Appendix

Proof of proposition 3.1

Proof. Let $N = \sum_{i=1}^n N_i$. Since, conditioned on Λ, $N_1, \dots, N_n$ are independent Poisson process with mean measure Λ, by the superposition proposition for Poisson processes, N is a Poisson process conditioned on Λ with mean measure $n\Lambda$. Since, $\mathbb{E}[e^{t_1 N(A)+t_2 N(B)}|\Lambda] = \mathbb{E}[e^{t_1 N(A)}|\Lambda(A)]\mathbb{E}[e^{t_2 N(B)}|\Lambda(B)]$, and $\Lambda(A)$ and $\Lambda(B)$ are independent, hence, $N(A)$ and $N(B)$ are also independent and therefore, N is a CRM. Hence, $N(\mathrm{d}x) = \int_{\mathbb{R}^+} s\bar{N}(\mathrm{d}z, \mathrm{d}x)$, for a Poisson process $\bar{N}$ on $S \times \mathbb{R}^+$. Moreover,

$$\begin{aligned}
&\mathbb{E}[e^{-tN(A)}] \\
&= \mathbb{E}\left[\mathbb{E}[e^{-tN(A)}|\Lambda]\right] \\
&= \mathbb{E}\left[\exp\left(-n\Lambda(A)(1-e^{-t})\right)\right] \\
&= \exp\left(-\mu(A)\int_{\mathbb{R}^+}(1-e^{-n(1-e^{-t})z})\rho(\mathrm{d}z)\right) \\
&= \exp\left(-\mu(A)\int_{\mathbb{R}^+}\left(\sum_{k=0}^{\infty}\frac{e^{-nz}(nz)^k}{k!}\right.\right. \\
&\qquad\left.\left.-\sum_{k=0}^{\infty}\frac{e^{-nz}e^{-kt}(nz)^k}{k!}\right)\rho(\mathrm{d}z)\right)
\end{aligned}$$

where, we have used the fact that $1 = \sum_{k=0}^{\infty}\frac{e^{-nz}(nz)^k}{k!}$. Rearranging the terms in the above equation, we get

$$\begin{aligned}
&\mathbb{E}[e^{-tN(A)}] \\
&= \exp\left(-\mu(A)\sum_{k=1}^{\infty}(1-e^{-kt})\int_{\mathbb{R}^+}\frac{e^{-nz}(nz)^k}{k!}\rho(\mathrm{d}z)\right).
\end{aligned}$$

Hence, the Poisson intensity measure of N, when viewed as a CRM is given by

$$\bar{\nu}(\mathrm{d}k, \mathrm{d}x) = \mu(\mathrm{d}x)\int_{\mathbb{R}^+}\frac{e^{-nz}(nz)^k}{k!}\rho(\mathrm{d}z)$$

when $k \in \{1, 2, 3, \dots\}$, and 0 otherwise. The distinct points of N can be obtained by projecting N on S. Hence, by the mapping proposition for Poisson processes (Kingman, 1992), the distinct points of N form a Poisson process with mean measure $\mu^*(\mathrm{d}x) = \bar{\nu}(f^{-1}(\mathrm{d}x))$, where f is the projection map on S. Hence $f^{-1}(\mathrm{d}x) = (\mathbb{R}^+, \mathrm{d}x)$, and

$$\begin{aligned}
\mu^*(\mathrm{d}x) &= \bar{\nu}(\mathbb{R}^+, \mathrm{d}x) \\
&= \mu(\mathrm{d}x)\int_{\mathbb{R}^+}\sum_{k=1}^{\infty}\frac{e^{-nz}(nz)^k}{k!}\rho(\mathrm{d}z) \\
&= \mu(\mathrm{d}x)\int_{\mathbb{R}^+}(1-e^{-nz})\rho(\mathrm{d}z).
\end{aligned}$$

Thus, the result follows. □

Proof of proposition 3.2

Proof. The proof relies on the simple fact, that conditioned on the number of points to be sampled, the points of a Poisson process are independent (Kingman, 1992). Thus, n point processes can be sampled from a measure Λ, by first sampling the number of points in each point process from a Poisson distribution with mean $\Lambda(S)$, and then sampling the points independently. Let $\Lambda = \sum_{i=1}^n \Delta_i\delta_{X_i}$. Let $(X_{l_1}, \dots, X_{l_k})$ be the features discovered by the n Poisson processes. Let the i^{th} point process N_i consist of m_{i1} occurrences of X_{l_1}, m_{i2} occurrences of X_{l_2} and m_{ik} occurrences of X_{l_k}. Then, the joint distribution of the n point processes conditioned on Λ is given by

$$\begin{aligned}
&\mathbb{P}(N_1, \dots, N_n|\Lambda) \\
&= \prod_{i=1}^n\frac{\exp(-T)T^{\sum_{j=1}^k m_{ij}}}{(\sum_{j=1}^k m_{ij})!}\prod_{j=1}^k\left(\frac{\Delta_{l_j}}{T}\right)^{m_{ij}},
\end{aligned}$$

where $T = \Lambda(S) = \sum_{i=1}^{\infty}\Delta_i\delta_{X_i}(S) = \sum_{i=1}^{\infty}\Delta_i$. Readjusting the outermost product in the above equation, we get,

$$\mathbb{P}(N_1, \dots, N_n|\Lambda) = \frac{\exp(-nT)}{\prod_{i=1}^n(\sum_{j=1}^k m_{ij})!}\prod_{j=1}^k\Delta_{l_j}^{\sum_{i=1}^n m_{ij}}.$$

Since, we are not interested in the actual points X_{l_i}'s, but only the number of occurrences of the different points in the point processes, that is, $[m_{ij}]_{(n,k)}$, we can sum over every k-tuple of distinct atoms in the random measure Λ. Hence,

$$\begin{aligned}
&P([m_{ij}]_{(n,k)}|\Lambda) \\
&= \frac{\exp(-nT)}{\prod_{i=1}^n(\sum_{j=1}^k m_{ij})!}\sum_{\Delta_{l_1}\neq\Delta_{l_2}\neq\cdots\neq\Delta_{l_k}}\prod_{j=1}^k\Delta_{l_j}^{\sum_{i=1}^n m_{ij}},
\end{aligned}$$

where the sum is over all subsets of length k of the set $\{\Delta_1, \Delta_2, \dots\}$. Finally, in order to compute the result, we need to take expectation with respect to the distribution of Λ. Towards that end, we note that only the weights of Λ appear in the above equation. From section 2.2, we know that the weights of a CRM with Poisson intensity measure $\rho(\mathrm{d}z)\mu(\mathrm{d}x)$form a Poisson process with mean measure $\mu(S)\rho(\mathrm{d}z)$. Hence, it is enough to take the expectation with respect to the Poisson process.

$$P([m_{ij}]_{(n,k)}) \tag{32}$$

$$= \frac{1}{\prod_{i=1}^n(\sum_{j=1}^k m_{ij})!}\mathbb{E}\Bigg[\exp(-nT) \tag{33}$$

$$\sum_{\Delta_{l_1}\neq\Delta_{l_2}\neq\cdots\neq\Delta_{l_k}}\prod_{j=1}^k\Delta_{l_j}^{\sum_{i=1}^n m_{ij}}\Bigg], \tag{34}$$

The expectation can further be simplified by applying Proposition 2.1 of (James, 2005).

Proposition 6.1 ((James, 2005)). *Let $\mathcal{N}$ be the space of all $\sigma-$finite counting measures on $\mathbb{R}^+$, equipped with an appropriate σ-field. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathcal{N} \to \mathbb{R}^+$ be measurable with respect to their σ-fields. Then, for a Poisson process N with mean measure $\mathbb{E}[N(\mathrm{d}x)] = \rho(\mathrm{d}x)$,*

$$\mathbb{E}\left[g(N)e^{-\sum_{\Delta\in N} f(\Delta)}\right] = \mathbb{E}\left[e^{-\sum_{\Delta\in N} f(\Delta)}\right]\mathbb{E}[g(\bar{N})]$$

where $\bar{N}$ is a Poisson process with mean measure $\mathbb{E}[N(\mathrm{d}x)] = e^{-f(x)}\rho(\mathrm{d}x)$.

Applying the above proposition to (**??**), we get

$$P([m_{ij}]_{(n,k)}) = \frac{\mathbb{E}\left[e^{-\sum_{i=1}^{\infty} n\Delta_i}\right]}{\prod_{i=1}^n(\sum_{j=1}^k m_{ij})!} \times \mathbb{E}\left[\sum_{\Delta_{l_1}\neq\Delta_{l_2}\neq\cdots\neq\Delta_{l_k}\in\bar{N}} \prod_{j=1}^k \Delta_{l_j}^{\sum_{i=1}^n m_{ij}}\right]$$

where $\bar{N}$ is a Poisson process with mean measure $\mathbb{E}[N(\mathrm{d}z)] = e^{-nz}\rho(\mathrm{d}z)\theta$. The first expectation can be evaluated using Campbell's proposition and is given by $\exp\left(-\theta\int_{\mathbb{R}^+}(1-e^{-nz})\rho(\mathrm{d}z)\right)$. In order to evaluate the second expectation, we construct a new point process from N^* on ${\mathbb{R}^+}^k$ by concatenating every set of k distinct points in $\bar{N}$. The expression in the second expectation can then be rewritten as

$$\sum_{(\Delta_{l_1},\ldots,\Delta_{l_k})\in N^*} \prod_{j=1}^k \Delta_{l_j}^{\sum_{i=1}^n m_{ij}}$$

By Campbell's proposition for point processes,

$$\mathbb{E}\left[\sum_{\Delta\in N} f(\Delta)\right] = \int_{z\in\mathbb{R}^+} f(z)\rho(\mathrm{d}z)\,,$$

where $\rho(\mathrm{d}z) = \mathbb{E}[N(\mathrm{d}z)]$. Moreover, since the point process N^* is obtained by concatenating distinct points in N, $\mathbb{E}[N^*(\mathrm{d}z_1,\ldots,\mathrm{d}z_k)] = \prod_{j=1}^k \mathbb{E}[\bar{N}(\mathrm{d}z_j)] = \prod_{j=1}^k \theta e^{-nz}\rho(\mathrm{d}z_j)$, whenever z_j's are distinct. Hence,

$$\mathbb{E}\left[\sum_{\Delta_{l_1}\neq\Delta_{l_2}\neq\cdots\neq\Delta_{l_k}\in\bar{N}} \prod_{j=1}^k \Delta_{l_j}^{\sum_{i=1}^n m_{ij}}\right] = \prod_{j=1}^k \int_{z\in\mathbb{R}^+} \theta e^{-nz} z^{\sum_{i=1}^n m_{ij}} \rho(\mathrm{d}z)\,.$$

Hence, the final expression for the marginal distribution of the set of counts for each latent feature is given by

$$P([m_{ij}]_{(n,k)}) = \frac{\exp\left(-\theta\int_{\mathbb{R}^+}(1-e^{-nz})\rho(\mathrm{d}z)\right)}{\prod_{i=1}^n(\sum_{j=1}^k m_{ij})!} \times \prod_{j=1}^k \int_{z\in\mathbb{R}^+} \theta e^{-nz} z^{\sum_{i=1}^n m_{ij}} \rho(\mathrm{d}z)$$

The above expression can be simplified by letting $\psi(t) = \theta\int_{\mathbb{R}^+}(1-e^{-tz})\rho(\mathrm{d}z)$. Hence, $\psi^{(l)}(t) = (-1)^{l-1}\int_{\mathbb{R}^+}\theta e^{-tz}z^l\rho(\mathrm{d}z)$. Hence, the above expression can be rewritten as

$$P([m_{ij}]_{(n,k)}) = (-1)^{\sum_{i=1}^n\sum_{j=1}^k m_{ij}-k} \frac{\theta^k e^{-\theta\psi(n)}}{\prod_{i=1}^n(\sum_{j=1}^k m_{ij})!} \times \prod_{j=1}^k \psi^{(\sum_{i=1}^n m_{ij})}(n)$$

□

Proof of Corollary 3.3

Proof. From proposition 3.1, the distinct points in the point processes $N_i, 1 \le i \le n$, form a Poisson process with mean measure $\frac{\mu(\mathrm{d}x)}{\mu(S)}\psi(n)$. Hence, the total number of distinct points k is distributed as Poisson$(\psi(n))$. Hence, conditioning equation (5) with respect to k, we get the desired result. □

Proof of Proposition 3.4

Proof. Let $N = \sum_{i=1}^n N_i$. From the arguments of proposition 3.1, N is a CRM, and hence, can be written as $N(\mathrm{d}x) = \int_{\mathbb{R}^+} z\bar{N}(\mathrm{d}z,\mathrm{d}x)$ for some Poisson process $\bar{N}$. Let Π be the random collection of points corrsponding to $\bar{N}$. Now define a map $f : \mathbb{R}^+ \times S \to S$ as the projection map on S, that is, $f(x,y) = y$ and $M = f(\Pi) = \{\{f(x,y) : (x,y)\in\Pi\}\}$, where the double brackets indicate that M is a multiset. The rest of the arguments remain same as in proposition 3.1 and proposition 3.2. □

Proof of Lemma 4.1

Proof. Using Proposition 3.4 to marginalize Λ_i from 8, we get that $[m_{ij}]_{1\le j\le r_{i\cdot}}$ is distributed as CRM-Poisson$(\Phi(S),\rho,1)$, that is,

$$P([m_{ij}]_{1\le j\le r_{i\cdot}}|\Phi(S)) = \frac{\exp\left(-\Phi(S)\int_{\mathbb{R}^+}(1-e^{-z})\bar{\rho}(\mathrm{d}z)\right)}{(\sum_{j=1}^{r_{i\cdot}} m_{ij})!} \times \prod_{j=1}^{r_{i\cdot}} \int_{z\in\mathbb{R}^+} \Phi(S)e^{-z}z^{m_{ij}}\bar{\rho}(\mathrm{d}z) \tag{35}$$

Let $m_{i\boldsymbol{\cdot}} = \sum_{j=1}^{r_{i\boldsymbol{\cdot}}} m_{ij}$. Taking expectation with respect to $\Phi(S)$, we get the marginal distribution of $[m_{ij}]_{1\leq j\leq r_{i\boldsymbol{\cdot}}}$, where $r_{i\boldsymbol{\cdot}}$ is also random.

$$\begin{aligned}
&P([m_{ij}]_{1\leq j\leq r_{i\boldsymbol{\cdot}}}) \\
&= \mathbb{E}\left[\exp\left(-\Phi(S)\int_{\mathbb{R}^+}(1-e^{-z})\bar{\rho}(\mathrm{d}z)\right)\Phi(S)^{r_{i\boldsymbol{\cdot}}}\right] \\
&\qquad \times \frac{\prod_{j=1}^{r_{i\boldsymbol{\cdot}}}\int_{z\in\mathbb{R}^+} e^{-z}z^{m_{ij}}\bar{\rho}(\mathrm{d}z)}{m_{i\boldsymbol{\cdot}}!} \qquad (36)
\end{aligned}$$

It is given that

$$h(u) = \mathbb{E}[e^{-u\Phi(S)}]$$

$$\bar{\psi}(u) = \int_{\mathbb{R}^+}(1-e^{-uz})\bar{\rho}(\mathrm{d}z)\,,$$

Hence

$$\frac{d^{r_{i\boldsymbol{\cdot}}}}{du^{r_{i\boldsymbol{\cdot}}}}h(u) = (-1)^{r_{i\boldsymbol{\cdot}}}\mathbb{E}\left[\Phi(S)^{r_{i\boldsymbol{\cdot}}}e^{-u\Phi(S)}\right]$$

$$\bar{\psi}^{(m_{ij})}(u) = (-1)^{m_{ij}-1}\int_{\mathbb{R}^+} e^{-uz}z^{m_{ij}}\bar{\rho}(\mathrm{d}z)$$

Using the above results with $u = \bar{\psi}(1)$, equation (**??**) can be rewritten as

$$\begin{aligned}
&P([m_{ij}]_{1\leq j\leq r_{i\boldsymbol{\cdot}}}) \\
&= (-1)^{m_{i\boldsymbol{\cdot}}}h^{(r_{i\boldsymbol{\cdot}})}(\bar{\psi}(1))\frac{\prod_{j=1}^{r_{i\boldsymbol{\cdot}}}\bar{\psi}^{(m_{ij})}(1)}{m_{i\boldsymbol{\cdot}}!} \qquad (37)
\end{aligned}$$

□