## A Proof of Theorem 1

**Proof** By employing the doubling trick described in the text, one can fix a budget $B$, run the algorithm with that budget and output the recommended arm, then repeat with $2B$, and so on. We consider a worst-case analysis in the sense that on the current budget round, we make no assumptions as to the number of times an arm has been pulled in the previous budget round. If a budget of $B'$ is necessary to succeed in finding the best arm, by performing the doubling trick one will have only had to use a budget of $2B'$ in the worst case without ever having to know $B'$ in the first place. Thus, in what follows $B$ is fixed and assumed to be sufficient to identify the best arm.

First, we verify that the algorithm never takes a total number of samples that exceeds the budget $B$:

$$\sum_{k=0}^{\lceil \log_2(n) \rceil - 1} |S_k| \left\lfloor \tfrac{B}{|S_k| \lceil \log(n) \rceil} \right\rfloor \le \sum_{k=0}^{\lceil \log_2(n) \rceil - 1} \tfrac{B}{\lceil \log(n) \rceil} \le B \, .$$

For notational ease, define $[\cdot] = \{\{\cdot\}_{t=1}\}_{i=1}^n$ so that $[\ell_{i,t}] = \{\{\ell_{i,t}\}_{t=1}^\infty\}_{i=1}^n$. Without loss of generality, we may assume that the $n$ infinitely long loss sequences $[\ell_{i,t}]$ with limits $\{\nu_i\}_{i=1}^n$ were fixed prior to the start of the game so that the $\gamma_i(t)$ envelopes are also defined for all time and are fixed. Let $\Omega$ be the set that contains all possible sets of $n$ infinitely long sequences of real numbers with limits $\{\nu_i\}_{i=1}^n$ and envelopes $[\bar{\gamma}(t)]$, that is,

$$\Omega = \left\{ [\ell'_{i,t}] : [\ |\ell'_{i,t} - \nu_i| \le \bar{\gamma}(t)\ ] \wedge \lim_{\tau \to \infty} \ell'_{i,\tau} = \nu_i \quad \forall i \ \right\}$$

where we recall that $\wedge$ is read as "and" and $\vee$ is read as "or." Clearly, $[\ell_{i,t}]$ is a single element of $\Omega$.

We present a proof by contradiction. We begin by considering the singleton set containing $[\ell_{i,t}]$ under the assumption that the Successive Halving algorithm fails to identify the best arm, i.e., $S_{\lceil \log_2(n) \rceil} \neq 1$. We then consider a sequence of subsets of $\Omega$, with each one contained in the next. The proof is completed by showing that the final subset in our sequence (and thus our original singleton set of interest) is empty when $B > z_{SH}$, which contradicts our assumption and proves the statement of our theorem.

To reduce clutter in the following arguments, it is understood that $S'_k$ for all $k$ in the following sets is a function of $[\ell'_{i,t}]$ in the sense that it is the state of $S_k$ in the algorithm when it is run with losses $[\ell'_{i,t}]$. We now present our argument in detail, starting with the singleton set of interest, and using the definition of $S_k$ in Figure 3.

$$\begin{aligned}
&\left\{ [\ell'_{i,t}] \in \Omega : [\ell'_{i,t} = \ell_{i,t}] \wedge S'_{\lceil \log_2(n) \rceil} \neq 1 \right\} \\
&= \left\{ [\ell'_{i,t}] \in \Omega : [\ell'_{i,t} = \ell_{i,t}] \wedge \bigvee_{k=1}^{\lceil \log_2(n) \rceil} \{ 1 \notin S'_k,\ 1 \in S'_{k-1} \} \right\} \\
&= \left\{ [\ell'_{i,t}] \in \Omega : [\ell'_{i,t} = \ell_{i,t}] \wedge \bigvee_{k=0}^{\lceil \log_2(n) \rceil - 1} \left\{ \sum_{i \in S'_k} \mathbf{1}\{ \ell'_{i,R_k} < \ell'_{1,R_k} \} > \lfloor |S'_k|/2 \rfloor \right\} \right\} \\
&= \left\{ [\ell'_{i,t}] \in \Omega : [\ell'_{i,t} = \ell_{i,t}] \wedge \bigvee_{k=0}^{\lceil \log_2(n) \rceil - 1} \left\{ \sum_{i \in S'_k} \mathbf{1}\{ \nu_i - \nu_1 < \ell'_{1,R_k} - \nu_1 - \ell'_{i,R_k} + \nu_i \} > \lfloor |S'_k|/2 \rfloor \right\} \right\} \\
&\subseteq \left\{ [\ell'_{i,t}] \in \Omega : \bigvee_{k=0}^{\lceil \log_2(n) \rceil - 1} \left\{ \sum_{i \in S'_k} \mathbf{1}\{ \nu_i - \nu_1 < |\ell'_{1,R_k} - \nu_1| + |\ell'_{i,R_k} - \nu_i| \} > \lfloor |S'_k|/2 \rfloor \right\} \right\} \\
&\subseteq \left\{ [\ell'_{i,t}] \in \Omega : \bigvee_{k=0}^{\lceil \log_2(n) \rceil - 1} \left\{ \sum_{i \in S'_k} \mathbf{1}\{ 2\bar{\gamma}(R_k) > \nu_i - \nu_1 \} > \lfloor |S'_k|/2 \rfloor \right\} \right\}, \qquad (1)
\end{aligned}$$

where the last set relaxes the original equality condition to just considering the maximum envelope $\bar{\gamma}$ that is encoded in $\Omega$. The summation in Eq. 1 only involves the $\nu_i$, and this summand is maximized if each $S'_k$ contains the first $|S_k'|$ arms. Hence we have,

$$\begin{aligned}
(1) &\subseteq \left\{ [\ell'_{i,t}] \in \Omega : \bigvee_{k=0}^{\lceil \log_2(n) \rceil - 1} \left\{ \sum_{i=1}^{|S_k'|} \mathbf{1}\{ 2\bar{\gamma}(R_k) > \nu_i - \nu_1 \} > \lfloor |S_k'|/2 \rfloor \right\} \right\} \\
&= \left\{ [\ell'_{i,t}] \in \Omega : \bigvee_{k=0}^{\lceil \log_2(n) \rceil - 1} \left\{ 2\bar{\gamma}(R_k) > \nu_{\lfloor |S_k'|/2 \rfloor + 1} - \nu_1 \right\} \right\} \\
&\subseteq \left\{ [\ell'_{i,t}] \in \Omega : \bigvee_{k=0}^{\lceil \log_2(n) \rceil - 1} \left\{ R_k < \bar{\gamma}^{-1}\left( \tfrac{\nu_{\lfloor |S_k'|/2 \rfloor + 1} - \nu_1}{2} \right) \right\} \right\}, \qquad (2)
\end{aligned}$$

where we use the definition of $\gamma^{-1}$ in Eq. 2. Next, we recall that $R_k = \sum_{j=0}^{k} \lfloor \frac{B}{|S_k| \lceil \log_2(n) \rceil} \rfloor \geq \frac{B/2}{(\lfloor |S_k|/2 \rfloor + 1)\lceil \log_2(n) \rceil} - 1$ since $|S_k| \leq 2(\lfloor |S_k|/2 \rfloor + 1)$. We note that we are underestimating by almost a factor of 2 to account for integer effects in favor of a simpler form. By plugging in this value for $R_k$ and rearranging we have that

$$\begin{aligned}
(2) &\subseteq \left\{ [\ell'_{i,t}] \in \Omega : \bigvee_{k=0}^{\lceil \log_2(n) \rceil - 1} \left\{ \tfrac{B/2}{\lceil \log_2(n) \rceil} < (\lfloor |S_k'|/2 \rfloor + 1)(1 + \bar{\gamma}^{-1}\left( \tfrac{\nu_{\lfloor |S_k'|/2 \rfloor + 1} - \nu_1}{2} \right)) \right\} \right\} \\
&= \left\{ [\ell'_{i,t}] \in \Omega : \tfrac{B/2}{\lceil \log_2(n) \rceil} < \max_{k=0,\dots,\lceil \log_2(n) \rceil - 1} (\lfloor |S_k'|/2 \rfloor + 1)(1 + \bar{\gamma}^{-1}\left( \tfrac{\nu_{\lfloor |S_k'|/2 \rfloor + 1} - \nu_1}{2} \right)) \right\} \\
&\subseteq \left\{ [\ell'_{i,t}] \in \Omega : B < 2\lceil \log_2(n) \rceil \max_{i=2,\dots,n} i\,(\bar{\gamma}^{-1}\left( \tfrac{\nu_i - \nu_1}{2} \right) + 1) \right\} = \emptyset
\end{aligned}$$

where the last equality holds if $B > z_{SH}$.

The second, looser, but perhaps more interpretable form of $z_{SH}$ is thanks to [16] who showed that

$$\max_{i=2,\dots,n} i\,\bar{\gamma}^{-1}\left( \tfrac{\nu_i - \nu_1}{2} \right) \leq \sum_{i=2,\dots,n} \bar{\gamma}^{-1}\left( \tfrac{\nu_i - \nu_1}{2} \right) \leq \log_2(2n) \max_{i=2,\dots,n} i\,\bar{\gamma}^{-1}\left( \tfrac{\nu_i - \nu_1}{2} \right)$$

where both inequalities are achievable with particular settings of the $\nu_i$ variables. ∎

## B Proof of Theorem 2

**Proof** Recall the notation from the proof of Theorem 1 and let $\widehat{i}([\ell'_{i,t}])$ be the output of the uniform allocation strategy at timestep $B$ with input losses $[\ell'_{i,t}]$.

$$\begin{aligned}
\left\{ [\ell'_{i,t}] \in \Omega : [\ell'_{i,t} = \ell_{i,t}] \wedge \widehat{i}([\ell'_{i,t}]) \neq 1 \right\} &= \left\{ [\ell'_{i,t}] \in \Omega : [\ell'_{i,t} = \ell_{i,t}] \wedge \ell'_{1,B/n} \geq \min_{i=2,\dots,n} \ell'_{i,B/n} \right\} \\
&\subseteq \left\{ [\ell'_{i,t}] \in \Omega : 2\bar{\gamma}(B/n) \geq \min_{i=2,\dots,n} \nu_i - \nu_1 \right\} \\
&= \left\{ [\ell'_{i,t}] \in \Omega : 2\bar{\gamma}(B/n) \geq \nu_2 - \nu_1 \right\} \\
&\subseteq \left\{ [\ell'_{i,t}] \in \Omega : B \leq n\bar{\gamma}^{-1}\left( \tfrac{\nu_2 - \nu_1}{2} \right) \right\} = \emptyset
\end{aligned}$$

where the last equality follows from the fact that $B > z_U$ which implies $\widehat{i}([\ell_{i,t}]) = 1$. ∎

# C Proof of Theorem 3

**Proof** Let $\beta(t)$ be an arbitrary, monotonically decreasing function of $t$ with $\lim_{t\to\infty} \beta(t) = 0$. Define $\ell_{1,t} = \nu_1 + \beta(t)$ and $\ell_{i,t} = \nu_i - \beta(t)$ for all $i$. Consider timestep $B$. Note that for all $i$, $\gamma_i(t) = \bar{\gamma}(t) = \beta(t)$ so that

$$\begin{aligned}
\widehat{i} = 1 &\iff \ell_{1,B/n} < \min_{i=2,\dots,n} \ell_{i,B/n} \\
&\iff \nu_1 + \bar{\gamma}(B/n) < \min_{i=2,\dots,n} \nu_i - \bar{\gamma}(B/n) \\
&\iff \nu_1 + \bar{\gamma}(B/n) < \nu_2 - \bar{\gamma}(B/n) \\
&\iff \bar{\gamma}(B/n) < \frac{\nu_2 - \nu_1}{2} \\
&\iff B \geq n\bar{\gamma}^{-1}\left(\tfrac{\nu_2-\nu_1}{2}\right).
\end{aligned}$$

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# D Proof of Theorem 4

Consider timestep $B$. We can guarantee for the Successive Halving algorithm of Figure 3 that the output arm $\widehat{i}$ satisfies

$$\begin{aligned}
\nu_{\widehat{i}} - \nu_1 &= \min_{i \in S_{\lceil \log_2(n) \rceil}} \nu_i - \nu_1 \\
&= \sum_{k=0}^{\lceil \log_2(n) \rceil - 1} \min_{i \in S_{k+1}} \nu_i - \min_{i \in S_k} \nu_i \\
&\leq \sum_{k=0}^{\lceil \log_2(n) \rceil - 1} \min_{i \in S_{k+1}} \ell_{i,R_k} - \min_{i \in S_k} \ell_{i,R_k} + 2\bar{\gamma}(R_k) \\
&= \sum_{k=0}^{\lceil \log_2(n) \rceil - 1} 2\bar{\gamma}(R_k) \leq \lceil \log_2(n) \rceil 2\bar{\gamma}\left(\lfloor \tfrac{B}{n\lceil \log_2(n) \rceil} \rfloor\right)
\end{aligned}$$

simply by inspecting how the algorithm eliminates arms and plugging in a trivial lower bound for $R_k$ for all $k$ in the last step.