Simple and Scalable Constrained Clustering: A Generalized Spectral Method

(Supplementary file)

1 Generalized Cheeger Inequality (Proof)

Definition. Let $G = (V, E, w)$ be a graph. The demand graph K_G of G is the graph with adjacency matrix $K_G(i, j) = d_i d_j / vol(V)$, where $d_u = \sum_{j \neq i} w_{i,j}$ and $vol(V) = \sum_{i \in V} d_i$.

We begin with two Lemmas.

Lemma 1.1. *For all $a_i, b_i > 0$ we have*

$$\frac{\sum_i a_i}{\sum_i b_i} \geq \min_i \left\{ \frac{a_i}{b_i} \right\}.$$

Lemma 1.2. *Let G be a graph, d be the vector containing the degrees of the vertices, and D be corresponding diagonal matrix. For all vectors x where $x^T d = 0$ we have*

$$x^T D x = x^T L_K x,$$

where K is the demand graph for G.

Proof. Let d be the vector consisting of the entries along the diagonal of D. By definition, we have

$$L_K = D - \frac{dd^T}{vol(V)}.$$

The lemma follows. □

We prove the following theorem.

Theorem 1.3. *Let G and H be any two weighted graphs and D be the vector containing the degrees of the vertices in G. For any vector x such that $x^T d = 0$, we have*

$$\frac{x^T L_G x}{x^T L_H x} \geq \phi(G, K) \cdot \phi(G, H)/4,$$

where K is the demand graph of G. A cut meeting the guarantee of the inequality can be obtained via a Cheeger sweep on x.

Let V^- denote the set of u such that $x_u \leq 0$ and V^+ denote the set such that $x_u > 0$. Then we can divide E_G into two sets: E_G^{same} consisting of edges with both endpoints in V^- or V^+, and E_G^{dif} consisting of edges with one endpoint in each. In other words:

$$E_G^{dif} = \delta_G\left(V^-, V^+\right), \text{ and}$$
$$E_G^{same} = E_G \setminus E_G^{dif}.$$

We also define E_H^{dif} and E_H^{same} similarly.

We first show a lemma which is identical to one used in the proof of Cheeger's inequality [Chung, 1997]:

Lemma 1.4. *Let G and H be any two weighted graphs on the same vertex set V partitioned into V^- and V^+. For any vector x we have*

$$\frac{\sum_{uv\in E_G^{same}} w_G\left(u,v\right)\left|x_u^2-x_v^2\right|+\sum_{uv\in E_G^{dif}} w_G(u,v)\left(x_u^2+x_v^2\right)}{x^T L_H x}\geq\frac{\phi(G,H)}{2}.$$

Proof. We begin with a few algebraic identities:

Note that $2x_u^2+2x_v^2-(x_u-x_v)^2=(x_u+x_v)^2\geq 0$ gives:

$$\left(x_u-x_v\right)^2\leq 2x_u^2+2x_v^2.$$

Also, suppose $uv\in E_H^{same}$ and without loss of generality that $|x_u|\geq|x_v|$. Then letting $y=|x_u|-|x_v|$, we get:

$$\begin{aligned}
|x_u^2-x_v^2| &= \left(|x_v|+y\right)^2-|x_v|^2\\
&= y^2+y|x_v|\\
&\geq y^2=\left(x_u-x_v\right)^2.
\end{aligned}$$

The last equality follows because x_u and x_v have the same sign.

We then use the above inequalities to decompose the $x^T L_H x$ term.

$$\begin{aligned}
x^T L_H &= \sum_{uv\in E_H^{same}} w_H(u,v)\left(x_u-x_v\right)^2+\sum_{uv\in E_H^{dif}} w_H(u,v)\left(x_u-x_v\right)^2\\
&\leq \sum_{uv\in E_H^{same}} w_H(u,v)\left(x_u-x_v\right)^2+\sum_{uv\in E_H^{dif}} w_H(u,v)\left(2x_u^2+2x_v^2\right)\\
&\leq 2\left(\sum_{uv\in E_H^{same}} w_H(u,v)\left(x_u-x_v\right)^2+\sum_{uv\in E_H^{dif}} w_H(u,v)\left(x_u^2+x_v^2\right)\right)\\
&\leq 2\left(\sum_{uv\in E_H^{same}} w_H(u,v)\left|x_u^2-x_v^2\right|+\sum_{uv\in E_H^{dif}} w_H(u,v)\left(x_u^2+x_v^2\right)\right). \qquad (1)
\end{aligned}$$

We can now decompose the summation further into parts for V^- and V^+:

$$\begin{aligned}
&\sum_{uv\in E_G^{same}} w_G\left(u,v\right)\left|x_u^2-x_v^2\right|+\sum_{uv\in E_G^{dif}} w_G\left(u,v\right)\left(x_u^2+x_v^2\right)\\
&=\sum_{u\in V^-,v\in V^-} w_G\left(u,v\right)\left|x_u^2-x_v^2\right|+\sum_{u\in V^-,v\in V^+} w_G\left(u,v\right)x_u^2\\
&\quad+\sum_{u\in V^+,v\in V^+} w_G\left(u,v\right)\left|x_u^2-x_v^2\right|+\sum_{u\in V^-,v\in V^+} w_G\left(u,v\right)x_u^2.
\end{aligned}$$

Doing the same for $\sum_{uv\in E_H^{same}} w_H(u,v)|x_u^2-x_v^2|+\sum_{uv\in E_H^{dif}} w_H(u,v)(x_u^2+x_v^2)$ we get:

$$\begin{aligned}
&\frac{\sum_{uv\in E_G^{same}} w_G(u,v)\left|x_u^2-x_v^2\right|+\sum_{uv\in E_G^{dif}} w_G(u,v)\left(x_u^2+x_v^2\right)}{x^T L_H x}\\
&\geq\min\left\{\frac{\sum_{u\in V^-,v\in V^-} w_G(u,v)\left|x_u^2-x_v^2\right|+\sum_{u\in V^-,v\in V^+} w_G(u,v)x_u^2}{\sum_{u\in V^-,v\in V^-} w_H(u,v)\left|x_u^2-x_v^2\right|+\sum_{u\in V^-,v\in V^+} w_H(u,v)x_u^2},\right.\\
&\qquad\left.\frac{\sum_{u\in V^+,v\in V^+} w_G(u,v)\left|x_u^2-x_v^2\right|+\sum_{u\in V^-,v\in V^+} w_G(u,v)x_v^2}{\sum_{u\in V^+,v\in V^+} w_H(u,v)\left|x_u^2-x_v^2\right|+\sum_{u\in V^-,v\in V^+} w_H(u,v)x_v^2}\right\}.
\end{aligned}$$

The inequality comes from applying of Lemma 1.1.

By symmetry in V^- and V^+, it suffices to show that

$$\frac{\sum_{u\in V^-,v\in V^-} w_G\left(u,v\right)\left|x_u^2-x_v^2\right| + \sum_{u\in V^-,v\in V^+} w_G(u,v)x_u^2}{\sum_{u\in V^-,v\in V^-} w_G\left(u,v\right)\left|x_u^2-x_v^2\right| + \sum_{u\in V^-,v\in V^+} w_G\left(u,v\right)x_u^2} \geq \phi(G,H). \tag{2}$$

We sort the x_u in increasing order of $|x_u|$ into such that $x_{u_1} \geq \ldots \geq x_{u_k}$, and let $S_k = \{x_{u_1},\ldots,x_{u_k}\}$. We have

$$\sum_{u\in V^-,v\in V^-} w_G(u,v)\left|x_u^2-x_v^2\right| + \sum_{u\in V^-,v\in V^+} w_G(u,v)x_u^2 = \sum_{i=1\ldots k}\left(x_{u_i}^2 - x_{u_{i-1}}^2\right) cap_G\left(S_k,\bar{S_k}\right),$$

and

$$\sum_{u\in V^-,v\in V^-} w_H(u,v)\left|x_u^2-x_v^2\right| + \sum_{u\in V^-,v\in V^+} w_H(u,v)x_u^2 = \sum_{i=1\ldots k}\left(x_{u_i}^2 - x_{u_{i-1}}^2\right) cap_H\left(S_k,\bar{S_k}\right).$$

Applying Lemma 1.1 we have

$$\frac{\sum_{u\in V^-,v\in V^-} w_G(u,v)|x_u^2-x_v^2| + \sum_{u\in V^-,v\in V^+} w_G(u,v)x_u^2}{\sum_{u\in V^-,v\in V^-} w_G\left(u,v\right)|x_u^2-x_v^2| + \sum_{u\in V^-,v\in V^+} w_G\left(u,v\right)x_u^2} \geq \min_k \frac{cap_H\left(S_G,\bar{S_i}\right)}{cap_H\left(S_i,\bar{S_i}\right)} \geq \phi(G,H),$$

where the second inequality is by definition of $\phi(G,H)$. This proves equation 2 and the Lemma follows. □

We now proceed with the proof of the main Theorem.

Proof. We have

$$\begin{aligned} x^T L_G x &= \sum_{uv\in E_G} w_G(u,v)(x_u-x_v)^2 \\ &= \sum_{uv\in E_G^{same}} w_G(u,v)(x_u-x_v)^2 + \sum_{uv\in E_G^{dif}} w_G(u,v)(x_u-x_v)^2 \\ &\geq \sum_{uv\in E_G^{same}} w_G(u,v)(x_u-x_v)^2 + \sum_{uv\in E_G^{dif}} w_G(u,v)(x_u^2+x_v^2). \end{aligned} \tag{3}$$

The last inequality follows by $x_u x_v \leq 0$ as $x_u \leq 0$ for all $u \in V^-$ and $x_v \geq 0$ for all $v \in V^+$.

We multiply both sides of the inequality by

$$\sum_{uv\in E_G^{same}} w_G(u,v)(x_u+x_v)^2 + \sum_{uv\in E_G^{dif}} w_G(u,v)(x_u^2+x_v^2).$$

We have

$$\begin{aligned} &\left(\sum_{uv\in E_G^{same}} w_G(u,v)(x_u-x_v)^2 + \sum_{uv\in E_G^{dif}} w_G(u,v)(x_u^2+x_v^2)\right) \\ &\cdot \left(\sum_{uv\in E_G^{same}} w_G(u,v)(x_u+x_v)^2 + \sum_{uv\in E_G^{dif}} w_G(u,v)(x_u^2+x_v^2)\right) \\ \geq\ &\left(\sum_{uv\in E_G^{same}} |x_u-x_v||x_u+x_v| + \sum_{uv\in E_G^{dif}} w_G(u,v)(x_u^2+x_v^2)\right)^2 \\ =\ &\left(\sum_{uv\in E_G^{same}} |x_u^2-x_v^2| + \sum_{uv\in E_G^{dif}} w_G(u,v)(x_u^2+x_v^2)\right)^2. \end{aligned}$$

Furthermore, notice that $(x_u + x_v)^2 \le 2x_u^2 + 2x_v^2$ since $2x_u^2 + 2x_v^2 - (x_u + x_v)^2 = (x_u - x_v)^2 \ge 0$. So, we have

$$\begin{aligned}
&\sum_{uv \in E_G^{same}} w_G(u,v)(x_u + x_v)^2 + \sum_{uv \in E_G^{dif}} w_G(u,v)(x_u^2 + x_v^2) \\
\le& 2\left(\sum_{uv \in E_G^{same}} w_G(u,v)(x_u^2 + x_v^2) + \sum_{uv \in E_G^{dif}} w_G(u,v)(x_u^2 + x_v^2) \right) \\
&= 2x^T D x \le 4x^T L_K x,
\end{aligned}$$

where D is the diagonal of L_G and the last inequality comes from Lemma 1.2. Combining the last two inequalities we get:

$$\begin{aligned}
\frac{x^T L_G x}{x^T L_H x} \ge \quad & \frac{1}{2} \cdot \left(\frac{\sum_{uv \in E_G^{same}} \left| x_u^2 - x_v^2 \right| + \sum_{uv \in E_G^{dif}} w_G(u,v)\left(x_u^2 + x_v^2\right)}{x^T L_H x} \right) \\
& \cdot \left(\frac{\sum_{uv \in E_G^{same}} \left| x_u^2 - x_v^2 \right| + \sum_{uv \in E_G^{dif}} w_G(u,v)\left(x_u^2 + x_v^2\right)}{x^T L_K x} \right).
\end{aligned}$$

By Lemma 1.4, we have that the first factor is bounded by $\frac{1}{2}\phi(G,H)$ and the second factor bounded by $\frac{1}{2}\phi(G,K)$. Hence we get

$$\frac{x^T L_G x}{x^T L_H x} \ge \frac{1}{4}\phi(G,H)\phi(G,K). \tag{4}$$

□

References

[Chung, 1997] Chung, F. (1997). *Spectral Graph Theory*, volume 92 of *Regional Conference Series in Mathematics*. American Mathematical Society.