Supplementary Material

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1 Additional Theorem

Theorem 1.1. *The estimator $\hat{\tau}$ is unbiased for τ under query randomization if all auctions are unconstrained.*

Proof. Without quota throttling, Assumption 2 and query randomization imply that

$$\begin{aligned} I(Z_i = 1)Y_i(Z) &= I(Z_i = 1)Y_i(Z_{q[i]}) \\ &= I(Z_i = 1)Y_i(1_{q[i]}) \\ &= I(Z_i = 1)Y_i(1). \end{aligned}$$

Therefore,

$$\begin{aligned} E(\hat{\tau}(Z)) &= 2E\left(\sum_i (I(Z_i = 1)Y_i(Z) - I(Z_i = 0)Y_i(Z))\right) \\ &= 2E\left(\sum_i (I(Z_i = 1)Y_i(1) - I(Z_i = 0)Y_i(0)\right) \\ &= 2\sum_i (P(Z_i = 1)Y_i(1) - P(Z_i = 0)Y_i(0)) \\ &= \tau. \end{aligned}$$

□

2 Proofs of main theorems

2.0.1 Proof of Theorem 5.1

Proof. First notice that for a *bid treatment*, we have:

$$W(Z)|Z \sim W(Z)|Z = \vec{1} \sim W(Z)|Z = \vec{0} \quad (1)$$

which leads to:

$$P(W(Z) = w|Z) = P(W(Z) = w|Z = \vec{1}) \quad \text{for all } Z \quad (2)$$

Under *joint throttling* we let $\mathcal{W}_b = \{w \in [0,1]^{N[b]} : \sum_i w_i = Q[b] \ \}$, and compute the following expectation:

$$\begin{aligned} E(Y_i(W(Z), Z)|Z) &= \sum_{w \in \mathcal{W}_b} P(W(Z) = w|Z) Y(w, Z) \\ &= \sum_{w \in \mathcal{W}_b} P(W(Z) = w|Z = \vec{1}) Y(w, Z) \\ &= E(Y_i(W(\vec{1}), Z)|Z) \end{aligned}$$

So combining this with the query randomization assumption, we have:

$$\begin{aligned} E\left(I(Z_i = 1) Y_i(W(Z), Z)|Z\right) &= E\left(I(Z_i = 1) Y_i(W(Z), Z_{q[i]} = \vec{1}_{q[i]})|Z\right) \\ &= I(Z_i = 1) E\left(Y_i(W(Z), Z_{q[i]} = \vec{1}_{q[i]})|Z\right) \\ &= I(Z_i = 1) E\left(Y_i(W(\vec{1}), Z_{q[i]} = \vec{1}_{q[i]})|Z\right) \end{aligned}$$

Finally, for a specific realization w, the Assumption 2 implies:

$$Y_i(w, Z_{q[i]} = \vec{1}_{q[i]}) = Y_i(w, Z = \vec{1}) \tag{3}$$

and so finally:

$$\begin{aligned} E\left(I(Z_i = 1) Y_i(W(Z), Z)|Z\right) &= I(Z_i = 1) E\left(Y_i(W(\vec{1}), Z_{q[i]} = \vec{1}_{q[i]})|Z\right) \\ &= I(Z_i = 1) E\left(Y_i(W(\vec{1}), Z = \vec{1}|Z\right) \\ &= I(Z_i = 1) E\left(Y_i(W(\vec{1}), \vec{1})\right) \end{aligned}$$

Using the same reasoning to establish

$$E(I(Z_i = 0) Y_i(W(Z), Z)|Z) = I(Z_i = 0) E(Y_i(W(\vec{0}), \vec{0})) \tag{4}$$

we can then write:

$$
\begin{aligned}
E(\hat{\tau}) &= E(E(\hat{\tau}|Z)) \\
&= E(2\sum_i (I(Z_i=1)E\left(Y_i(W(\vec{1}),\vec{1})\right) - I(Z_i=0)E\left(Y_i(W(\vec{0}),\vec{0})\right))) \\
&= 2\sum_i (P(Z_i=1)E\left(Y_i(W(\vec{1}),\vec{1})\right) - P(Z_i=0)E\left(Y_i(W(\vec{0}),\vec{0})\right)) \\
&= \sum_i (E\left(Y_i(W(\vec{1}),\vec{1})\right) - E\left(Y_i(W(\vec{0}),\vec{0})\right)) \\
&= E(\sum_i (Y_i(W(\vec{1}),\vec{1}) - Y_i(W(\vec{0}),\vec{0}))) \\
&= E(\tau) \\
&= \tau^*
\end{aligned}
$$

which completes the proof. □

2.0.2 Proof of Theorem 5.2

Proof. Denote by $W(Z)_{[x=1]}$ the sub vector of $W(Z)$ corresponding to the units satisfying $x_i = 1$. We have that:

$$
\begin{aligned}
P(W(Z)_{[q[i],x=1]} = w_{[x=1]}|Z_i=0, Z_{-i}) = &\prod_{j:q[j]=q[i],x_j=1} \min(\frac{Q^{(0)}_{b[j]}}{N^{(0)}_{b[j]}(Z)},1)^{(1-Z_j)w_j} \\
&\times \min(1-\frac{Q^{(0)}_{b[j]}}{N^{(0)}_{b[j]}(Z)},1)^{(1-Z_j)(1-w_j)} \\
&\times \min(\frac{Q^{(1)}_{b[j]}}{N^{(1)}_{b[j]}(x=1)(Z)},1)^{Z_j w_j} \\
&\times \min(1-\frac{Q^{(1)}_{b[j]}}{N^{(1)}_{b[j]}(x=1)(Z)},1)^{Z_j(1-w_j)} \\
= &\prod_{j:q[j]=q[i]\&x_j=1} (\min(\frac{Q^{(0)}_{b[j]}}{N^{(0)}_{b[j]}(Z)},1)^{w_j} \\
&\times \min(1-\frac{Q^{(0)}_{b[j]}}{N^{(0)}_{b[j]}(Z)},1)^{(1-w_j)}
\end{aligned}
$$

since *query randomization* implies $Z_i = 0 \Rightarrow Z_{q[i]} = 0_{q[i]}$. So we have:

$$
\begin{aligned}
P(W(Z)_{[q[i],x=1]} = w_{[x=1]}|Z_i = 0, Z_{-i}) = & \prod_{j:q[j]=q[i],x_j=1} (\min(\frac{Q^{(0)}_{b[j]}}{N^{(0)}_{b[j]}(Z)}, 1)^{w_j} \\
& \times \min(1 - \frac{Q^{(0)}_{b[j]}}{N^{(0)}_{b[j]}(Z)}, 1)^{(1-w_j)} \\
= & \prod_{j:q[j]=q[i],x_j=1} (\min(\frac{Q^{(0)}_b + Q^{(1)}_b}{N_b}, 1)^{w_j} \\
& \times \min(1 - \frac{Q^{(0)}_b + Q^{(1)}_b}{N_b}, 1)^{(1-w_j)} \\
= & P(W(0)_{[q[i],x=1]} = w_{[x=1]})
\end{aligned}
$$

Using the conditions stated in the theorem. So we have shown that:

$$
\begin{aligned}
P(W(Z)_{[q[i],x=1]} = w_{[x=1]}|Z_i = 0, Z_{-i}) \\
= P(W(0)_{[q[i],x=1]} = w_{[x=1]})
\end{aligned}
$$

By a similar reasoning, we can show that:

$$
\begin{aligned}
P(W(Z)_{[q[i],x=1]} = w_{[x=1]}|Z_i = 1, Z_{-i}) \\
= P(W(1)_{[q[i],x=1]} = w_{[x=1]})
\end{aligned}
$$

Then, after noticing the obvious fact that:

$$
Y_i(W(Z), Z) = Y_i(W(Z)_{q[i],x=1}, Z) \tag{5}
$$

we have:

$$
\begin{aligned}
E[Y_i(W(Z), Z)|Z_i = 1, Z_{-i}] &= E[Y_i(W(Z)_{q[i],x=1}, Z)|Z_i = 1, Z_{-i}] \\
&= \sum_{w_{q[i],x=1} \in \mathcal{W}_{q[i],x=1}} P(W(Z)_{q[i],x=1} = w_{q[i],x=1}|Z_i = 1, Z_{-i}) \times Y_i(w_{q[i],x=1}, Z) \\
&= \sum_{w_{q[i],x=1} \in \mathcal{W}_{q[i],x=1}} P(W(1)_{q[i],x=1} = w_{q[i],x=1}) \times Y_i(w_{q[i],x=1}, Z) \\
&= E[Y_i(W(Z), Z)|Z = 1]
\end{aligned}
$$

Similarly, we can show that

$$
E[Y_i(W(Z), Z)|Z_i = 0, Z_{-i}] = E[Y_i(W(Z), Z)|Z = 0]
$$

So following the same technique as in the proof of Theorem 5.1 we have:

$$E(I(Z_i = 1)Y_i(W(Z), Z)) = I(Z_i = 1)E(Y_i(W(1), Z = 1))$$

and

$$E(I(Z_i = 0)Y_i(W(Z), Z)) = I(Z_i = 0)E(Y_i(W(0), Z = 0))$$

the rest of the proof being identical to that of Theorem 5.1

□

3 Detailed derivations for Section 4

3.1 Scenario 2

The estimated treatment effect has expectation

$$\begin{aligned} E(\hat{\tau}) &= \frac{1}{2^K}\tau_q + \frac{1}{2^K}\left(-B_1(1) + \sum_{i=1}^{K} 2^{K-i} B_i(1)\right) \\ &\geq \frac{1}{2^K}\tau + \frac{1}{2^K}\left(-B_1(1) + 2^K B_K(1)\sum_{i=1}^{K}(\frac{1}{2})^i\right) \\ &\geq \frac{1}{2^K}\tau + \frac{1}{2^K}(-B_1(1) \\ &+ 2^{K+1}B_K(1)(1 - (\frac{1}{2})^{K+1} - \frac{1}{2})) \\ &= \frac{1}{2^K}\tau + A_K \end{aligned}$$

where

$$A_K = \frac{1}{2^K}(-B_1(1) + 2^{K+1}B_K(1)(1 - (\frac{1}{2})^{K+1} - \frac{1}{2}))$$

4 Details and additional simulations

4.1 Correlation between potential bids

In the simulations in section 6.1, we induce a correlation between the treatment and control potential bids. The complete generating process is as follows:

$$\begin{aligned} B_i(0) &= \exp(B_i^*(0)) \\ B_i(1) &= \exp(B_i^*(1)) \\ (B_i^*(0), B_i^*(1)) &\sim MVN(\mu, \Sigma) \end{aligned}$$

where $MVN(\mu, \Sigma)$ denotes the multivariate distribution with mean $\mu = (\mu_0, \mu_1)^t$ and covariance matrix $\Sigma = \left(\begin{smallmatrix} 0.1 & 0.09 \\ 0.09 & 0.1 \end{smallmatrix}\right)$. In particular, the correlation between $B_i^*(1)$ and $B_i^*(0)$ is $\rho = 0.9$

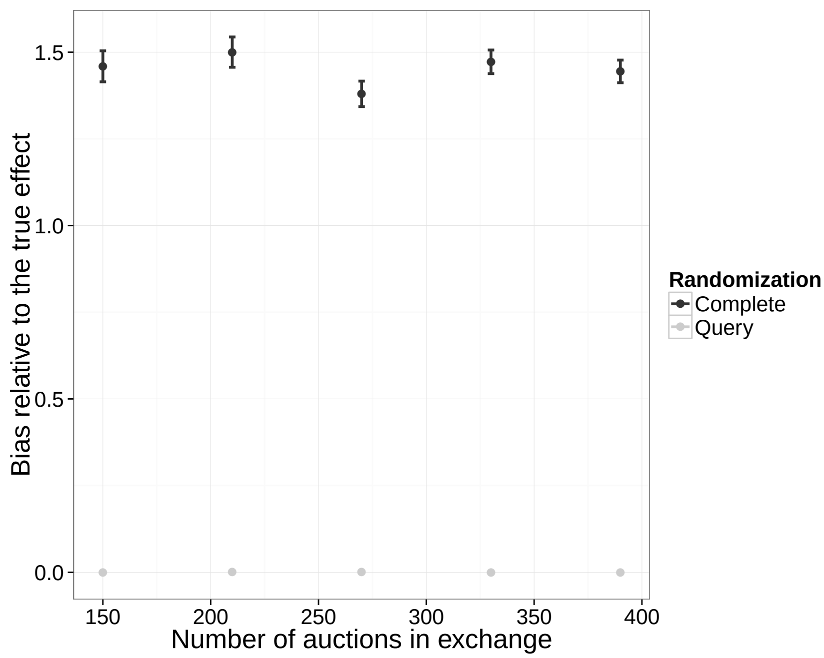

Figure 1: Simulated bias of the effect of a bid treatment relative to the true effect under balanced query randomization and balanced (query, bidder) randomization, for $\mu_1 = 1.05$ and $Q = \frac{2}{3}N_q$. The dots show the mean relative bias, and the endpoints show the mean $\pm$ twice the simulated standard deviation of the relative effect size.

4.2 Experiments with larger auctions

Due to computational limitations, we were unable to simulate an exchange with a large number of auctions. In the paper, we considered a maximum of $N_q = 150$ auctions. Here, we extend the simulations to $N_q = 390$ auctions, for $\mu_1 = 1.05$ and $Q = \frac{2}{3}N_q$. Figure 1 and Figure 2 show the results for the relative biases and ratio of variances. They give weight to our previous observation that the relative biases remain flat as the number of auctions increase, as does the ratio of variances.

4.3 Results for balanced complete randomization estimator under split and joint quota

In the main paper, we report the results for *balanced query randomization* under split and joint quota. We focused on those since our main purpose was to show that even when the conditions of theorem 5.2 where violated, the combination of *query randomization* and *split quota* still resulted in very low bias. In this section, we show empirically that *split quota* is also preferable to *joint quota* under *query randomization*, although the bias is larger than under *query randomization*. Figure 3 in particular show that the relative bias under *joint quota* is -1, as was the case for *query randomization*. The relative bias under *split quota* is i close to zero when $p_x = 0.9$, but is overall high otherwise.

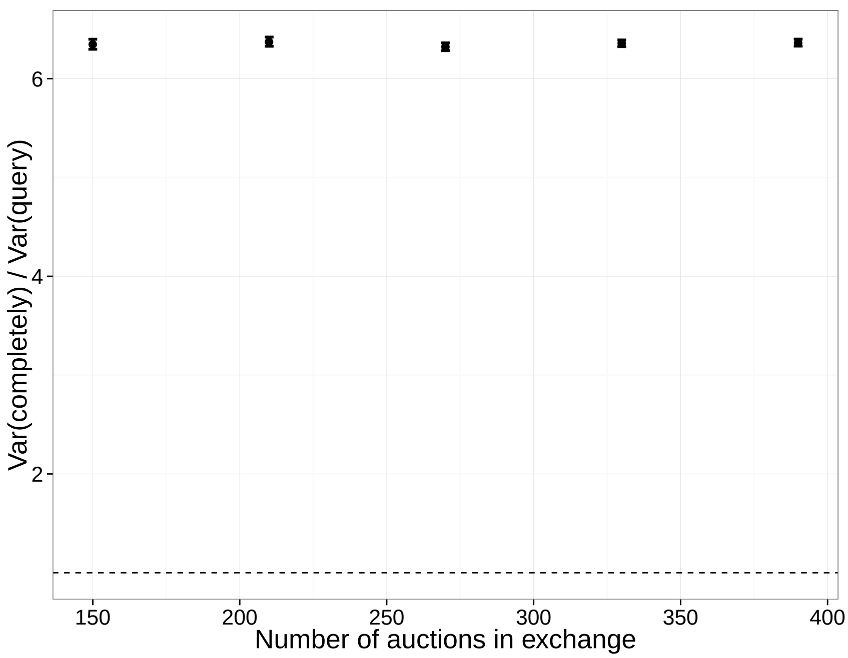

Figure 2: Ratio of variance of balanced complete randomization and balanced query randomization estimators for different combinations of parameters. The dots show the mean ratios, and the endpoints show the mean $\pm$ twice the simulated standard deviation of the ratio. The horizontal dotted line lies at one.

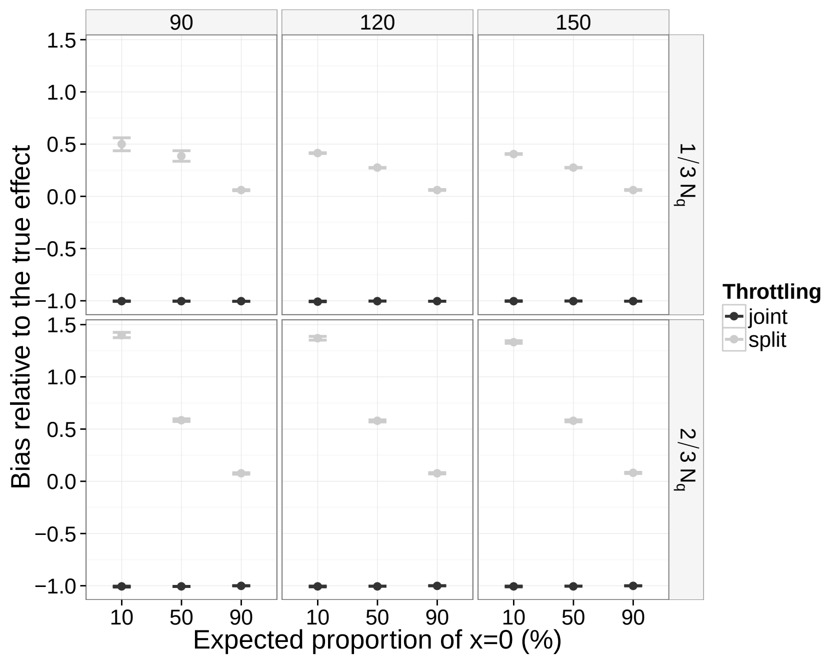

Figure 3: Bias of joint quota throttling and split quota throttling estimators relative tot he true effect under complete randomization. The dots show the mean ratios, and the endpoints show the mean ± twice the simulated standard deviation of the relative effect size.