Appendix: supplementary material

A. Proof of Theorem 2

We prove a more general result for an arbitrary choice of the parameter $\lambda_{j,a} > 0$ for all $j \in [n]$ and $a \in [\ell_j]$. The following theorem proves the (near)-optimality of the choice of $\lambda_{j,a}$'s proposed in (14), and implies the corresponding error bound as a corollary.

Theorem 5. *Under the hypotheses of Theorem 2 and any $\lambda_{j,a}$'s, the rank-breaking estimator achieves*

$$\frac{1}{\sqrt{d}}\big\| \widehat{\theta} - \theta^* \big\|_2 \;\leq\; \frac{4\sqrt{2}e^{4b}(1+e^{2b})^2\sqrt{d\log d}}{\alpha\,\gamma}\frac{\sqrt{\sum_{j=1}^n\sum_{a=1}^{\ell_j}\big(\lambda_{j,a}\big)^2\big(\kappa_j - p_{j,a}\big)\big(\kappa_j - p_{j,a}+1\big)}}{\sum_{j=1}^n\sum_{a=1}^{\ell_j}\lambda_{j,a}(\kappa_j - p_{j,a})}\;, \tag{17}$$

with probability at least $1-3e^3d^{-3}$, if

$$\sum_{j=1}^{n}\sum_{a=1}^{\ell_j}\lambda_{j,a}(\kappa_j - p_{j,a}) \;\geq\; 2^6e^{18b}\frac{\eta\delta}{\alpha^2\beta\gamma^2\tau}d\log d\;, \tag{18}$$

where γ, η, τ, δ, α, β, are now functions of $\lambda_{j,a}$'s and defined in (11), (12), (20), (22) and (25).

We first claim that $\lambda_{j,a} = 1/(\kappa_j - p_{j,a}+1)$ is the optimal choice for minimizing the above upper bound on the error. From Cauchy-Schwartz inequality and the fact that all terms are non-negative, we have that

$$\frac{\sqrt{\sum_{j=1}^n\sum_{a=1}^{\ell_j}\big(\lambda_{j,a}\big)^2(\kappa_j - p_{j,a})(\kappa_j - p_{j,a}+1)}}{\sum_{j=1}^n\sum_{a=1}^{\ell_j}\lambda_{j,a}(\kappa_j - p_{j,a})} \;\geq\; \frac{1}{\sqrt{\sum_{j=1}^n\sum_{a=1}^{\ell_j}\frac{(\kappa_j-p_{j,a})}{(\kappa_j-p_{j,a}+1)}}}\;, \tag{19}$$

where $\lambda_{j,a} = 1/(\kappa_j - p_{j,a}+1)$ achieves the universal lower bound on the right-hand side with an equality. Since $\sum_{j=1}^n\sum_{a=1}^{\ell_j}\frac{(\kappa_j-p_{j,a})}{(\kappa_j-p_{j,a}+1)} \geq \sum_{j=1}^n \ell_j$, substituting this into (17) gives the desired error bound in (15). Although we have identified the optimal choice of $\lambda_{j,a}$'s, we choose a slightly different value of $\lambda = 1/(\kappa_j - p_{j,a})$ for the analysis. This achieves the same desired error bound in (15), and significantly simplifies the notations of the sufficient conditions.

We first define all the parameters in the above theorem for general $\lambda_{j,a}$. With a slight abuse of notations, we use the same notations for $\mathcal{H}$, L, α and β for both the general $\lambda_{j,a}$'s and also the specific choice of $\lambda_{j,a} = 1/(\kappa_j - p_{j,a})$. It should be clear from the context what we mean in each case. Define

$$\tau \;\equiv\; \min_{j\in[n]}\tau_j\;, \quad \text{where}\;\; \tau_j \equiv \frac{\sum_{a=1}^{\ell_j}\lambda_{j,a}(\kappa_j - p_{j,a})}{\ell_j} \tag{20}$$

$$\delta_{j,1} \;\equiv\; \Big\{\max_{a\in[\ell_j]}\Big\{\lambda_{j,a}(\kappa_j - p_{j,a})\Big\} + \sum_{a=1}^{\ell_j}\lambda_{j,a}\Big\}\;, \text{ and } \quad \delta_{j,2} \equiv \sum_{a=1}^{\ell_j}\lambda_{j,a} \tag{21}$$

$$\delta \;\equiv\; \max_{j\in[n]}\bigg\{4\delta_{j,1}^2 + \frac{2\big(\delta_{j,1}\delta_{j,2} + \delta_{j,2}^2\big)\kappa_j}{\eta_j\ell_j}\bigg\}\;. \tag{22}$$

Note that $\delta \geq \delta_{j,1}^2 \geq \max_a \lambda_{j,a}^2(\kappa_j - p_{j,a})^2 \geq \tau^2$, and for the choice of $\lambda_{j,a} = 1/(\kappa_j - p_{j,a})$ it simplifies as $\tau = \tau_j = 1$. We next define a comparison graph $\mathcal{H}$ for general $\lambda_{j,a}$, which recovers the proposed comparison graph for the optimal choice of $\lambda_{j,a}$'s

Definition 6. *(Comparison graph $\mathcal{H}$). Each item $i \in [d]$ corresponds to a vertex i. For any pair of vertices i, i', there is a weighted edge between them if there exists a set S_j such that $i, i' \in S_j$; the weight equals $\sum_{j:i,i'\in S_j}\frac{\tau_j\ell_j}{\kappa_j(\kappa_j-1)}$.*

Let A denote the weighted adjacency matrix, and let $D = \mathrm{diag}(A\mathbf{1})$. Define,

$$D_{\max} \;\equiv\; \max_{i\in[d]}D_{ii} \;=\; \max_{i\in[d]}\bigg\{\sum_{j:i\in S_j}\frac{\tau_j\ell_j}{\kappa_j}\bigg\} \;\geq\; \tau_{\min}\max_{i\in[d]}\bigg\{\sum_{j:i\in S_j}\frac{\ell_j}{\kappa_j}\bigg\}\,. \tag{23}$$

Define graph Laplacian L as $L = D - A$, i.e.,

$$L \;=\; \sum_{j=1}^{n} \frac{\tau_j \ell_j}{\kappa_j(\kappa_j - 1)} \sum_{i<i' \in S_j} (e_i - e_{i'})(e_i - e_{i'})^\top . \tag{24}$$

Let $0 = \lambda_1(L) \leq \lambda_2(L) \leq \cdots \leq \lambda_d(L)$ denote the sorted eigenvalues of L. Note that $\mathrm{Tr}(L) = \sum_{i=1}^{d} \sum_{j:i \in S_j} \tau_j \ell_j / \kappa_j = \sum_{j=1}^{n} \tau_j \ell_j$. Define α and β such that

$$\alpha \equiv \frac{\lambda_2(L)(d-1)}{\mathrm{Tr}(L)} = \frac{\lambda_2(L)(d-1)}{\sum_{j=1}^{n} \tau_j \ell_j} \;\;\text{and}\;\; \beta \equiv \frac{\mathrm{Tr}(L)}{dD_{\max}} = \frac{\sum_{j=1}^{n} \tau_j \ell_j}{dD_{\max}} . \tag{25}$$

For the proposed choice of $\lambda_{j,a} = 1/(\kappa_j - p_{j,a})$, we have $\tau_j = 1$ and the definitions of $\mathcal{H}$, L, α, and β reduce to those defined in Definition 1. We are left to prove an upper bound, $\delta \leq 32(\log(\ell_{\max}+2))^2$, which implies the sufficient condition in (13) and finishes the proof of Theorem 2. We have,

$$\begin{aligned} \delta_{j,1} = \max_{a \in [\ell_j]} \Big\{ \lambda_{j,a}(\kappa_j - p_{j,a}) \Big\} + \sum_{a=1}^{\ell_j} \lambda_{j,a} \;\; &= \;\; 1 + \sum_{a=1}^{\ell_j} \frac{1}{\kappa_j - p_{j,a}} \\ &\leq \;\; 1 + \sum_{a=1}^{\ell_j} \frac{1}{a} \\ &\leq \;\; 2\log(\ell_j + 2)\,, \end{aligned} \tag{26}$$

where in the first inequality follows from taking the worst case for the positions, i.e. $p_{j,a} = \kappa_j - \ell_j + a - 1$ Using the fact that for any integer x, $\sum_{a=0}^{\ell-1} 1/(x+a) \leq \log((x+\ell-1)/(x-1))$, we also have

$$\begin{aligned} \frac{\delta_{j,2}\kappa_j}{\eta_j \ell_j} \;\; &\leq \;\; \sum_{a=1}^{\ell_j} \frac{1}{\kappa_j - p_{j,a}} \frac{\max\{\ell_j, \kappa_j - p_{j,\ell_j}\}}{\ell_j} \\ &\leq \;\; \min\Big\{ \log(\ell_j + 2)\,,\, \log\Big(\frac{\kappa_j - p_{j,\ell_j} + \ell_j - 1}{\kappa_j - p_{j,\ell_j} - 1}\Big) \Big\} \frac{\max\{\ell_j, \kappa_j - p_{j,\ell_j}\}}{\ell_j} \\ &\leq \;\; \frac{\log(\ell_j+2)\ell_j}{\max\{\ell_j, \kappa_j - p_{j,\ell_j} - 1\}} \frac{\max\{\ell_j, \kappa_j - p_{j,\ell_j}\}}{\ell_j} \\ &\leq \;\; 2\log(\ell_j + 2)\,, \end{aligned} \tag{27}$$

where the first inequality follows from the definition of η_j, Equation (12). From (26), (27), and the fact that $\delta_{j,2} \leq \log(\ell_j + 2)$, we have

$$\delta = \max_{j \in [n]} \left\{ 4\delta_{j,1}^2 + \frac{2\big(\delta_{j,1}\delta_{j,2} + \delta_{j,2}^2\big)\kappa_j}{\eta_j \ell_j} \right\} \;\; \leq \;\; 28(\log(\ell_{\max}+2))^2 \,. \tag{28}$$

B. Proof of Theorem 5

We first introduce two key technical lemmas. In the following lemma we show that $\mathbb{E}_{\theta^*}[\nabla \mathcal{L}_{\mathrm{RB}}(\theta^*)] = 0$ and provide a bound on the deviation of $\nabla \mathcal{L}_{\mathrm{RB}}(\theta^*)$ from its mean. The expectation $\mathbb{E}_{\theta^*}[\cdot]$ is with respect to the randomness in the samples drawn according to θ^*. The log likelihood Equation (5) can be rewritten as

$$\mathcal{L}_{\mathrm{RB}}(\theta) = \sum_{j=1}^{n} \sum_{a=1}^{\ell_j} \sum_{i<i' \in S_j} \mathbb{I}_{\big\{(i,i') \in G_{j,a}\big\}} \lambda_{j,a} \Big(\theta_i \mathbb{I}_{\big\{\sigma_j^{-1}(i) < \sigma_j^{-1}(i')\big\}} + \theta_{i'} \mathbb{I}_{\big\{\sigma_j^{-1}(i) > \sigma_j^{-1}(i')\big\}} - \log\Big(e^{\theta_i} + e^{\theta_{i'}}\Big) \Big) . \tag{29}$$

We use $(i,i') \in G_{j,a}$ to mean either (i,i') or (i',i) belong to $E_{j,a}$. Taking the first-order partial derivative of $\mathcal{L}_{\mathrm{RB}}(\theta)$, we get

$$\nabla_i \mathcal{L}_{\mathrm{RB}}(\theta^*) \;\; = \;\; \sum_{j:i \in S_j} \sum_{a=1}^{\ell_j} \sum_{\substack{i' \in S_j \\ i' \neq i}} \lambda_{j,a}\, \mathbb{I}_{\big\{(i,i') \in G_{j,a}\big\}} \left(\mathbb{I}_{\big\{\sigma_j^{-1}(i) < \sigma_j^{-1}(i')\big\}} - \frac{\exp(\theta_i^*)}{\exp(\theta_i^*) + \exp(\theta_{i'}^*)} \right) . \tag{30}$$

Lemma 7. *Under the hypotheses of Theorem 2, with probability at least* $1 - 2e^3 d^{-3}$,

$$\big\|\nabla\mathcal{L}_{\mathrm{RB}}(\theta^*)\big\|_2 \;\leq\; \sqrt{6\log d \sum_{j=1}^{n}\sum_{a=1}^{\ell_j}\big(\lambda_{j,a}\big)^2\big(\kappa_j - p_{j,a}\big)\big(\kappa_j - p_{j,a}+1\big)}\,.$$

The Hessian matrix $H(\theta) \in \mathcal{S}^d$ with $H_{ii'}(\theta) = \frac{\partial^2 \mathcal{L}_{\mathrm{RB}}(\theta)}{\partial\theta_i \partial\theta_{i'}}$ is given by

$$H(\theta) = -\sum_{j=1}^{n}\sum_{a=1}^{\ell_j}\sum_{i<i'\in S_j}\mathbb{I}_{\left\{(i,i')\in G_{j,a}\right\}}\lambda_{j,a}\left((e_i - e_{i'})(e_i - e_{i'})^\top \frac{\exp(\theta_i + \theta_{i'})}{[\exp(\theta_i)+\exp(\theta_{i'})]^2}\right). \tag{31}$$

It follows from the definition that $-H(\theta)$ is positive semi-definite for any $\theta \in \mathbb{R}^d$. The smallest eigenvalue of $-H(\theta)$ is equal to zero and the corresponding eigenvector is all-ones vector. The following lemma lower bounds its second smallest eigenvalue $\lambda_2(-H(\theta))$.

Lemma 8. *Under the hypotheses of Theorem 2, if*

$$\sum_{j=1}^{n}\sum_{a=1}^{\ell_j}\lambda_{j,a}(\kappa_j - p_{j,a}) \geq 2^6 e^{18b}\frac{\eta\delta}{\alpha^2\beta\gamma^2\tau} d\log d \tag{32}$$

then with probability at least $1 - d^{-3}$, *the following holds for any* $\theta \in \Omega_b$*:*

$$\lambda_2(-H(\theta)) \;\geq\; \frac{e^{-4b}}{(1+e^{2b})^2}\frac{\alpha\gamma}{d-1}\sum_{j=1}^{n}\sum_{a=1}^{\ell_j}\lambda_{j,a}(\kappa_j - p_{j,a})\,. \tag{33}$$

Define $\Delta = \widehat{\theta} - \theta^*$. It follows from the definition that Δ is orthogonal to the all-ones vector. By the definition of $\hat{\theta}$ as the optimal solution of the optimization (6), we know that $\mathcal{L}_{\mathrm{RB}}(\widehat{\theta}) \geq \mathcal{L}_{\mathrm{RB}}(\theta^*)$ and thus

$$\mathcal{L}_{\mathrm{RB}}(\widehat{\theta}) - \mathcal{L}_{\mathrm{RB}}(\theta^*) - \langle\nabla\mathcal{L}_{\mathrm{RB}}(\theta^*),\Delta\rangle \;\geq\; -\langle\nabla\mathcal{L}_{\mathrm{RB}}(\theta^*),\Delta\rangle \;\geq\; -\|\nabla\mathcal{L}_{\mathrm{RB}}(\theta^*)\|_2\|\Delta\|_2, \tag{34}$$

where the last inequality follows from the Cauchy-Schwartz inequality. By the mean value theorem, there exists a $\theta = a\widehat{\theta} + (1-a)\theta^*$ for some $a \in [0,1]$ such that $\theta \in \Omega_b$ and

$$\mathcal{L}_{\mathrm{RB}}(\widehat{\theta}) - \mathcal{L}_{\mathrm{RB}}(\theta^*) - \langle\nabla\mathcal{L}_{\mathrm{RB}}(\theta^*),\Delta\rangle \;=\; \frac{1}{2}\Delta^\top H(\theta)\Delta \leq -\frac{1}{2}\lambda_2(-H(\theta))\|\Delta\|_2^2, \tag{35}$$

where the last inequality holds because the Hessian matrix $-H(\theta)$ is positive semi-definite with $H(\theta)\mathbf{1} = \mathbf{0}$ and $\Delta^\top\mathbf{1} = 0$. Combining (34) and (35),

$$\|\Delta\|_2 \;\leq\; \frac{2\|\nabla\mathcal{L}_{\mathrm{RB}}(\theta^*)\|_2}{\lambda_2(-H(\theta))}. \tag{36}$$

Note that $\theta \in \Omega_b$ by definition. Theorem 5 follows by combining Equation (36) with Lemma 7 and Lemma 8.

B.1. Proof of Lemma 7

The idea of the proof is to view $\nabla\mathcal{L}_{\mathrm{RB}}(\theta^*)$ as the final value of a discrete time vector-valued martingale with values in $\mathbb{R}^d$. Define $\nabla\mathcal{L}_{G_{j,a}}(\theta^*)$ as the gradient vector arising out of each rank-breaking graph $\{G_{j,a}\}_{j\in[n],a\in[\ell_j]}$ that is

$$\nabla_i\mathcal{L}_{G_{j,a}}(\theta^*) \equiv \sum_{\substack{i'\in S_j\\ i'\neq i}}\lambda_{j,a}\,\mathbb{I}_{\left\{(i,i')\in G_{j,a}\right\}}\left(\mathbb{I}_{\left\{\sigma_j^{-1}(i)<\sigma_j^{-1}(i')\right\}} - \frac{\exp(\theta_i^*)}{\exp(\theta_i^*)+\exp(\theta_{i'}^*)}\right). \tag{37}$$

Consider $\nabla\mathcal{L}_{G_{j,a}}(\theta^*)$ as the incremental random vector in a martingale of $\sum_{j=1}\ell_j$ time steps. Lemma 9 shows that the expectation of each incremental vector is zero. Observe that the conditioning event $\{i'' \in S : \sigma^{-1}(i'') < p_{j,a}\}$ given in

(39) is equivalent to conditioning on the history $\{G_{j,a'}\}_{a'<a}$. Therefore, using the assumption that the rankings $\{\sigma_j\}_{j\in[n]}$ are mutually independent, we have that the conditional expectation of $\nabla\mathcal{L}_{G_{j,a}}(\theta^*)$ conditioned on $\{G_{j',a''}\}_{j'<j,a''\in[\ell_{j'}]}$ is zero. Further, the conditional expectation of $\nabla\mathcal{L}_{G_{j,a}}(\theta^*)$ is zero even when conditioned on the rank breaking due to previous separators $\{G_{j,a'}\}_{a'<a}$ that are ranked higher (i.e. $a' < a$), which follows from the next lemma.

Lemma 9. *For a position-p rank breaking graph G_p, defined over a set of items S, where $p \in [|S|-1]$,*

$$\mathbb{P}\Big[\sigma^{-1}(i) < \sigma^{-1}(i') \;\Big|\; (i,i') \in G_p\Big] \;=\; \frac{\exp(\theta_i^*)}{\exp(\theta_i^*)+\exp(\theta_{i'}^*)}\;, \tag{38}$$

for all $i, i' \in S$ and also

$$\mathbb{P}\Big[\sigma^{-1}(i) < \sigma^{-1}(i') \;\Big|\; (i,i') \in G_p \text{ and } \{i'' \in S \,:\, \sigma^{-1}(i'') < p\}\Big] \;=\; \frac{\exp(\theta_i^*)}{\exp(\theta_i^*)+\exp(\theta_{i'}^*)}\;. \tag{39}$$

This is one of the key technical lemmas since it implies that the proposed rank-breaking is consistent, i.e. $\mathbb{E}_{\theta^*}[\nabla\mathcal{L}_{\text{RB}}(\theta^*)] = 0$. Throughout the proof of Theorem 2, this is the only place where the assumption on the proposed (consistent) rank-breaking is used. According to a companion theorem in Azari Soufiani et al. (2014, Theorem 2), it also follows that any rank-breaking that is not union of position-p rank-breakings results in inconsistency, i.e. $\mathbb{E}_{\theta^*}[\nabla\mathcal{L}_{\text{RB}}(\theta^*)] \neq 0$. We claim that for each rank-breaking graph $G_{j,a}$, $\|\nabla\mathcal{L}_{G_{j,a}}(\theta^*)\|_2^2 \leq (\lambda_{j,a})^2(\kappa_j - p_{j,a})(\kappa_j - p_{j,a}+1)$. By Lemma 10 which is a generalization of the vector version of the Azuma-Hoeffding inequality found in (Hayes, 2005, Theorem 1.8), we have

$$\mathbb{P}\big[\big\|\nabla\mathcal{L}_{\text{RB}}(\theta^*)\big\|_2 \geq \delta\big] \;\;\leq\;\; 2e^3 \exp\left(\frac{-\delta^2}{2\sum_{j=1}^n\sum_{a=1}^{\ell_j}\big(\lambda_{j,a}\big)^2\big(\kappa_j - p_{j,a}\big)\big(\kappa_j - p_{j,a}+1\big)}\right), \tag{40}$$

which implies the result.

Lemma 10. *Let $(X_1, X_2, \cdots, X_n)$ be real-valued martingale taking values in $\mathbb{R}^d$ such that $X_0 = 0$ and for every $1 \leq i \leq n$, $\|X_i - X_{i-1}\|_2 \leq c_i$, for some non-negative constant c_i. Then for every $\delta > 0$,*

$$\mathbb{P}[\|X_n\|_2 \geq \delta] \;\;\leq\;\; 2e^3 e^{-\frac{\delta^2}{2\sum_{i=1}^n c_i^2}}\;. \tag{41}$$

It follows from the upper bound on $\|\nabla\mathcal{L}_{G_{j,a}}(\theta^*)\|_2^2 \leq c_i^2$ with $c_i^2 = \lambda^2\big((k_j - p_{j,a})^2 + (k_j - p_{j,a})\big)$. In the expression (37), $\nabla\mathcal{L}_{G_{j,a}}(\theta^*)$ has one entry at $p_{j,a}$-th position that is compared to $(k_j - p_{j,a})$ other items and $(k_j - p_{j,a})$ entries that is compared only once, giving the bound

$$\|\nabla\mathcal{L}_{G_{j,a}}(\theta^*)\|_2^2 \;\;\leq\;\; \lambda_{j,a}^2(k_j - p_{j,a})^2 + \lambda_{j,a}^2(k_j - p_{j,a})\;.$$

B.2. Proof of Lemma 9

Define event $E \equiv \{(i,i') \in G_p\}$. Observe that

$$E = \Big\{\Big(\mathbb{I}_{\{(\sigma^{-1}(i)=p\}} + \mathbb{I}_{\{\sigma^{-1}(i'))=p\}} = 1\Big) \wedge \Big(\sigma^{-1}(i), \sigma^{-1}(i') \geq p\Big)\Big\}\;.$$

Consider any set $\Omega \subset S \setminus \{i,i'\}$ such that $|\Omega| = p-1$. Let M denote an event that items of the set Ω are ranked in top-$(p-1)$ positions in a particular order. It is easy to verify the following:

$$\begin{aligned}
\mathbb{P}\Big[\sigma^{-1}(i) < \sigma^{-1}(i')\Big|E, M\Big] = \frac{\mathbb{P}\Big[\big(\sigma^{-1}(i) < \sigma^{-1}(i')\big), E, M\Big]}{\mathbb{P}\Big[E, M\Big]} \;\;&=\;\; \frac{\mathbb{P}\Big[\big(\sigma^{-1}(i) = p\big), M\Big]}{\mathbb{P}\Big[\big(\sigma^{-1}(i) = p\big), M\Big] + \mathbb{P}\Big[\big(\sigma^{-1}(i') = p\big), M\Big]} \\
&=\;\; \frac{\exp(\theta_i^*)}{\exp(\theta_i^*)+\exp(\theta_{i'}^*)} \\
&=\;\; \mathbb{P}\Big[\sigma^{-1}(i) < \sigma^{-1}(i')\Big]\;.
\end{aligned}$$

Since M is any particular ordering of the set Ω and Ω is any subset of $S \setminus \{i,i'\}$ such that $|\Omega| = p-1$, conditioned on event E probabilities of all the possible events M over all the possible choices of set Ω sum to 1.

B.3. Proof of Lemma 10

It follows exactly along the lines of proof of Theorem 1.8 in (Hayes, 2005).

B.4. Proof of Lemma 8

The Hessian $H(\theta)$ is given in (31). For all $j \in [n]$, define $M^{(j)} \in \mathcal{S}^d$ as

$$M^{(j)} \;\; \equiv \;\; \sum_{a=1}^{\ell_j} \lambda_{j,a} \sum_{i<i' \in S_j} \mathbb{I}_{\big\{(i,i')\,\in\, G_{j,a}\big\}} (e_i - e_{i'})(e_i - e_{i'})^\top , \tag{42}$$

and let $M \equiv \sum_{j=1}^n M^{(j)}$. Observe that M is positive semi-definite and the smallest eigenvalue of M is zero with the corresponding eigenvector given by the all-ones vector. If $|\theta_i| \leq b$, for all $i \in [d]$, $\frac{\exp(\theta_i + \theta_{i'})}{[\exp(\theta_i)+\exp(\theta_{i'})]^2} \geq \frac{e^{2b}}{(1+e^{2b})^2}$. Recall the definition of $H(\theta)$ from Equation (31). It follows that $-H(\theta) \succeq \frac{e^{2b}}{(1+e^{2b})^2} M$ for $\theta \in \Omega_b$. Since, $-H(\theta)$ and M are symmetric matrices, from Weyl's inequality we have, $\lambda_2(-H(\theta)) \geq \frac{e^{2b}}{(1+e^{2b})^2}\lambda_2(M)$. Again from Weyl's inequality, it follows that

$$\lambda_2(M) \;\; \geq \;\; \lambda_2(\mathbb{E}[M]) - \|M - \mathbb{E}[M]\| \, , \tag{43}$$

where $\|\cdot\|$ denotes the spectral norm. We will show in (48) that $\lambda_2(\mathbb{E}[M]) \geq 2\gamma e^{-6b}(\alpha/(d-1)) \sum_{j=1}^n \tau_j \ell_j$, and in (60) that $\|M - \mathbb{E}[M]\| \leq 8e^{3b}\sqrt{\frac{\eta\delta \log d}{\beta\tau d} \sum_{j=1}^n \tau_j \ell_j}$.

$$\lambda_2(M) \;\; \geq \;\; \frac{2e^{-6b}\alpha\gamma}{d-1}\sum_{j=1}^n \tau_j\ell_j - 8e^{3b}\sqrt{\frac{\eta\delta\log d}{\beta\tau d}\sum_{j=1}^n \tau_j\ell_j} \;\; \geq \;\; \frac{e^{-6b}\alpha\gamma}{d-1}\sum_{j=1}^n \tau_j\ell_j \; , \tag{44}$$

where the last inequality follows from the assumption that $\sum_{j=1}^n \tau_j\ell_j \geq 2^6 e^{18b}\frac{\eta\delta}{\alpha^2\beta\gamma^2\tau} d\log d$. This proves the desired claim.

To prove the lower bound on $\lambda_2(\mathbb{E}[M])$, notice that

$$\mathbb{E}[M] \;\; = \;\; \sum_{j=1}^n \sum_{a=1}^{\ell_j} \lambda_{j,a} \sum_{i<i' \in S_j} \mathbb{P}\Big[(i,i') \in G_{j,a}\Big| (i,i' \in S_j)\Big] (e_i - e_{i'})(e_i - e_{i'})^\top \, . \tag{45}$$

The following lemma provides a lower bound on $\mathbb{P}[(i,i') \in G_{j,a} | (i,i' \in S_j)]$.

Lemma 11. *Consider a ranking σ over a set $S \subseteq [d]$ such that $|S| = \kappa$. For any two items $i, i' \in S$, $\theta \in \Omega_b$, and $1 \leq \ell \leq \kappa - 1$,*

$$\mathbb{P}_\theta\Big[\sigma^{-1}(i) = \ell, \sigma^{-1}(i') > \ell\Big] \;\; \geq \;\; \frac{e^{-6b}(\kappa-\ell)}{\kappa(\kappa-1)}\left(1 - \frac{\ell}{\kappa}\right)^{\alpha_{i,i',\ell,\theta}-2} \, , \tag{46}$$

where the probability $\mathbb{P}_\theta$ is with respect to the sampled ranking resulting from PL weights $\theta \in \Omega_b$, and $\alpha_{i,i',\ell,\theta}$ is defined as $1 \leq \alpha_{i,i',\ell,\theta} = \lceil \widetilde{\alpha}_{i,i',\ell,\theta} \rceil$, and $\widetilde{\alpha}_{i,i',\ell,\theta}$ is,

$$\widetilde{\alpha}_{i,i',\ell,\theta} \;\; \equiv \;\; \max_{\ell' \in [\ell]} \max_{\substack{\Omega \subseteq S \setminus \{i,i'\} \\ : |\Omega| = \kappa - \ell'}} \left\{ \frac{\exp(\theta_i) + \exp(\theta_{i'})}{\left(\sum_{j \in \Omega} \exp(\theta_j)\right)/|\Omega|} \right\} . \tag{47}$$

Note that we do not need $\max_{\ell' \in [\ell]}$ in the above equation as the expression achieves its maxima at $\ell' = \ell$, but we keep the definition to avoid any confusion. In the worst case, $2e^{-2b} \leq \widetilde{\alpha}_{i,i',\ell,\theta} \leq 2e^{2b}$. Therefore, using definition of rank breaking graph $G_{j,a}$, and Equations (45) and (46) we have,

$$\begin{aligned}
\mathbb{E}[M] &\succeq \gamma e^{-6b} \sum_{j=1}^{n} \sum_{a=1}^{\ell_j} \lambda_{j,a} \frac{2(\kappa_j - p_{j,a})}{\kappa_j(\kappa_j - 1)} \sum_{i<i' \in S_j} (e_i - e_{i'})(e_i - e_{i'})^\top \\
&\succeq 2\gamma e^{-6b} \sum_{j=1}^{n} \frac{1}{\kappa_j(\kappa_j - 1)} \sum_{a=1}^{\ell_j} \lambda_{j,a}(\kappa_j - p_{j,a}) \sum_{i<i' \in S_j} (e_i - e_{i'})(e_i - e_{i'})^\top \\
&= 2\gamma e^{-6b} L, \qquad (48)
\end{aligned}$$

where we used $\gamma \le (1 - p_{j,\ell_j}/\kappa_j)^{\alpha_1 - 2}$ which follows for the definition in (11). (48) follows from the definition of Laplacian L, defined for the comparison graph $\mathcal{H}$ in Definition 6. Using $\lambda_2(L) = (\alpha/(d-1)) \sum_{j=1}^n \tau_j \ell_j$ from (25), we get the desired bound $\lambda_2(\mathbb{E}[M]) \ge 2\gamma e^{-6b} (\alpha/(d-1)) \sum_{j=1}^n \tau_j \ell_j$.

Next we need to upper bound $\| \sum_{j=1}^n \mathbb{E}[(M^j)^2] \|$ to bound the deviation of M from its expectation. To this end, we prove an upper bound on $\mathbb{P}[\sigma_j^{-1}(i) = p_{j,a} \mid i \in S_j]$ in the following lemma.

Lemma 12. *Under the hypotheses of Lemma 11,*

$$\mathbb{P}_\theta\Big[\sigma^{-1}(i) = \ell\Big] \;\le\; \frac{e^{6b}}{\kappa}\left(1 - \frac{\ell}{\kappa + \alpha_{i,\ell,\theta}}\right)^{\alpha_{i,\ell,\theta} - 1} \;\le\; \frac{e^{6b}}{\kappa - \ell}\,, \qquad (49)$$

where $0 \le \alpha_{i,\ell,\theta} = \lfloor \widetilde{\alpha}_{i,\ell,\theta} \rfloor$*, and* $\widetilde{\alpha}_{i,\ell,\theta}$ *is,*

$$\widetilde{\alpha}_{i,\ell,\theta} \;\equiv\; \min_{\ell' \in [\ell]} \min_{\substack{\Omega \in S \setminus \{i\} \\ : |\Omega| = \kappa - \ell' + 1}} \left\{ \frac{\exp(\theta_i)}{\big(\sum_{j \in \Omega} \exp(\theta_j)\big)/|\Omega|} \right\}. \qquad (50)$$

In the worst case, $e^{-2b} \le \widetilde{\alpha}_{i,\ell,\theta} \le e^{2b}$*. Note that* $\alpha_{i,\ell,\theta} = 0$ *gives the worst upper bound.*

Therefore using Equation (49), for all $i \in [d]$, we have,

$$\mathbb{P}\Big[\sigma_j^{-1}(i) \in \mathcal{P}_j\Big] \le \min\left\{1, \frac{e^{6b}\ell_j}{\kappa_j - p_{j,\ell_j}}\right\} \;\le\; \frac{e^{6b}\ell_j}{\max\{\ell_j, \kappa_j - p_{j,\ell_j}\}} \le \frac{e^{6b}\eta\ell_j}{\kappa_j}\,, \qquad (51)$$

where we used η defined in Equation (12). Define a diagonal matrix $D^{(j)} \in \mathcal{S}^d$ and a matrix $A^{(j)} \in \mathcal{S}^d$,

$$A^{(j)}_{ii'} \;\equiv\; \mathbb{I}_{\big\{i,i' \in S_j\big\}} \sum_{a=1}^{\ell_j} \lambda_{j,a} \mathbb{I}_{\big\{(i,i') \in G_{j,a}\big\}}\;, \text{ for all } \; i, i' \in [d]\,, \qquad (52)$$

and $D^{(j)}_{ii} = \sum_{i' \ne i} A^{(j)}_{ii'}$. Observe that $M^{(j)} = D^{(j)} - A^{(j)}$. For all $i \in [d]$, we have,

$$\begin{aligned}
D^{(j)}_{ii} &= \mathbb{I}_{\big\{i \in S_j\big\}} \sum_{i'=1}^{\kappa_j} \mathbb{I}_{\big\{\sigma_j^{-1}(i) = i'\big\}} \sum_{a=1}^{\ell_j} \lambda_{j,a} \mathrm{deg}_{G_{j,a}}(\sigma_j^{-1}(i')) \\
&\le \mathbb{I}_{\big\{i \in S_j\big\}} \left\{ \mathbb{I}_{\big\{\sigma_j^{-1}(i) \in \mathcal{P}_j\big\}} \left(\max_{a \in [\ell_j]} \Big\{ \lambda_{j,a}(\kappa_j - p_{j,a}) \Big\} + \sum_{a=1}^{\ell_j} \lambda_{j,a} \right) + \mathbb{I}_{\big\{\sigma_j^{-1}(i) \notin \mathcal{P}_j\big\}} \left(\sum_{a=1}^{\ell_j} \lambda_{j,a} \right) \right\} \\
&= \mathbb{I}_{\big\{i \in S_j\big\}} \left\{ \mathbb{I}_{\big\{\sigma_j^{-1}(i) \in \mathcal{P}_j\big\}} \delta_{j,1} \;+\; \mathbb{I}_{\big\{\sigma_j^{-1}(i) \notin \mathcal{P}_j\big\}} \delta_{j,2} \right\}, \qquad (53)
\end{aligned}$$

where the last equality follows from the definition of $\delta_{j,1}$ and $\delta_{j,2}$ in Equation (21). Note that $\max_{i \in [d]}\{D_{ii}\} = \delta_{j,1}$. Using (51) and (53), we have,

$$\mathbb{E}\Big[D^{(j)}_{ii}\Big] \;\le\; \mathbb{I}_{\big\{i \in S_j\big\}} \left\{ \frac{e^{6b}\eta\ell_j}{\kappa_j} \left(\delta_{j,1} + \frac{\delta_{j,2}\kappa_j}{\eta\ell_j} \right) \right\}. \qquad (54)$$

Similarly we have,

$$\mathbb{E}\Big[\big(D_{ii}^{(j)}\big)^2\Big] \;\; \le \;\; \mathbb{I}_{\{i\in S_j\}}\left\{\frac{e^{6b}\eta\ell_j}{\kappa_j}\left(\delta_{j,1}^2 + \frac{\delta_{j,2}^2\kappa_j}{\eta\ell_j}\right)\right\} \tag{55}$$

For all $i \in [d]$, we have,

$$\begin{aligned}
\mathbb{E}\left[\sum_{i'=1}^{d}\big(\big(A^{(j)}\big)^2\big)_{ii'}\right] &\le \mathbb{E}\left[\left(\sum_{i'=1}^{d}A_{ii'}^{(j)}\right)\max_{i\in[d]}\left\{\sum_{i'=1}^{d}A_{ii'}^{(j)}\right\}\right] \\
&\le \mathbb{E}\Big[D_{ii}^{(j)}\delta_{j,1}\Big] \\
&\le \mathbb{I}_{\{i\in S_j\}}\left\{\frac{e^{6b}\eta\ell_j}{\kappa_j}\left(\delta_{j,1}^2 + \frac{\delta_{j,1}\delta_{j,2}\kappa_j}{\eta\ell_j}\right)\right\}. \qquad (56)
\end{aligned}$$

Using (55) and (56), we have, for all $i \in [d]$,

$$\begin{aligned}
\sum_{i'=1}^{d}\left|\mathbb{E}\Big[\big(\big(M^{(j)}\big)^2\big)_{ii'}\Big]\right| &= \sum_{i'=1}^{d}\left|\mathbb{E}\Big[\big(\big(D^{(j)}\big)^2\big)_{ii'}\Big] - \mathbb{E}\Big[\big(D^{(j)}A^{(j)}\big)_{ii'}\Big] - \mathbb{E}\Big[\big(A^{(j)}D^{(j)}\big)_{ii'}\Big] + \mathbb{E}\Big[\big(\big(A^{(j)}\big)^2\big)_{ii'}\Big]\right| \\
&\le 2\mathbb{E}\Big[\big(D_{ii}^{(j)}\big)^2\Big] + \sum_{i'=1}^{d}\left(\mathbb{E}\Big[\delta_{j,1}\big(A^{(j)}\big)_{ii'}\Big] + \mathbb{E}\Big[\big(\big(A^{(j)}\big)^2\big)_{ii'}\Big]\right) \\
&\le \mathbb{I}_{\{i\in S_j\}}\left\{\frac{e^{6b}\eta\ell_j}{\kappa_j}\left(4\delta_{j,1}^2 + \frac{2\big(\delta_{j,1}\delta_{j,2}+\delta_{j,2}^2\big)\kappa_j}{\eta\ell_j}\right)\right\} \\
&= \mathbb{I}_{\{i\in S_j\}}\left\{\frac{e^{6b}\delta\eta\ell_j}{\kappa_j}\right\}, \qquad (57)
\end{aligned}$$

where the last equality follows from the definition of δ, Equation (22).

To bound $\|\sum_{j=1}^{n}\mathbb{E}[(M^{(j)})^2]\|$, we use the fact that for $J \in \mathbb{R}^{d\times d}$, $\|J\| \le \max_{i\in[d]}\sum_{i'=1}^{d}|J_{ii'}|$. Therefore, we have

$$\begin{aligned}
\left\|\sum_{j=1}^{n}\mathbb{E}\Big[(M^{(j)})^2\Big]\right\| &\le e^{6b}\delta\eta\max_{i\in[d]}\left\{\sum_{j:i\in S_j}\frac{\ell_j}{\kappa_j}\right\} \\
&= \frac{e^{6b}\eta\delta}{\tau}D_{\max} \qquad (58)\\
&= \frac{e^{6b}\eta\delta}{\beta\tau d}\sum_{j=1}^{n}\tau_j\ell_j\;, \qquad (59)
\end{aligned}$$

where (58) follows from the definition of $D_{\max}$ in Equation(23) and (59) follows from the definition of β in (25). Observe that from Equation (53), $\|M^{(j)}\| \le 2\delta_{j,1} \le 2\sqrt{\delta}$. Applying matrix Bernstien inequality, we have,

$$\mathbb{P}\Big[\big\|M - \mathbb{E}[M]\big\| \ge t\Big] \le d\,\exp\left(\frac{-t^2/2}{\frac{e^{6b}\eta\delta}{\beta\tau d}\sum_{j=1}^{n}\tau_j\ell_j + 4\sqrt{\delta}t/3}\right).$$

Therefore, with probability at least $1 - d^{-3}$, we have,

$$\big\|M - \mathbb{E}[M]\big\| \le 4e^{3b}\sqrt{\frac{\eta\delta\log d}{\beta\tau d}\sum_{j=1}^{n}\tau_j\ell_j} + \frac{64\sqrt{\delta}\log d}{3} \le 8e^{3b}\sqrt{\frac{\eta\delta\log d}{\beta\tau d}\sum_{j=1}^{n}\tau_j\ell_j}\;, \tag{60}$$

where the second inequality uses $\sum_{j=1}^{n}\tau_j\ell_j \ge 2^6(\beta\tau/\eta)d\log d$ which follows from the assumption that $\sum_{j=1}^{n}\tau_j\ell_j \ge 2^6e^{18b}\frac{\eta\delta}{\tau\gamma^2\alpha^2\beta}d\log d$ and the fact that $\alpha, \beta \le 1$, $\gamma \le 1$, $\eta \ge 1$, and $\delta > \tau^2$.

B.5. Proof of Lemma 11

Since providing a lower bound on $\mathbb{P}_\theta\big[\sigma^{-1}(i) = \ell, \sigma^{-1}(i') > \ell\big]$ for arbitrary θ is challenging, we construct a new set of parameters $\{\widetilde{\theta}_j\}_{j\in[d]}$ from the original θ. These new parameters are constructed such that it is both easy to compute the probability and also provides a lower bound on the original distribution. We denote the sum of the weights by $W \equiv \sum_{j\in S}\exp(\theta_j)$. We define a new set of parameters $\{\widetilde{\theta}_j\}_{j\in S}$:

$$\widetilde{\theta}_j \quad = \quad \begin{cases} \log(\widetilde{\alpha}_{i,i',\ell,\theta}/2) & \text{for } j = i \text{ or } i' \,, \\ 0 & \text{otherwise} \,. \end{cases} \tag{61}$$

Similarly define $\widetilde{W} \equiv \sum_{j\in S}\exp(\widetilde{\theta}_j) = \kappa - 2 + \widetilde{\alpha}_{i,i',\ell,\theta}$. We have,

$$\begin{aligned} & \mathbb{P}_\theta\Big[\sigma^{-1}(i) = \ell, \sigma^{-1}(i') > \ell\Big] \\ = \; & \sum_{\substack{j_1\in S\\ j_1\neq i,i'}} \left(\frac{\exp(\theta_{j_1})}{W} \sum_{\substack{j_2\in S\\ j_2\neq i,i',j_1}} \left(\frac{\exp(\theta_{j_2})}{W - \exp(\theta_{j_1})} \cdots \left(\sum_{\substack{j_{\ell-1}\in S\\ j_{\ell-1}\neq i,i',\\ j_1,\cdots,j_{\ell-2}}} \frac{\exp(\theta_{j_{\ell-1}})}{W - \sum_{k=j_1}^{j_{\ell-2}}\exp(\theta_k)} \frac{\exp(\theta_i)}{W - \sum_{k=j_1}^{j_{\ell-1}}\exp(\theta_k)}\right)\cdots\right)\right) \\ = \; & \frac{\exp(\theta_i)}{W} \sum_{\substack{j_1\in S\\ j_1\neq i,i'}} \left(\frac{\exp(\theta_{j_1})}{W - \exp(\theta_{j_1})} \sum_{\substack{j_2\in S\\ j_2\neq i,i',j_1}} \left(\frac{\exp(\theta_{j_2})}{W - \exp(\theta_{j_1}) - \exp(\theta_{j_2})} \cdots \sum_{\substack{j_{\ell-1}\in S\\ j_{\ell-1}\neq i,i',\\ j_1,\cdots,j_{\ell-2}}} \left(\frac{\exp(\theta_{j_{\ell-1}})}{W - \sum_{k=j_1}^{j_{\ell-1}}\exp(\theta_k)}\right)\cdots\right)\right) \end{aligned} \tag{62}$$

Consider the last summation term in the above equation and let $\Omega_\ell = S \setminus \{i, i', j_1, \ldots, j_{\ell-2}\}$. Observe that, $|\Omega_\ell| = \kappa - \ell$ and from equation (47), $\frac{\exp(\theta_i)+\exp(\theta_{i'})}{\sum_{j\in\Omega_\ell}\exp(\theta_j)} \leq \frac{\widetilde{\alpha}_{i,i',\ell,\theta}}{\kappa-\ell}$. We have,

$$\begin{aligned} \sum_{j_{\ell-1}\in\Omega_\ell} \frac{\exp(\theta_{j_{\ell-1}})}{W - \sum_{k=j_1}^{j_{\ell-1}}\exp(\theta_k)} \; &= \; \sum_{j_{\ell-1}\in\Omega_\ell} \frac{\exp(\theta_{j_{\ell-1}})}{W - \sum_{k=j_1}^{j_{\ell-2}}\exp(\theta_k) - \exp(\theta_{j_{\ell-1}})} \\ &\geq \; \frac{\sum_{j_{\ell-1}\in\Omega_\ell}\exp(\theta_{j_{\ell-1}})}{W - \sum_{k=j_1}^{j_{\ell-2}}\exp(\theta_k) - \big(\sum_{j_{\ell-1}\in\Omega_\ell}\exp(\theta_{j_{\ell-1}})\big)/|\Omega_\ell|} \qquad (63) \\ &= \; \frac{\sum_{j_{\ell-1}\in\Omega_\ell}\exp(\theta_{j_{\ell-1}})}{\exp(\theta_i) + \exp(\theta_{i'}) + \sum_{j_{\ell-1}\in\Omega_\ell}\exp(\theta_{j_{\ell-1}}) - \big(\sum_{j_{\ell-1}\in\Omega_\ell}\exp(\theta_{j_{\ell-1}})\big)/|\Omega_\ell|} \\ &= \; \left(\frac{\exp(\theta_i) + \exp(\theta_{i'})}{\sum_{j_{\ell-1}\in\Omega_\ell}\exp(\theta_{j_{\ell-1}})} + 1 - \frac{1}{\kappa-\ell}\right)^{-1} \\ &\geq \; \left(\frac{\widetilde{\alpha}_1}{\kappa-\ell} + 1 - \frac{1}{\kappa-\ell}\right)^{-1} \qquad (64) \\ &= \; \frac{\kappa-\ell}{\widetilde{\alpha}_1 + \kappa - \ell - 1} \\ &= \; \sum_{j_{\ell-1}\in\Omega_\ell} \frac{\exp(\widetilde{\theta}_{j_{\ell-1}})}{\widetilde{W} - \sum_{k=j_1}^{j_{\ell-2}}\exp(\widetilde{\theta}_k) - \exp(\widetilde{\theta}_{j_{\ell-1}})} \,, \qquad (65) \end{aligned}$$

where (63) follows from the Jensen's inequality and the fact that for any $c > 0$, $0 < x < c$, $\frac{x}{c-x}$ is convex in x. Equation (64) follows from the definition of $\widetilde{\alpha}_{i,i',\ell,\theta}$, (47), and the fact that $|\Omega_\ell| = \kappa - \ell$. Equation (65) uses the definition of $\{\widetilde{\theta}_j\}_{j\in S}$.

Consider $\{\Omega_{\widetilde{\ell}}\}_{2\leq\widetilde{\ell}\leq\ell-1}$, $|\Omega_{\widetilde{\ell}}| = \kappa - \widetilde{\ell}$, corresponding to the subsequent summation terms in (62). Observe that $\frac{\exp(\theta_i)+\exp(\theta_{i'})}{\sum_{j\in\Omega_{\widetilde{\ell}}}\exp(\theta_j)} \leq \widetilde{\alpha}_{i,i',\ell,\theta}/|\Omega_{\widetilde{\ell}}|$. Therefore, each summation term in equation (62) can be lower bounded by the corresponding term where $\{\theta_j\}_{j\in S}$ is replaced by $\{\widetilde{\theta}_j\}_{j\in S}$. Hence, we have

$$\begin{aligned}
&\mathbb{P}_\theta\Big[\sigma^{-1}(i)=\ell,\sigma^{-1}(i')>\ell\Big] \\
\geq\;& \frac{\exp(\theta_i)}{W}\sum_{\substack{j_1\in S\\ j_1\neq i,i'}}\left(\frac{\exp(\widetilde{\theta}_{j_1})}{\widetilde{W}-\exp(\widetilde{\theta}_{j_1})}\sum_{\substack{j_2\in S\\ j_2\neq i,i',j_1}}\left(\frac{\exp(\widetilde{\theta}_{j_2})}{\widetilde{W}-\exp(\widetilde{\theta}_{j_1})-\exp(\widetilde{\theta}_{j_2})}\cdots\sum_{\substack{j_{\ell-1}\in S\\ j_{\ell-1}\neq i,i',\\ j_1,\cdots,j_{\ell-2}}}\left(\frac{\exp(\widetilde{\theta}_{j_{\ell-1}})}{\widetilde{W}-\sum_{k=j_1}^{j_{\ell-1}}\exp(\widetilde{\theta}_k)}\right)\right)\right) \\
\geq\;& \frac{e^{-4b}\exp(\widetilde{\theta}_i)}{\widetilde{W}}\sum_{\substack{j_1\in S\\ j_1\neq i,i'}}\left(\frac{\exp(\widetilde{\theta}_{j_1})}{\widetilde{W}-\exp(\widetilde{\theta}_{j_1})}\sum_{\substack{j_2\in S\\ j_2\neq i,i',j_1}}\left(\frac{\exp(\widetilde{\theta}_{j_2})}{\widetilde{W}-\exp(\widetilde{\theta}_{j_1})-\exp(\widetilde{\theta}_{j_2})}\cdots\sum_{\substack{j_{\ell-1}\in S\\ j_{\ell-1}\neq i,i',\\ j_1,\cdots,j_{\ell-2}}}\left(\frac{\exp(\widetilde{\theta}_{j_{\ell-1}})}{\widetilde{W}-\sum_{k=j_1}^{j_{\ell-1}}\exp(\widetilde{\theta}_k)}\right)\right)\right) \\
=\;& \left(e^{-4b}\right)\mathbb{P}_{\widetilde{\theta}}\Big[\sigma^{-1}(i)=\ell,\sigma^{-1}(i')>\ell\Big]\,. \qquad (66)
\end{aligned}$$

The second inequality uses $\frac{\exp(\theta_i)}{W}\geq e^{-2b}/\kappa$ and $\frac{\exp(\widetilde{\theta}_i)}{\widetilde{W}}\leq e^{2b}/\kappa$. Observe that $\exp(\widetilde{\theta}_j)=1$ for all $j\neq i,i'$ and $\exp(\widetilde{\theta}_i)+\exp(\widetilde{\theta}_{i'})=\widetilde{\alpha}_{i,i',\ell,\theta}\leq\lceil\widetilde{\alpha}_{i,i',\ell,\theta}\rceil=\alpha_{i,i',\ell,\theta}\geq 1$. Therefore, we have

$$\begin{aligned}
\mathbb{P}_{\widetilde{\theta}}\Big[\sigma^{-1}(i)=\ell,\sigma^{-1}(i')>\ell\Big] &= \binom{\kappa-2}{\ell-1}\frac{(\widetilde{\alpha}_{i,i',\ell,\theta}/2)(\ell-1)!}{(\kappa-2+\widetilde{\alpha}_{i,i',\ell,\theta})(\kappa-2+\widetilde{\alpha}_{i,i',\ell,\theta}-1)\cdots(\kappa-2+\widetilde{\alpha}_{i,i',\ell,\theta}-(\ell-1))} \\
&\geq \frac{(\kappa-2)!}{(\kappa-\ell-1)!}\frac{e^{-2b}}{(\kappa+\alpha_{i,i',\ell,\theta}-2)(\kappa+\alpha_{i,i',\ell,\theta}-3)\cdots(\kappa+\alpha_{i,i',\ell,\theta}-(\ell+1))} \qquad (67) \\
&= \frac{e^{-2b}(\kappa-\ell+\alpha_{i,i',\ell,\theta}-2)(\kappa-\ell+\alpha_{i,i',\ell,\theta}-3)\cdots(\kappa-\ell)}{(\kappa+\alpha_{i,i',\ell,\theta}-2)(\kappa+\alpha_{i,i',\ell,\theta}-3)\cdots(\kappa-1)} \\
&= \frac{e^{-2b}}{(\kappa-1)}\frac{(\kappa-\ell+\alpha_{i,i',\ell,\theta}-2)(\kappa-\ell+\alpha_{i,i',\ell,\theta}-3)\cdots(\kappa-\ell)}{(\kappa+\alpha_{i,i',\ell,\theta}-2)(\kappa+\alpha_{i,i',\ell,\theta}-3)\cdots(\kappa)} \\
&\geq \frac{e^{-2b}}{(\kappa-1)}\left(1-\frac{\ell}{\kappa}\right)^{\alpha_{i,i',\ell,\theta}-1} \\
&= \frac{e^{-2b}(\kappa-\ell)}{\kappa(\kappa-1)}\left(1-\frac{\ell}{\kappa}\right)^{\alpha_{i,i',\ell,\theta}-2}, \qquad (68)
\end{aligned}$$

where (67) follows from the fact that $\widetilde{\alpha}_{i,i',\ell,\theta}\geq 2e^{-2b}$. Claim (46) follows by combining Equations (66) and (68).

B.6. Proof of Lemma 12

Analogous to the proof of Lemma 11, we construct a new set of parameters $\{\widetilde{\theta}_j\}_{j\in[d]}$ from the original θ. We denote the sum of the weights by $W\equiv\sum_{j\in S}\exp(\theta_j)$. We define a new set of parameters $\{\widetilde{\theta}_j\}_{j\in S}$:

$$\widetilde{\theta}_j \;=\; \begin{cases} \log(\widetilde{\alpha}_{i,\ell,\theta}) & \text{for } j=i\,, \\ 0 & \text{otherwise}\,. \end{cases} \qquad (69)$$

Similarly define $\widetilde{W}\equiv\sum_{j\in S}\exp(\widetilde{\theta}_j)=\kappa-1+\widetilde{\alpha}_{i,\ell,\theta}$. We have,

$$\begin{aligned}
&\mathbb{P}_\theta\Big[\sigma^{-1}(i)=\ell\Big] \\
&=\sum_{\substack{j_1\in S\\ j_1\neq i}}\left(\frac{\exp(\theta_{j_1})}{W}\sum_{\substack{j_2\in S\\ j_2\neq i,j_1}}\left(\frac{\exp(\theta_{j_2})}{W-\exp(\theta_{j_1})}\cdots\left(\sum_{\substack{j_{\ell-1}\in S\\ j_{\ell-1}\neq i,\\ j_1,\cdots,j_{\ell-2}}}\frac{\exp(\theta_{j_{\ell-1}})}{W-\sum_{k=j_1}^{j_{\ell-2}}\exp(\theta_k)}\frac{\exp(\theta_i)}{W-\sum_{k=j_1}^{j_{\ell-1}}\exp(\theta_k)}\right)\right)\right) \\
&\leq\sum_{\substack{j_1\in S\\ j_1\neq i}}\left(\frac{\exp(\theta_{j_1})}{W}\sum_{\substack{j_2\in S\\ j_2\neq i,j_1}}\left(\frac{\exp(\theta_{j_2})}{W-\exp(\theta_{j_1})}\cdots\left(\sum_{\substack{j_{\ell-1}\in S\\ j_{\ell-1}\neq i,\\ j_1,\cdots,j_{\ell-2}}}\frac{\exp(\theta_{j_{\ell-1}})}{W-\sum_{k=j_1}^{j_{\ell-2}}\exp(\theta_k)}\right)\right)\right)\frac{e^{2b}}{\kappa-\ell+1} \qquad (70)
\end{aligned}$$

Consider the last summation term in the equation (70), and let $\Omega_\ell = S \setminus \{i, j_1, \ldots, j_{\ell-2}\}$, such that $|\Omega_\ell| = \kappa - \ell + 1$. Observe that from equation (50), $\frac{\exp(\theta_i)}{\sum_{j\in\Omega_\ell}\exp(\theta_j)} \geq \frac{\widetilde{\alpha}_{i,\ell,\theta}}{\kappa-\ell+1}$. We have,

$$\begin{aligned}
\sum_{j_{\ell-1}\in\Omega_\ell} \frac{\exp(\theta_{j_{\ell-1}})}{W - \sum_{k=j_1}^{j_{\ell-2}}\exp(\theta_k)} &= \frac{\sum_{j_{\ell-1}\in\Omega_\ell}\exp(\theta_{j_{\ell-1}})}{\exp(\theta_i) + \sum_{j_{\ell-1}\in\Omega_\ell}\exp(\theta_{j_{\ell-1}})} \\
&\leq \left(\frac{\widetilde{\alpha}_{i,\ell,\theta}}{\kappa-\ell+1} + 1\right)^{-1} \\
&= \frac{\kappa-\ell+1}{\widetilde{\alpha}_{i,\ell,\theta} + \kappa - \ell + 1} \\
&= \sum_{j_{\ell-1}\in\Omega_\ell} \frac{\exp(\widetilde{\theta}_{j_{\ell-1}})}{\widetilde{W} - \sum_{k=j_1}^{j_{\ell-2}}\exp(\widetilde{\theta}_k)}, \qquad (71)
\end{aligned}$$

where (71) follows from the definition of $\{\widetilde{\theta}\}_{j\in S}$.

Consider $\{\Omega_{\widetilde{\ell}}\}_{2\leq\widetilde{\ell}\leq\ell-1}$, $|\Omega_{\widetilde{\ell}}| = \kappa - \widetilde{\ell} + 1$, corresponding to the subsequent summation terms in (70). Observe that $\frac{\exp(\theta_i)}{\sum_{j\in\Omega_{\widetilde{\ell}}}\exp(\theta_j)} \geq \widetilde{\alpha}_{i,\ell,\theta}/|\Omega_{\widetilde{\ell}}|$. Therefore, each summation term in equation (62) can be lower bounded by the corresponding term where $\{\theta_j\}_{j\in S}$ is replaced by $\{\widetilde{\theta}_j\}_{j\in S}$. Hence, we have

$$\begin{aligned}
&\mathbb{P}_\theta\Big[\sigma^{-1}(i) = \ell\Big] \\
\leq\ & \sum_{\substack{j_1\in S\\ j_1\neq i}}\left(\frac{\exp(\widetilde{\theta}_{j_1})}{\widetilde{W}}\sum_{\substack{j_2\in S\\ j_2\neq i,j_1}}\left(\frac{\exp(\widetilde{\theta}_{j_2})}{\widetilde{W}-\exp(\widetilde{\theta}_{j_1})}\cdots\left(\sum_{\substack{j_{\ell-1}\in S\\ j_{\ell-1}\neq i,\\ j_1,\cdots,j_{\ell-2}}}\frac{\exp(\widetilde{\theta}_{j_{\ell-1}})}{\widetilde{W}-\sum_{k=j_1}^{j_{\ell-2}}\exp(\widetilde{\theta}_k)}\right)\right)\right)\frac{e^{2b}}{\kappa-\ell+1} \\
\leq\ & e^{4b}\sum_{\substack{j_1\in S\\ j_1\neq i}}\left(\frac{\exp(\widetilde{\theta}_{j_1})}{\widetilde{W}}\sum_{\substack{j_2\in S\\ j_2\neq i,j_1}}\left(\frac{\exp(\widetilde{\theta}_{j_2})}{\widetilde{W}-\exp(\widetilde{\theta}_{j_1})}\cdots\left(\sum_{\substack{j_{\ell-1}\in S\\ j_{\ell-1}\neq i,\\ j_1,\cdots,j_{\ell-2}}}\frac{\exp(\widetilde{\theta}_{j_{\ell-1}})}{\widetilde{W}-\sum_{k=j_1}^{j_{\ell-2}}\exp(\widetilde{\theta}_k)}\frac{\exp(\widetilde{\theta}_i)}{\widetilde{W}-\sum_{k=j_1}^{j_{\ell-1}}\exp(\widetilde{\theta}_k)}\right)\right)\right) \\
\leq\ & e^{4b}\mathbb{P}_{\widetilde{\theta}}\Big[\sigma^{-1}(i) = \ell\Big] \qquad (72)
\end{aligned}$$

The second inequality uses $\widetilde{\alpha}_2/(\kappa - \ell + \widetilde{\alpha}_{i,\ell,\theta}) \geq e^{-2b}/(\kappa - \ell + 1)$. Observe that $\exp(\widetilde{\theta}_j) = 1$ for all $j \neq i$ and $\exp(\widetilde{\theta}_i) = \widetilde{\alpha}_{i,\ell,\theta} \geq \lfloor\widetilde{\alpha}_{i,\ell,\theta}\rfloor = \alpha_{i,\ell,\theta} \geq 0$. Therefore, we have

$$\begin{aligned}
\mathbb{P}_{\widetilde{\theta}}\Big[\sigma^{-1}(i) = \ell\Big] &= \binom{\kappa-1}{\ell-1}\frac{\widetilde{\alpha}_{i,\ell,\theta}(\ell-1)!}{(\kappa-1+\widetilde{\alpha}_{i,\ell,\theta})(\kappa-2+\widetilde{\alpha}_{i,\ell,\theta})\cdots(\kappa-\ell+\widetilde{\alpha}_{i,\ell,\theta})} \\
&\leq \frac{(\kappa-1)!}{(\kappa-\ell)!}\frac{e^{2b}}{(\kappa-1+\alpha_{i,\ell,\theta})(\kappa-2+\alpha_{i,\ell,\theta})\cdots(\kappa-\ell+\alpha_{i,\ell,\theta})} \\
&\leq \frac{e^{2b}}{\kappa}\left(1 - \frac{\ell}{\kappa+\alpha_{i,\ell,\theta}}\right)^{\alpha_{i,\ell,\theta}-1}, \qquad (73)
\end{aligned}$$

Note that equation (73) holds for all values of $\alpha_{i,\ell,\theta} \geq 0$. Claim 49 follows by combining Equations (72) and (73).

C. Proof of Theorem 4

Let $H(\theta) \in \mathcal{S}^d$ be Hessian matrix such that $H_{ii'}(\theta) = \frac{\partial^2\mathcal{L}(\theta)}{\partial\theta_i\partial\theta_{i'}}$. The Fisher information matrix is defined as $I(\theta) = -\mathbb{E}_\theta[H(\theta)]$. Fix any unbiased estimator $\widehat{\theta}$ of $\theta \in \Omega_b$. Since, $\widehat{\theta} \in \mathcal{U}$, $\widehat{\theta} - \theta$ is orthogonal to $\mathbf{1}$. The Cramér-Rao lower bound then implies that $\mathbb{E}[\|\widehat{\theta} - \theta^*\|^2] \geq \sum_{i=2}^d \frac{1}{\lambda_i(I(\theta))}$. Taking the supremum over both sides gives

$$\sup_\theta \mathbb{E}[\|\widehat{\theta} - \theta\|^2] \;\geq\; \sup_\theta \sum_{i=2}^d \frac{1}{\lambda_i(I(\theta))} \geq \sum_{i=2}^d \frac{1}{\lambda_i(I(\mathbf{0}))}\ .$$

The following lemma provides a lower bound on $\mathbb{E}_\theta[H(\mathbf{0})]$, where $\mathbf{0}$ indicates the all-zeros vector.

Lemma 13. *Under the hypotheses of Theorem 4,*

$$\mathbb{E}_\theta[H(\mathbf{0})] \;\succeq\; -\sum_{j=1}^{n} \frac{2p\log(\kappa_j)^2}{\kappa_j(\kappa_j-1)} \sum_{i'<i\in S_j} (e_i - e_{i'})(e_i - e_{i'})^\top \,. \tag{74}$$

Observe that $I(\mathbf{0})$ is positive semi-definite. Moreover, $\lambda_1(I(\mathbf{0}))$ is zero and the corresponding eigenvector is the all-ones vector. It follows that

$$\begin{aligned} I(0) \;&\preceq\; \sum_{j=1}^{n} \frac{2p\log(\kappa_j)^2}{\kappa_j(\kappa_j-1)} \sum_{i'<i\in S_j} (e_i - e_{i'})(e_i - e_{i'})^\top \\ &\preceq\; 2p\log(\kappa_{\max})^2 \underbrace{\sum_{j=1}^{n} \frac{1}{\kappa_j(\kappa_j-1)} \sum_{i'<i\in S_j} (e_i - e_{i'})(e_i - e_{i'})^\top}_{=L} \,, \end{aligned}$$

where L is the Laplacian defined for the comparison graph $\mathcal{H}$, Definition 1, as $\ell_j = 1$ for all $j \in [n]$ in this setting. By Jensen's inequality, we have

$$\sum_{i=2}^{d} \frac{1}{\lambda_i(L)} \geq \frac{(d-1)^2}{\sum_{i=2}^{d} \lambda_i(L)} = \frac{(d-1)^2}{\mathrm{Tr}(L)} = \frac{(d-1)^2}{n}.$$

C.1. Proof of Lemma 13

Define $\mathcal{L}_j(\theta)$ for $j \in [n]$ such that $\mathcal{L}(\theta) = \sum_{j=1}^{n} \mathcal{L}_j(\theta)$. Let $H^{(j)}(\theta) \in \mathcal{S}^d$ be the Hessian matrix such that $H^{(j)}_{ii'}(\theta) = \frac{\partial^2 \mathcal{L}_j(\theta)}{\partial\theta_i \partial\theta_{i'}}$ for $i, i' \in S_j$. We prove that for all $j \in [n]$,

$$\mathbb{E}_\theta[H^{(j)}(\mathbf{0})] \;\succeq\; -\frac{2p\log(\kappa_j)^2}{\kappa_j(\kappa_j-1)} \sum_{i'<i\in S_j} (e_i - e_{i'})(e_i - e_{i'})^\top \,. \tag{75}$$

In the following, we omit superscript/subscript j for brevity. With a slight abuse of notation, we use $\mathbb{I}_{\{\Omega^{-1}(i)=a\}} = 1$ if item i is ranked at the a-th position in all the orderings $\sigma \in \Omega$. Let $\mathbb{P}[\theta]$ be the likelihood of observing $\Omega^{-1}(p) = i^{(p)}$ and the set Λ (the set of the items that are ranked before the p-th position). We have,

$$\mathbb{P}(\theta) = \sum_{\sigma\in\Omega} \left(\frac{\exp\left(\sum_{m=1}^{p} \theta_{\sigma(m)}\right)}{\prod_{a=1}^{p} \left(\sum_{m'=a}^{\kappa} \exp\left(\theta_{\sigma(m')}\right)\right)} \right). \tag{76}$$

For $i, i' \in S_j$, we have

$$H_{ii'}(\theta) = \frac{1}{\mathbb{P}(\theta)} \frac{\partial^2 \mathbb{P}(\theta)}{\partial\theta_i \partial\theta_{i'}} - \frac{\nabla_i \mathbb{P}(\theta) \nabla_{i'} \mathbb{P}(\theta)}{\left(\mathbb{P}(\theta)\right)^2} \tag{77}$$

We claim that at $\theta = \mathbf{0}$,

$$-H_{ii'}(\mathbf{0}) = \begin{cases} C_1 & \text{if } i = i',\ \left\{\Omega^{-1}(i) \geq p\right\} \\ C_2 + A_3^2 - C_3 & \text{if } i = i',\ \left\{\Omega^{-1}(i) < p\right\} \\ -B_1 & \text{if } i \neq i',\ \left\{\Omega^{-1}(i) \geq p,\ \Omega^{-1}(i') \geq p\right\} \\ -B_2 & \text{if } i \neq i',\ \left\{\Omega^{-1}(i) \geq p,\ \Omega^{-1}(i') < p\right\} \\ -B_2 & \text{if } i \neq i',\ \left\{\Omega^{-1}(i) < p,\ \Omega^{-1}(i') \geq p\right\} \\ -(B_3 + B_4 - A_3^2) & \text{if } i \neq i',\ \left\{\Omega^{-1}(i) < p,\ \Omega^{-1}(i') < p\right\} . \end{cases} \tag{78}$$

where constants $A_3, B_1, B_2, B_3, B_4, C_1, C_2$ and C_3 are defined in Equations (85), (87), (88), (89), (90), (92), (93) and (94) respectively. From this computation of the Hessian, note that we have

$$H(\mathbf{0}) = \sum_{i'<i\in S} (e_i - e_{i'})(e_i - e_{i'})^\top \Big(H_{ii'}(\mathbf{0})\Big) \, . \tag{79}$$

which follows directly from the fact that the diagonal entries are summations of the off-diagonals, i.e. $C_1 = B_1(\kappa - p) + B_2(p-1)$ and $C_2 + A_3^2 - C_3 = B_2(\kappa - p + 1) + (B_3 + B_4 - A_3^2)(p-2)$. The second equality follows from the fact that $C_2 = B_2(\kappa - p + 1) + B_3(p-2)$ and $A_3^2(p-1) = B_4(p-2) + C_3$. Note that since $\theta = \mathbf{0}$, all items are exchangeable. Hence, $\mathbb{E}[H_{ii'}(\mathbf{0})] = \mathbb{E}[H_{ii}(\mathbf{0})]/(\kappa - 1)$, and substituting this into (79) and using Equations (78), we get

$$\begin{aligned}
& \mathbb{E}\Big[H(\mathbf{0})\Big] \\
= \;& -\frac{1}{\kappa-1}\Big(\mathbb{P}\big[\Omega^{-1}(i) \ge p\big] C_1 + \mathbb{P}\big[\Omega^{-1}(i) < p\big](C_2 + A_3^2 - C_3)\Big) \sum_{i'<i\in S}(e_i - e_{i'})(e_i - e_{i'})^\top \\
\succeq \;& -\frac{1}{\kappa(\kappa-1)}\left((\kappa - p + 1)\log\left(\frac{\kappa}{\kappa-p}\right) + (p-1)\left(\log\left(\frac{\kappa}{\kappa-p+1}\right) + \log\left(\frac{\kappa}{\kappa-p+1}\right)^2\right)\right) \sum_{i'<i\in S}(e_i - e_{i'})(e_i - e_{i'})^\top \qquad (80) \\
\succeq \;& -\frac{2p\log(\kappa)^2}{\kappa(\kappa-1)} \sum_{i'<i\in S}(e_i - e_{i'})(e_i - e_{i'})^\top \, , \qquad (81)
\end{aligned}$$

where (80) uses $\sum_{a=1}^{p} \frac{1}{\kappa - a + 1} \le \log\left(\frac{\kappa}{\kappa - p}\right)$ and $C_3 \ge 0$. Equation (81) follows from the fact that for any $x > 0$, $\log(1+x) \le x$. To prove (78), we have the first order partial derivative of $\mathbb{P}(\theta)$ given by

$$\nabla_i \mathbb{P}(\theta) \;=\; \mathbb{I}_{\{\Omega^{-1}(i)\le p\}}\mathbb{P}(\theta) - \sum_{\sigma\in\Omega}\left(\frac{\exp\left(\sum_{m=1}^{p}\theta_{\sigma(m)}\right)}{\prod_{a=1}^{p}\left(\sum_{m'=a}^{\kappa}\exp\left(\theta_{\sigma(m')}\right)\right)}\left(\sum_{a=1}^{p}\frac{\mathbb{I}_{\{\sigma^{-1}(i)\ge a\}}\exp(\theta_i)}{\sum_{m'=a}^{\kappa}\exp\left(\theta_{\sigma(m')}\right)}\right)\right). \tag{82}$$

Define constants A_1, A_2 and A_3 such that

$$A_1 \;\equiv\; \mathbb{P}(\theta)\big|_{\{\theta=\mathbf{0}\}} = \frac{(p-1)!}{\kappa(\kappa-1)\cdots(\kappa-p+1)}, \tag{83}$$

$$A_2 \;\equiv\; \left(\sum_{a=1}^{p}\frac{\exp(\theta_i)}{\sum_{m'=a}^{\kappa}\exp\left(\theta_{\sigma(m')}\right)}\right)\Bigg|_{\{\theta=\mathbf{0}\}} = \left(\frac{1}{\kappa} + \frac{1}{\kappa-1} + \cdots + \frac{1}{\kappa-p+1}\right), \tag{84}$$

$$A_3 \;\equiv\; \left(\frac{(p-1)(p-2)!}{(p-1)!(\kappa)} + \frac{(p-2)(p-2)!}{(p-1)!(\kappa-1)} + \cdots + \frac{(p-2)!}{(p-1)!(\kappa-p+2)}\right). \tag{85}$$

Observe that, for all $i \in [d]$,

$$\nabla_i \mathbb{P}(\theta)\big|_{\{\theta=\mathbf{0}\}} = A_1\Big(\mathbb{I}_{\{\Omega_j^{-1}(i)=p\}}(1-A_2) + \mathbb{I}_{\{\Omega_j^{-1}(i)<p\}}(1-A_3) - \mathbb{I}_{\{\Omega_j^{-1}(i)>p\}}A_2\Big) \;\; . \tag{86}$$

Further define constants B_1, B_2, B_3 and B_4 such that

$$B_1 \;\equiv\; \left(\frac{1}{\kappa^2} + \frac{1}{(\kappa-1)^2} + \cdots + \frac{1}{(\kappa-p+1)^2}\right), \tag{87}$$

$$B_2 \;\equiv\; \left(\frac{p-1}{(p-1)\kappa^2} + \frac{p-2}{(p-1)(\kappa-1)^2} + \cdots + \frac{1}{(p-1)(\kappa-p+2)^2}\right), \tag{88}$$

$$B_3 \;\equiv\; \left(\frac{(p-1)(p-2)(p-3)!}{(p-1)!\kappa^2} + \frac{(p-2)(p-3)(p-3)!}{(p-1)!(\kappa-1)^2} + \cdots + \frac{2(p-3)!}{(p-1)!(\kappa-p+3)^2}\right), \tag{89}$$

$$B_4 \;\equiv\; \frac{(p-3)!}{(p-1)!}\left(\sum_{a,b\in[p-1],b\neq a}\left(\frac{1}{\kappa} + \frac{1}{\kappa-1} + \cdots + \frac{1}{\kappa-a+1}\right)\left(\frac{1}{\kappa} + \frac{1}{\kappa-1} + \cdots + \frac{1}{\kappa-b+1}\right)\right). \tag{90}$$

Observe that,

$$
\begin{aligned}
\left.\frac{\partial^2 \mathbb{P}(\theta)}{\partial\theta_i \partial\theta_{i'}}\right|_{\theta=\mathbf{0}} \quad = \quad & \mathbb{I}_{\left\{\Omega^{-1}(i),\Omega^{-1}(i')>p\right\}} A_1\Big((-A_2)(-A_2)+B_1\Big) \\
& + \Big(\mathbb{I}_{\left\{\Omega^{-1}(i)>p,\Omega^{-1}(i')=p\right\}} + \mathbb{I}_{\left\{\Omega^{-1}(i)=p,\Omega^{-1}(i')>p\right\}}\Big) A_1\Big((-A_2)(1-A_2)+B_1\Big) \\
& + \Big(\mathbb{I}_{\left\{\Omega^{-1}(i)=p,\Omega^{-1}(i')<p\right\}} + \mathbb{I}_{\left\{\Omega^{-1}(i)<p,\Omega^{-1}(i')=p\right\}}\Big) A_1\Big((1-A_3)+(-A_2)(1-A_3)+B_2\Big) \\
& + \Big(\mathbb{I}_{\left\{\Omega^{-1}(i)>p,\Omega^{-1}(i')<p\right\}} + \mathbb{I}_{\left\{\Omega^{-1}(i)<p,\Omega^{-1}(i')>p\right\}}\Big) A_1\Big((-A_2)(1-A_3)+B_2\Big) \\
& + \mathbb{I}_{\left\{\Omega^{-1}(i)<p,\Omega^{-1}(i')<p\right\}} A_1\Big((1-A_3)+(-A_3)+B_4+B_3\Big) \,. \qquad (91)
\end{aligned}
$$

The claims in (78) are easy to verify by combining Equations (86) and (91) with (77). Also, define constants C_1, C_2 and C_3 such that,

$$
C_1 \quad \equiv \quad \left(\frac{\kappa-1}{(\kappa)^2} + \frac{\kappa-2}{(\kappa-1)^2} + \cdots + \frac{\kappa-p}{(\kappa-p+1)^2}\right), \qquad (92)
$$

$$
C_2 \quad \equiv \quad \left(\frac{(p-1)(p-2)!(\kappa-1)}{(p-1)!(\kappa)^2} + \frac{(p-2)(p-2)!(\kappa-2)}{(p-1)!(\kappa-1)^2} + \cdots + \frac{(p-2)!(\kappa-p+1)}{(p-1)!(\kappa-p+2)^2}\right), \qquad (93)
$$

$$
C_3 \quad \equiv \quad \frac{(p-2)!}{(p-1)!}\left(\sum_{a,b\in[p-1],b=a}\left(\frac{1}{\kappa}+\frac{1}{\kappa-1}+\cdots+\frac{1}{\kappa-a+1}\right)\left(\frac{1}{\kappa}+\frac{1}{\kappa-1}+\cdots+\frac{1}{\kappa-b+1}\right)\right), \qquad (94)
$$

such that,

$$
\begin{aligned}
\left.\frac{\partial^2 \mathbb{P}(\theta)}{\partial\theta_i^2}\right|_{\theta=\mathbf{0}} \quad = \quad & \mathbb{I}_{\{\Omega^{-1}(i)>p\}} A_1\Big((-A_2)(-A_2)-C_1\Big) + \mathbb{I}_{\{\Omega^{-1}(i)=p\}} A_1\Big((1-A_2)-A_2(1-A_2)-C_1\Big) \\
& + \mathbb{I}_{\{\Omega^{-1}(i)<p\}} A_1\Big((1-A_3)-A_3-C_2+C_3\Big) \,. \qquad (95)
\end{aligned}
$$

The claims (78) is easy to verify by combining Equations (86) and (95) with (77).