A. Proof of Exponential SQR Normalization

The basic intuition is clear by looking at the asymptotic growth of each term. However, we specifically outline the possibilities:

1. $\eta_2 > 0$: $\mathrm{A}(\eta_1, \eta_2) \to \infty$.
2. $\eta_2 < 0$: $\mathrm{A}(\eta_1, \eta_2) < \infty$.
3. $\eta_2 = 0$: if $\eta_1 < 0$, then $\mathrm{A}(\eta_1, \eta_2) < \infty$, otherwise $\mathrm{A}(\eta_1, \eta_2) \to \infty$.

In summary, we need that $\eta_2 < 0$ *or* ($\eta_2 = 0$ *and* $\eta_1 < 0$).

Case 1: $\eta_2 > 0$ Let $\hat{\eta}_2 = \eta_2/2$. First, we seek an exponential lower bound on the partition function. In particular, we want to find an $\bar{z}$ such that for all $z > \bar{z}$, $\exp(\hat{\eta}_2 z) \leq \exp(\eta_1\sqrt{z} + \eta_2 z)$. Taking the log of both sides and solving, we find that the critical points of the above inequality are at 0 and $(-2\frac{\eta_1}{\eta_2})^2$. We take the non-trivial solution of $\bar{z} = (-2\frac{\eta_1}{\eta_2})^2$. Now we need to check if the region to the right of $\bar{z}$ is possible by plugging into the original equation. Let us try a point $\tilde{z} = a\bar{z}$ where $a > 1$:

$$\exp(\hat{\eta}_2 \tilde{z}) \overset{?}{\leq} \exp(\eta_1\sqrt{\tilde{z}} + \eta_2 \tilde{z}) \tag{16}$$

$$\Rightarrow \hat{\eta}_2 \tilde{z} \overset{?}{\leq} \eta_1\sqrt{\tilde{z}} + \eta_2 \tilde{z} \tag{17}$$

$$\Rightarrow (\eta_2/2 - \eta_2)\tilde{z} \overset{?}{\leq} \eta_1\sqrt{\tilde{z}} \tag{18}$$

$$\Rightarrow -\frac{\eta_2}{2}\left(-2a\frac{\eta_1}{\eta_2}\right)^2 \overset{?}{\leq} \eta_1\sqrt{\left(-2a\frac{\eta_1}{\eta_2}\right)^2} \tag{19}$$

$$\Rightarrow a\frac{-2\eta_1^2}{\eta_2} \overset{?}{\leq} \sqrt{a}\frac{-2\eta_1^2}{\eta_2} \tag{20}$$

$$\Rightarrow a \geq \sqrt{a}\,, \tag{21}$$

where the last line is because we assumed $a > 1$ and $\eta_2 > 0$. Thus, we can lower bound the log partition function as follows:

$$\begin{aligned}\mathrm{A}(\eta_1, \eta_2) &= \int_0^{\bar{z}} \exp(\eta_1\sqrt{z} + \eta_2 z)\mathrm{d}z \\ &\qquad + \int_{\bar{z}}^{\infty} \exp(\eta_1\sqrt{z} + \eta_2 z)\mathrm{d}z \\ &\geq \int_0^{\bar{z}} \exp(\eta_1\sqrt{z} + \eta_2 z)\mathrm{d}z + \underbrace{\int_{\bar{z}}^{\infty} \exp(\hat{\eta}_2 z)\mathrm{d}z}_{\to\infty,\ \text{since } \hat{\eta}_2 > 0} \\ &= \infty\,.\end{aligned}$$

Therefore, if $\eta_2 > 0$, the log partition function diverges and hence the joint distribution is not consistent.

Case 2: $\eta_2 < 0$ Now we will find an exponential upper bound and show that this upper bound converges—and hence the log partition function converges. In a similar manner to case 1, let $\hat{\eta}_2 = \eta_2/2$. We want to find an $\bar{z}$ such that for all $z > \bar{z}$, $\exp(\hat{\eta}_2 z) \geq \exp(\eta_1\sqrt{z} + \eta_2 z)$—the only difference from case 1 is the direction of the inequality. Thus, using the same reasoning as case 1, we have that $\bar{z} = (-2\frac{\eta_1}{\eta_2})^2$. Similarly, we need to check if the region to the right of $\bar{z}$ is possible by plugging into the original equation. In a analogous derivation, we arrive at the same equation as Eqn. 20 except with the inequality is flipped:

$$\Rightarrow a\frac{-2\eta_1^2}{\eta_2} \overset{?}{\geq} \sqrt{a}\frac{-2\eta_1^2}{\eta_2} \tag{22}$$

$$\Rightarrow a \geq \sqrt{a}\,, \tag{23}$$

where the last step is because we assumed $\eta_2 < 0$ and $a > 1$—note that we do not flip the inequality because $\frac{-2\eta_1^2}{\eta_2}$ is overall a positive number. Thus, this is an upper bound on the interval $[\bar{z}, \infty]$:

$$\begin{aligned}\mathrm{A}(\eta_1, \eta_2) &= \int_0^{\bar{z}} \exp(\eta_1\sqrt{z} + \eta_2 z)\mathrm{d}z \\ &\qquad + \int_{\bar{z}}^{\infty} \exp(\eta_1\sqrt{z} + \eta_2 z)\mathrm{d}z \\ &\leq \int_0^{\bar{z}} \exp(\eta_1\sqrt{z} + \eta_2 z)\mathrm{d}z + \underbrace{\int_{\bar{z}}^{\infty} \exp(\hat{\eta}_2 z)\mathrm{d}z}_{\text{Upper bound}} \\ &\leq \underbrace{\int_0^{\bar{z}} \exp(\eta_1\sqrt{z} + \eta_2 z)\mathrm{d}z}_{<\infty} + \underbrace{\int_0^{\infty} \exp(\hat{\eta}_2 z)\mathrm{d}z}_{\text{Exp. log partition}} \\ &< \infty,\end{aligned}$$

where the last step is based on the fact that a bounded integral of a finite smooth function is bounded away from ∞ and the second term is merely the log partition function of a standard exponential distribution.

Case 3: $\eta_2 = 0$ This gives the log partition function simply as:

$$\mathrm{A}(\eta_1, \eta_2) = \int_0^{\infty} \exp(\eta_1\sqrt{z})\mathrm{d}z,$$

which has the closed form solution:

$$\mathrm{A}(\eta_1, \eta_2) = 2\eta_1^{-2}(\eta_1\sqrt{z} - 1)\exp(\eta_1\sqrt{z})\Big|_0^{\infty} \tag{24}$$

$$= \lim_{z\to\infty} 2\eta_1^{-2}(\eta_1\sqrt{z} - 1)\exp(\eta_1\sqrt{z}) - \left(-2\eta_1^{-2}\right) \tag{25}$$

$$= 2\eta_1^{-2} + \lim_{z\to\infty} 2\eta_1^{-2}(\eta_1\sqrt{z} - 1)\exp(\eta_1\sqrt{z})\,. \tag{26}$$

The convergence critically depends on the limit in Eqn. 26. This limit diverges to ∞ if $\eta_1 \geq 0$ but converges to 0 if

$\eta_1 < 0$. Thus, if $\eta_2 = 0$, then $\eta_1 < 0$ for the log partition function to be finite.

B. Proof of Poisson SQR Normalization (Eqn. 15)

First, we take an upper bound by absorbing the $\sqrt{z}$ term:

$$\mathrm{A}(\eta_1, \eta_2) \leq \sum_{z \in \mathbb{Z}_+} \exp\Big(\underbrace{\eta_1\sqrt{z} + \eta_2 z}_{O(z)} \tag{27}$$

$$\underbrace{- \sum_s \ln(\Gamma(zv_s + 1))}_{O(z \ln z)}\Big) \tag{28}$$

$$\leq \sum_{z \in \mathbb{Z}_+} \exp\Big(\eta z - \sum_s \ln(\Gamma(zv_s + 1))\Big), \tag{29}$$

where $\eta = \eta_2 + |\eta_1|$. We continue the bound as follows:

$$A(\eta_1, \eta_2) \leq \sum_{z \in \mathbb{Z}_+} \exp\Big(\eta z - \max_s \ln(\Gamma(zv_s + 1))\Big) \tag{30}$$

$$\leq \sum_{z \in \mathbb{Z}_+} \exp\left(\eta z - \ln(\Gamma(z/p + 1))\right), \tag{31}$$

where Eqn. 31 comes from the fact that $\arg\max_{v_s} \ln(\Gamma(zv_s + 1)) \geq 1/p$ (simple proof by contradiction).

Now let us use the ratio test for convergent series where $a_z = \exp\Big(\eta(\mathbf{v})z - \ln(\Gamma(\frac{z}{p} + 1))\Big)$:

$$\lim_{z\to\infty} \frac{|a_{z+1}|}{|a_z|} = \exp(\eta(z+1) - \ln(\Gamma((z+1)/p + 1))$$

$$- [\eta z - \ln(\Gamma(z/p + 1))]) \tag{32}$$

$$= \lim_{z\to\infty} \exp\left(\eta + \ln\left(\frac{\Gamma(z/p + 1))}{\Gamma((z+1)/p + 1))}\right)\right) \tag{33}$$

$$= \exp(\eta) \lim_{z\to\infty} \frac{\Gamma(z/p + 1))}{\Gamma((z/p + 1 + 1/p))} \frac{(z/p + 1)^{1/p}}{(z/p + 1)^{1/p}} \tag{34}$$

$$= \exp(\eta) \lim_{z\to\infty} \frac{1}{(z/p + 1)^{1/p}}$$

$$\times \lim_{z\to\infty} \frac{\Gamma(z/p + 1))(z/p + 1)^{1/p}}{\Gamma((z/p + 1 + 1/p))} \tag{35}$$

$$= \exp(\eta) \lim_{z\to\infty} \frac{1}{(z/p + 1)^{1/p}}(1) \tag{36}$$

$$= \exp(\eta)(0)(1) = 0 < 1, \tag{37}$$

where Eqn. 35 is by the product of limits rule and Eqn. 36 is by the well-known asymptotic properties of gamma functions. Therefore, by the ratio test, the radial conditional log partition function is bounded for any $\eta_1 < \infty$ and $\eta_2 < \infty$.