leq A \leq \mathbb{I}$ and any $p_1, p_2 \in \mathbb{R}$, Jensen’s inequality for matrix exponentials states that: $e^{\lambda p_1 + (1-\lambda)p_2} \leq \lambda e^{p_1} + (1 - \lambda) e^{p_2}$.

The following lemma is a straightforward consequence of Theorem 2 of (Warmuth & Kuzmin, 2006):

**Lemma 5.1.** Let $\{C_t\}_{t=1}^T$ be an arbitrary sequence of dilations of instantaneous covariance matrices after appropriate spectrum-shift, such that $0 \leq C_t \leq \mathbb{I}$. For arbitrary density matrix $U$ and learning rate $\eta > 0$, the following bound holds on the MEG iterates (Algorithm 2):

$$
\begin{align*}
\frac{r \Delta(U, M_0) - r \Delta(U, M_T) + r \eta \sum_{t=1}^T \text{Tr}(UC_t)}{1 - e^{-r \eta}} & \geq \sum_{t=1}^T \text{Tr}(M_{t-1}C_t). \quad (17)
\end{align*}
$$

**Proof.** We start by bounding the difference in divergences of two consecutive iterates of Algorithm 2 from the reference $U$.

$$
\begin{align*}
\Delta(U, M_{t-1}) - \Delta(U, \tilde{M}_t) &= \text{Tr}(U \log \tilde{M}_t) - U \log (M_{t-1}) \\
&= \text{Tr}(U \log (e^{\log(M_{t-1}) - \eta C_t})) \\
&- \text{Tr}(U \log (\text{Tr}(e^{\log(M_{t-1}) - \eta C_t}))) \\
&- \text{Tr}(U \log (M_{t-1})) \\
&= - \eta \text{Tr}(UC_t) - \log (\text{Tr}(e^{\log(M_{t-1}) - \eta C_t})). \quad (18)
\end{align*}
$$

Using Golden-Thompson inequality with $B = -\eta C_t$ and $A = \log(M_{t-1})$ we get:

$$
\text{Tr}(e^{\log(M_{t-1}) - \eta C_t}) \leq \text{Tr}(M_{t-1} e^{-\eta C_t}).
$$

Now using Jensen’s inequality for $0 \leq C_t \leq 1$ with $p_1 = -\eta r, p_2 = 0$ we get: $e^{-\eta r} \leq 1 - \frac{1 - e^{-\eta r}}{r} C_t$. Multiplying both sides by $M_{t-1}$, using the fact that $\text{Tr}(AB) \leq \text{Tr}(AC)$ if $A$ is positive definite and $B \preceq C$, and taking logarithms of both sides we get:

$$
\begin{align*}
\log \left( \text{Tr}(M_{t-1} e^{-\eta r C_t}) \right) &\leq \log \left( 1 - \frac{1 - e^{-\eta r}}{r} \text{Tr}(M_{t-1}C_t) \right) \\
&\leq - \frac{1 - e^{-\eta r}}{r} \text{Tr}(M_{t-1}C_t),
\end{align*}
$$

where we used the inequality $\log(1 - x) \leq -x$ to get the second inequality. Thus, we have:

$$
\log \left( \text{Tr}(e^{\log(M_{t-1}) - \eta C_t}) \right) \leq - \frac{1 - e^{-\eta r}}{r} \text{Tr}(M_{t-1}C_t).
$$

Plugging the above back in equation (18) and rearranging we get:

$$
\frac{r \Delta(U, M_{t-1}) - r \Delta(U, \tilde{M}_t) + r \eta \text{Tr}(UC_t)}{1 - e^{-r \eta}} \geq \text{Tr}(M_{t-1}C_t). \quad (19)
$$

From the Generalized Pythagorean Theorem we have $\Delta(U, \tilde{M}_t) \geq \Delta(U, M_t)$. Hence, inequality (19) holds when we replace $\tilde{M}_t$ by $M_t$. Now summing over $T$ completes the proof. $\square$

Note that lemma 5.1 holds for any density matrix $U$, and specifically for $U = M^*$, the optimum of Problem (12). We can now tune the learning rate using the following lemma, which we state without the proof:

**Lemma 5.2.** (Lemma 4 in (Freund & Schapire, 1997)) Suppose $0 \leq \mu \leq \tilde{\mu}$ and $0 < \rho \leq \tilde{\rho}$. Let $\beta = g(\frac{\mu}{\rho})$, where $g(z) = \frac{1}{1 + \sqrt{2/z}}$. Then

$$
\frac{-\mu \log (\beta) + \rho}{1 - \beta} \leq \mu + \sqrt{2 \tilde{\rho}} \beta + \rho \quad (20)
$$

Letting $\rho := r \Delta(M^*, M_0) - r \Delta(M^*, M_T)$, it is easy to verify $\beta := r \log (d) \geq \rho$. Also, by assumptions of Theorem 3.3, we know $\mu := \sum_{t=1}^T \text{Tr}(M^* C_t)$ is bounded above by $\tilde{\mu} := LT$. Setting $\beta = e^{-\eta r}$, the learning rate

$$
\eta = \frac{1}{r} \log \frac{1}{\beta} = \frac{1}{r} \log \left( 1 + \sqrt{2r \log (d) / LT} \right).
$$

Substituting $\mu$, $\tilde{\mu}$, $\rho$, and $\tilde{\rho}$ in (20), moving $\mu$ to the left hand side, and noting that replacing $\rho$ with $\tilde{\rho}$ only makes the right hand side bigger completes the proof.