A. Proofs

Lemma 1. *Let $x, y \in [0,1]^K$ satisfy $x \geq y$. Then*

$$V(x) - V(y) \leq \sum_{k=1}^{K} x_k - \sum_{k=1}^{K} y_k \,.$$

Proof. Let $x = (x_1, \dots, x_K)$ and

$$d(x) = \sum_{k=1}^{K} x_k - V(x) = \sum_{k=1}^{K} x_k - \left[1 - \prod_{k=1}^{K}(1 - x_k)\right] .$$

Our claim can be proved by showing that $d(x) \geq 0$ and $\frac{\partial}{\partial x_i} d(x) \geq 0$, for any $x \in [0,1]^K$ and $i \in [K]$. First, we show that $d(x) \geq 0$ by induction on K. The claim holds trivially for $K = 1$. For any $K \geq 2$,

$$d(x) = \sum_{k=1}^{K-1} x_k - \left[1 - \prod_{k=1}^{K-1}(1 - x_k)\right] + \underbrace{x_K - x_K \prod_{k=1}^{K-1}(1 - x_k)}_{\geq 0} \geq 0 \,,$$

where $\sum_{k=1}^{K-1} x_k - \left[1 - \prod_{k=1}^{K-1}(1 - x_k)\right] \geq 0$ holds by our induction hypothesis. Second, we note that

$$\frac{\partial}{\partial x_i} d(x) = 1 - \prod_{k \neq i}(1 - x_k) \geq 0 \,.$$

This concludes our proof. ■

Lemma 2. *Let $x, y \in [0, p_{\max}]^K$ satisfy $x \geq y$. Then*

$$\alpha \left[\sum_{k=1}^{K} x_k - \sum_{k=1}^{K} y_k\right] \leq V(x) - V(y) \,,$$

where $\alpha = (1 - p_{\max})^{K-1}$.

Proof. Let $x = (x_1, \dots, x_K)$ and

$$d(x) = V(x) - \alpha \sum_{k=1}^{K} x_k = 1 - \prod_{k=1}^{K}(1 - x_k) - (1 - p_{\max})^{K-1} \sum_{k=1}^{K} x_k \,.$$

Our claim can be proved by showing that $d(x) \geq 0$ and $\frac{\partial}{\partial x_i} d(x) \geq 0$, for any $x \in [0, p_{\max}]^K$ and $i \in [K]$. First, we show that $d(x) \geq 0$ by induction on K. The claim holds trivially for $K = 1$. For any $K \geq 2$,

$$d(x) = 1 - \prod_{k=1}^{K-1}(1 - x_k) - (1 - p_{\max})^{K-1} \sum_{k=1}^{K-1} x_k + \underbrace{x_K \prod_{k=1}^{K-1}(1 - x_k) - x_K(1 - p_{\max})^{K-1}}_{\geq 0} \geq 0 \,,$$

where $1 - \prod_{k=1}^{K-1}(1 - x_k) - (1 - p_{\max})^{K-1} \sum_{k=1}^{K-1} x_k \geq 0$ holds because $1 - \prod_{k=1}^{K-1}(1 - x_k) - (1 - p_{\max})^{K-2} \sum_{k=1}^{K-1} x_k \geq 0$, which holds by our induction hypothesis; and the remainder is non-negative because $1 - x_k \geq 1 - p_{\max}$ for any $k \in [K]$. Second, note that

$$\frac{\partial}{\partial x_i} d(x) = \prod_{k \neq i}(1 - x_k) - (1 - p_{\max})^{K-1} \geq 0 \,.$$

This concludes our proof. ■

Lemma 3. *Let $x \in [0,1]^K$ and x' be the permutation of x whose entries are in decreasing order, $x'_1 \geq \ldots \geq x'_K$. Let the entries of $c \in [0,1]^K$ be in decreasing order. Then*

$$V(c \odot x') - V(c \odot x) \leq \sum_{k=1}^{K} c_k x'_k - \sum_{k=1}^{K} c_k x_k \,.$$

Proof. Note that our claim is equivalent to proving

$$1 - \prod_{k=1}^{K}(1 - c_k x'_k) - \left[1 - \prod_{k=1}^{K}(1 - c_k x_k)\right] \leq \sum_{k=1}^{K} c_k x'_k - \sum_{k=1}^{K} c_k x_k \,.$$

If $x = x'$, our claim holds trivially. If $x \neq x'$, there must exist indices i and j such that $i < j$ and $x_i < x_j$. Let $\tilde{x}$ be the same vector as x where entries x_i and x_j are exchanged, $\tilde{x}_i = x_j$ and $\tilde{x}_j = x_i$. Since $i < j$, $c_i \geq c_j$. Let

$$X_{\text{-}i,\text{-}j} = \prod_{k \neq i,j} (1 - c_k x_k) \,.$$

Then

$$\begin{aligned}
1 - \prod_{k=1}^{K}(1 - c_k x'_k) - \left[1 - \prod_{k=1}^{K}(1 - c_k x_k)\right] &= X_{\text{-}i,\text{-}j} \left((1 - c_i x_i)(1 - c_j x_j) - (1 - c_i \tilde{x}_i)(1 - c_j \tilde{x}_j)\right) \\
&= X_{\text{-}i,\text{-}j} \left((1 - c_i x_i)(1 - c_j x_j) - (1 - c_i x_j)(1 - c_j x_i)\right) \\
&= X_{\text{-}i,\text{-}j} \left(-c_i x_i - c_j x_j + c_i x_j + c_j x_i\right) \\
&= X_{\text{-}i,\text{-}j} (c_i - c_j)(x_j - x_i) \\
&\leq (c_i - c_j)(x_j - x_i) \\
&= c_i x_j + c_j x_i - c_i x_i - c_j x_j \\
&= c_i \tilde{x}_i + c_j \tilde{x}_j - c_i x_i - c_j x_j \\
&= \sum_{k=1}^{K} c_k \tilde{x}_k - \sum_{k=1}^{K} c_k x_k \,,
\end{aligned}$$

where the inequality is by our assumption that $(c_i - c_j)(x_j - x_i) \geq 0$. If $\tilde{x} = x'$, we are finished. Otherwise, we repeat the above argument until $x = x'$. ∎

Lemma 4. *Let $x, y \in [0,1]^K$ satisfy $x \geq y$. Let $\gamma \in [0,1]$. Then*

$$V(\gamma x) - V(\gamma y) \geq \gamma [V(x) - V(y)] \,.$$

Proof. Note that our claim is equivalent to proving

$$\prod_{k=1}^{K}(1 - \gamma y_k) - \prod_{k=1}^{K}(1 - \gamma x_k) \geq \gamma \left[\prod_{k=1}^{K}(1 - y_k) - \prod_{k=1}^{K}(1 - x_k)\right] .$$

The proof is by induction on K. To simplify exposition, we define the following shorthands

$$X_i = \prod_{k=1}^{i}(1 - x_k) \,, \quad X_i^\gamma = \prod_{k=1}^{i}(1 - \gamma x_k) \,, \quad Y_i = \prod_{k=1}^{i}(1 - y_k) \,, \quad Y_i^\gamma = \prod_{k=1}^{i}(1 - \gamma y_k) \,.$$

Our claim holds trivially for $K = 1$ because

$$(1 - \gamma y_1) - (1 - \gamma x_1) = \gamma[(1 - y_1) - (1 - x_1)] \,.$$

To prove that the claim holds for any K, we first rewrite $Y_K^\gamma - X_K^\gamma$ in terms of $Y_{K-1}^\gamma - X_{K-1}^\gamma$ as

$$\begin{aligned}
Y_K^\gamma - X_K^\gamma &= (1 - \gamma y_K) Y_{K-1}^\gamma - (1 - \gamma x_K) X_{K-1}^\gamma \\
&= Y_{K-1}^\gamma - \gamma y_K Y_{K-1}^\gamma - X_{K-1}^\gamma + \gamma y_K X_{K-1}^\gamma + \gamma (x_K - y_K) X_{K-1}^\gamma \\
&= (1 - \gamma y_K)(Y_{K-1}^\gamma - X_{K-1}^\gamma) + \gamma (x_K - y_K) X_{K-1}^\gamma \,.
\end{aligned}$$

By our induction hypothesis, $Y_{K-1}^\gamma - X_{K-1}^\gamma \geq \gamma (Y_{K-1} - X_{K-1})$. Moreover, $X_{K-1}^\gamma \geq X_{K-1}$ and $1 - \gamma y_K \geq 1 - y_K$. We apply these lower bounds to the right-hand side of the above equality and then rearrange it as

$$\begin{aligned}
Y_K^\gamma - X_K^\gamma &\geq \gamma (1 - y_K)(Y_{K-1} - X_{K-1}) + \gamma (x_K - y_K) X_{K-1} \\
&= \gamma [(1 - y_K) Y_{K-1} - (1 - y_K + y_K - x_K) X_{K-1}] \\
&= \gamma [Y_K - X_K] \,.
\end{aligned}$$

This concludes our proof. ■