Appendix

A Configurations of the test function

In this section, for the test functions used in Table 1, we provide the detailed configurations in Table 2 below. We show the mathematical form of these four test functions. We also include the distributions to sample the truths $\boldsymbol{\alpha} \sim \mathcal{N}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}^{\boldsymbol{\vartheta}})$ and the distributions of the priors $\boldsymbol{\alpha} \sim \mathcal{N}(\boldsymbol{\vartheta}^0, \boldsymbol{\Sigma}^{\boldsymbol{\vartheta},0})$. Note all the test functions chosen here can be written as linear expansions with respect to basis functions only containing $\boldsymbol{x}$. Also, these functions were designed to be minimized, so all the coefficients were achieved by taking the negative of the functions. Here each function is scaled to have a range of 100, so that the measurement noises are given on the same scale. For HDGPO, we spend the initial 25 samples on dimension reduction and the other 25 samples to run GP-UCB algorithm.

B Technical proofs

In this section, we show the detailed proof of Theorem 4.1. We first provide a sketch of the proof. As one can see from the updating equations in (12) and (11), the posterior mean estimate $\boldsymbol{\vartheta}_{\mathcal{S}}^{n+1}$ is the weighted sum of prior $\boldsymbol{\vartheta}_{\mathcal{S}}^{n}$ and the current Lasso estimate $\widehat{\boldsymbol{\vartheta}}_{\mathcal{S}}^{n+1}$. Therefore, if the Lasso estimate has an error bound as described in Lemma B.1 Zhang and Huang (2008), then the posterior estimate should also have a similar bound under certain conditions of the weighted covariance matrix. One should note that both the mean and the covariance are updated on some support $\mathcal{S}$ from the current Lasso estimate. Thus we work on a sequence of Lasso solutions and prove the bound on the intersection support set as large enough samples are made. Also note that in order to use the bound in Lemma B.1, we need to make sure that assumptions 4.1 and 4.2 are satisfied for every Lasso problem in such a sequence. Assumption 4.1 is easy to satisfy. To show that all the sequential Lasso problems satisfy Assumption 4.2, we work from a "warm" start at time N'. Then Proposition B.1 actually verifies that if the design matrix at time N' satisfies Assumption 4.2, then the following ones should also satisfy the SRC, only with a slight loose up to some constant. We begin by introducing the bound of Lasso estimates as proved in Zhang and Huang (2008) (refer to Theorem 1, 2 and 3 of the paper).

Lemma B.1 (Zhang and Huang (2008))**.** *Suppose Assumption 4.1 is satisfied and the design matrix X^{n-1} satisfies SRC(s, c_*, c^*). Let $\{c_*, c^*, c_0, s\}$ be fixed and $p \to \infty$. If we solve the Lasso given in (9) with $\lambda^n = \lambda(n, p)$, then the following properties hold with probability converging to 1 as $n \to \infty$:*

(1) $|\bar{\mathcal{S}}| \le C_3 |\mathcal{S}^*|$ for some finite positive constant C_3 defined as $C_3 = 2 + 4\widehat{c}$;

(2) Any optimal solution $\widehat{\boldsymbol{\vartheta}}^n$ to (9) satisfies the following error bound

$$\|\widehat{\boldsymbol{\vartheta}}^n - \boldsymbol{\vartheta}\|_2^2 \le \frac{C_4 \sigma_\epsilon^2 s^* \log p}{n},$$

for some positive constant C_4 depending only on c_*, c^*, and c_0.

Then the following proposition proves that all the following design matrix satisfy SRC only with looser spectrum bounds.

Proposition B.1. *Let Assumption 4.3 be satisfied. Besides, assume for some large enough N', the design matrix $\mathbf{X}^{N'-1}$ satisfies condition SRC (s, c_*, c^*). Then, for all $N' < n' \le cN'$, of which $c > 1$ is some constant, the design matrix $\mathbf{X}^{n'-1}$ can satisfy SRC $(s, c_*/c, B)$.*

Proof. To begin with, let us define the following notation. For any square matrix $\mathbf{M}$, let $\Lambda_{\max}(\mathbf{M})$ and $\Lambda_{\min}(\mathbf{M})$ be the largest eigenvalue and smallest eigenvalue of $\mathbf{M}$. Let us define $\boldsymbol{\Sigma}^{\mathbf{X},n-1}$ be the sample covariance matrix, that is $\boldsymbol{\Sigma}^{\mathbf{X},n-1} = \frac{(\mathbf{X}^{n-1})^T \mathbf{X}^{n-1}}{n}$. For any $N' < n' \le cN'$, let us divide the design matrix $\mathbf{X}^{n'-1}$ as

$$\mathbf{X}^{n'-1} = \begin{bmatrix} \mathbf{X}^{N'-1} \\ \mathbf{X}^{+} \end{bmatrix}.$$

We need to prove $\mathbf{X}^{n'-1}$ satisfies condition SRC $(s, c_*/c, B)$. Note that $\mathbf{X}^{N'-1}$ satisfies SRC (s, c_*, c^*) is equivalent to

$$c_* \le \Lambda_{\min}(\Sigma_{\mathcal{S}}^{\mathbf{X},N'-1}) \le \Lambda_{\max}(\Sigma_{\mathcal{S}}^{\mathbf{X},N'-1}) \le c^*, \quad \forall \mathcal{S} \text{ with } s = |\mathcal{S}| \text{ and } \boldsymbol{\nu} \in \mathbb{R}^s.$$

Then we have that for $\forall \mathcal{S}$ with $s = |\mathcal{S}|$

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathcal{S}}^{\mathbf{X},n'-1} &= \frac{(\mathbf{X}_{*\mathcal{S}}^{n'-1})^T \mathbf{X}_{*\mathcal{S}}^{n'-1}}{n'} \\ &= \frac{(\mathbf{X}_{*\mathcal{S}}^{N'-1})^T \mathbf{X}_{*\mathcal{S}}^{N'-1} + (\mathbf{X}_{*\mathcal{S}}^{+})^T \mathbf{X}_{*\mathcal{S}}^{+}}{n'} \\ &= \frac{N' \boldsymbol{\Sigma}_{\mathcal{S}}^{\mathbf{X},N'-1} + (\mathbf{X}_{*\mathcal{S}}^{+})^T \mathbf{X}_{*\mathcal{S}}^{+}}{n'}. \end{aligned}$$

This implies that

$$\Lambda_{\min}(\Sigma_{\mathcal{S}}^{\mathbf{X},n'-1}) \ge \frac{N'}{n'} \Lambda_{\min}(\Sigma_{\mathcal{S}}^{\mathbf{X},N'-1}) \ge \frac{c_*}{c} \tag{15}$$

and

$$\Lambda_{\max}(\Sigma_{\mathcal{S}}^{\mathbf{X},n'-1}) \le \frac{N'}{n'} \Lambda_{\max}(\Sigma_{\mathcal{S}}^{\mathbf{X},N'-1}) + \frac{\Lambda_{\max}[(\mathbf{X}_{*\mathcal{S}}^{+})^T \mathbf{X}_{*\mathcal{S}}^{+}]}{n'}.$$

Table 2: Detailed configurations for test functions in Table 1.

Test function	**Mean**
Matyas $\mu(\boldsymbol{x}) = 0.26(x_1^2 + x_2^2) - 0.48x_1x_2$, $\mathcal{X} = [-10, 10]^2$	$\boldsymbol{\vartheta} = [-0.26, -0.26, 0.48, 0, \ldots, 0]$ $\boldsymbol{\vartheta}^0 = [-0.18, -0.34, 0.3, 0, \ldots, 0]$
Six-hump Camel $\mu(\boldsymbol{x}) = (4 - 2.1x_1^2 + x_1^4/3)x_1^2$ $+x_1x_2 + (-4 + 4x_2^2)x_2^2$, $\mathcal{X} = [-3, 3] \times [-2, 2]$	$\boldsymbol{\vartheta} = [-4, 2.1, -1/3, -1, 4, -4, 0, \ldots, 0]$ $\boldsymbol{\vartheta}^0 = [-3.2, 1.5, -0.1, -1.5, 4.5, -3.6, 0, \ldots, 0]$
Trid $\mu(\boldsymbol{x}) = \sum_{i=1}^d (x_i - 1)^2 - \sum_{i=2}^d x_i x_{i-1}$, $d = 6, \mathcal{X} = [-36, 36]^6$	$\boldsymbol{\vartheta} = [\underbrace{-1, \ldots, -1}_{6} \underbrace{2, \ldots, 2}_{6}, \underbrace{1, \ldots, 1}_{5}, -6, 0, \ldots, 0]$ $\boldsymbol{\vartheta}^0 = [\underbrace{-0.6, \ldots, -0.6}_{6}, \underbrace{2.3, \ldots, 2.3}_{6}, \underbrace{1.5, \ldots, 1.5}_{5}, -4, 0, \ldots, 0]$

Since

$$(\mathbf{X}_{*\mathcal{S}}^+)^T \mathbf{X}_{*\mathcal{S}}^+ = \boldsymbol{x}_{\mathcal{S}}^{N'} (\boldsymbol{x}_{\mathcal{S}}^{N'})^T + \boldsymbol{x}_{\mathcal{S}}^{N'+1} (\boldsymbol{x}_{\mathcal{S}}^{N'+1})^T + \cdots + \boldsymbol{x}_{\mathcal{S}}^{n'-1} (\boldsymbol{x}_{\mathcal{S}}^{n'-1})^T$$

and

$$\Lambda_{\max}[\boldsymbol{x}_{\mathcal{S}}^n (\boldsymbol{x}_{\mathcal{S}}^n)^T] = \|\boldsymbol{x}_{\mathcal{S}}^n\|_2^2 \leq B, \quad \forall n,$$

we can get that

$$\begin{aligned} \Lambda_{\max}(\Sigma_{\mathcal{S}}^{\mathbf{X}, n'-1}) &\leq \frac{N'}{n'} c^* + \frac{n' - N'}{n'} B \\ &\leq \max(c^*, B) = B. \end{aligned} \tag{16}$$

Combining (15) and (16) completes the proof. $\square$

Thus we have all the ingredients to complete the proof of Theorem 4.1.

Proof of Theorem 4.1

Proof. We begin with the proof of part (1).

Theorem 4.1 assumes that $\mathbf{X}^{N'-1}$ satisfies $\mathrm{SRC}(C_1 s^*, c_*, c^*)$. By Proposition B.1, we know that for all $\underline{c}N' \leq n' \leq \bar{c}N'$, the design matrix $\mathbf{X}^{n'-1}$ can satisfy $\mathrm{SRC}(C_1 s^*, c_*/\bar{c}, B)$. Thus the result of part (1) directly follows from part(1) of Lemma B.1.

We now proceed to prove part (2). Throughout the proof, we let $c_*, c^*, c_0, \underline{c}, \bar{c}$, and B be fixed. We also let the bounds $[C_{\min}, C_{\max}]$ for truncating the eigenvalues of $\widehat{\mathrm{Cov}}(\boldsymbol{z}_{\mathcal{S}})^{(n+1)}$ be fixed positive constants, so in the following, the C_is are some positive constants depending only on these quantities. If we let $\bar{\mathcal{S}} := \bigcap_{n'=N'}^{n} \mathcal{S}^{n'}$, then from the updating formula in (12) and (11), we have

$$\boldsymbol{\vartheta}_{\bar{\mathcal{S}}}^n = \boldsymbol{\Sigma}_{\bar{\mathcal{S}}}^{\boldsymbol{\vartheta}, n} \Big[(\boldsymbol{\Sigma}_{\bar{\mathcal{S}}}^{\boldsymbol{\vartheta}, N'-1})^{-1} \boldsymbol{\vartheta}_{\bar{\mathcal{S}}}^{N'-1} + [(\widehat{\boldsymbol{\Sigma}}_{\mathcal{S}^{N'}}^{\boldsymbol{\vartheta}, N'})^{-1}]_{\bar{\mathcal{S}}} \widehat{\boldsymbol{\vartheta}}_{\bar{\mathcal{S}}}^{N'} + \cdots + [(\widehat{\boldsymbol{\Sigma}}_{\mathcal{S}^n}^{\boldsymbol{\vartheta}, n})^{-1}]_{\bar{\mathcal{S}}} \widehat{\boldsymbol{\vartheta}}_{\bar{\mathcal{S}}}^n \Big],$$

$$\boldsymbol{\Sigma}_{\bar{\mathcal{S}}}^{\boldsymbol{\vartheta}, n} = \Big[(\boldsymbol{\Sigma}_{\bar{\mathcal{S}}}^{\boldsymbol{\vartheta}, N'-1})^{-1} + [(\widehat{\boldsymbol{\Sigma}}_{\mathcal{S}^{N'}}^{\boldsymbol{\vartheta}, N'})^{-1}]_{\bar{\mathcal{S}}} + \cdots + [(\widehat{\boldsymbol{\Sigma}}_{\mathcal{S}^n}^{\boldsymbol{\vartheta}, n})^{-1}]_{\bar{\mathcal{S}}} \Big]^{-1}.$$

Then if we define

$$\begin{aligned} \boldsymbol{\delta}_{\bar{\mathcal{S}}}^{n'} &:= \boldsymbol{\vartheta}_{\bar{\mathcal{S}}}^{n'} - \boldsymbol{\vartheta}_{\bar{\mathcal{S}}} \\ \widehat{\boldsymbol{\delta}}_{\bar{\mathcal{S}}}^{n'} &:= \widehat{\boldsymbol{\vartheta}}_{\bar{\mathcal{S}}}^{n'} - \boldsymbol{\vartheta}_{\bar{\mathcal{S}}}, \end{aligned}$$

for all $N' - 1 \leq n' \leq n$ to simplify notation, we have

$$\boldsymbol{\delta}_{\bar{\mathcal{S}}}^n = \boldsymbol{\Sigma}_{\bar{\mathcal{S}}}^{\boldsymbol{\vartheta}, n} \Big[(\boldsymbol{\Sigma}_{\bar{\mathcal{S}}}^{\boldsymbol{\vartheta}, N'-1})^{-1} \boldsymbol{\delta}_{\bar{\mathcal{S}}}^{N'-1} + [(\widehat{\boldsymbol{\Sigma}}_{\mathcal{S}^{N'}}^{\boldsymbol{\vartheta}, N'})^{-1}]_{\bar{\mathcal{S}}} \widehat{\boldsymbol{\delta}}_{\bar{\mathcal{S}}}^{N'} + \cdots + [(\widehat{\boldsymbol{\Sigma}}_{\mathcal{S}^n}^{\boldsymbol{\vartheta}, n})^{-1}]_{\bar{\mathcal{S}}} \widehat{\boldsymbol{\delta}}_{\bar{\mathcal{S}}}^n \Big].$$

This gives us the following bound on $\boldsymbol{\delta}_{\bar{\mathcal{S}}}^n$,

$$\|\boldsymbol{\delta}_{\bar{\mathcal{S}}}^n\|_2 \leq \|\boldsymbol{\Sigma}_{\bar{\mathcal{S}}}^{\boldsymbol{\vartheta}, n}\|_2 \Big[\|(\boldsymbol{\Sigma}_{\bar{\mathcal{S}}}^{\boldsymbol{\vartheta}, N'-1})^{-1}\|_2 \|\boldsymbol{\delta}_{\bar{\mathcal{S}}}^{N'-1}\|_2 + \|[(\widehat{\boldsymbol{\Sigma}}_{\mathcal{S}^{N'}}^{\boldsymbol{\vartheta}, N'})^{-1}]_{\bar{\mathcal{S}}}\|_2 \|\widehat{\boldsymbol{\delta}}_{\bar{\mathcal{S}}}^{N'}\|_2 + \cdots + \|[(\widehat{\boldsymbol{\Sigma}}_{\mathcal{S}^n}^{\boldsymbol{\vartheta}, n})^{-1}]_{\bar{\mathcal{S}}}\|_2 \|\widehat{\boldsymbol{\delta}}_{\bar{\mathcal{S}}}^n\|_2 \Big].$$

We now proceed to bound each of the quantities. Let us for now assume that $N' \leq n' \leq n$. As we suppose the design matrix for the Lasso solution $\widehat{\boldsymbol{\vartheta}}_{\mathcal{S}}^{N'}$ satisfies $\mathrm{SRC}(C_1 s^*, c_*, c^*)$, by Proposition B.1 and Lemma B.1, if we choose $\lambda^{n'} = \lambda(n, p)$ such that

$$\lambda^{n'} = O(\sqrt{n' \log p}), \tag{17}$$

then there exists some constant C_4 such that

$$\|\widehat{\boldsymbol{\delta}}_{\bar{\mathcal{S}}}^{n'}\|_2 \leq C_4 \sigma_\epsilon \sqrt{\frac{s^* \log p}{n'}}, \quad \text{for all } \underline{c}N' \leq n' \leq n, \tag{18}$$

with probability converging to 1. We know from (10) that $\widehat{\boldsymbol{\Sigma}}_{\mathcal{S}^{n'}}^{\boldsymbol{\vartheta}, n'}$ is computed by:

$$\widehat{\boldsymbol{\Sigma}}_{\mathcal{S}^{n'}}^{\boldsymbol{\vartheta}, n'} = \mathbf{M}_{\mathcal{S}^{n'}}^{n'-1} \sigma_\epsilon^2 + (\lambda^{n'})^2 \mathbf{M}_{\mathcal{S}^{n'}}^{n'-1} \widehat{\mathrm{Cov}}(\boldsymbol{z}_{\mathcal{S}^{n'}})^{(n')} \mathbf{M}_{\mathcal{S}^{n'}}^{n'-1},$$

where

$$\mathbf{M}_{\mathcal{S}^{n'}}^{n'-1} = \left[(\mathbf{X}_{*\mathcal{S}^{n'}}^{n'-1})^T \mathbf{X}_{*\mathcal{S}^{n'}}^{n'-1}\right]^{-1}.$$

The SRC$(C_1 s, c_*, c^*)$ gives us

$$\begin{aligned}\Lambda_{\max}(\mathbf{M}_{\mathcal{S}}^{N'-1}) &\leq \frac{1}{N' c_*} < \infty,\\ \Lambda_{\min}(\mathbf{M}_{\mathcal{S}}^{N'-1}) &\geq \frac{1}{N' c^*} > 0,\end{aligned}$$

for any $\mathcal{S}$ with $|\mathcal{S}| = C_1 s^*$. Therefore, since $|\mathcal{S}^{n'}| \leq C_1 s^*$, which is proved in part (1), by Proposition B.1, we can show that for all $N' \leq n' \leq n$, there exist positive constants C_5 and C_6, such that

$$\Lambda_{\max}(\mathbf{M}_{\mathcal{S}^{n'}}^{n'-1}) \leq \frac{C_5}{n'} < \infty, \tag{19}$$

$$\Lambda_{\min}(\mathbf{M}_{\mathcal{S}^{n'}}^{n'-1}) \geq \frac{C_6}{n'} > 0. \tag{20}$$

It is not hard to prove

$$\Lambda_{\min}(\mathbf{MN}) \geq \Lambda_{\min}(\mathbf{M})\Lambda_{\min}(\mathbf{N})$$

for any positive semidefinite matrices $\mathbf{M}$ and $\mathbf{N}$, so using Weyl's inequality in matrix theory, (10), and (20), we have the following bound,

$$\begin{aligned}&\|[(\widehat{\mathbf{\Sigma}}_{\mathcal{S}^{n'}}^{\boldsymbol{\vartheta},n'})^{-1}]_{\bar{\mathcal{S}}}\|_2 \leq \|(\widehat{\mathbf{\Sigma}}_{\mathcal{S}^{n'}}^{\boldsymbol{\vartheta},n'})^{-1}\|_2 = \Lambda_{\min}^{-1}(\widehat{\mathbf{\Sigma}}_{\mathcal{S}^{n'}}^{\boldsymbol{\vartheta},n'})\\ &\leq \frac{1}{\Lambda_{\min}(\sigma_\epsilon^2 \mathbf{M}_{\mathcal{S}^{n'}}^{n'-1}) + (\lambda^{n'})^2 \Lambda_{\min}(\widehat{\mathrm{Cov}}(\boldsymbol{z}_{\mathcal{S}^{n'}}^{n'}))\Lambda_{\min}^2(\mathbf{M}_{\mathcal{S}^{n'}}^{n'-1})}\\ &\leq \frac{C_7 n'}{\sigma_\epsilon^2 \log p},\end{aligned} \tag{21}$$

for some constant C_7. Similarly, by (17), (19), and (10), we can also get

$$\begin{aligned}&\|\widehat{\mathbf{\Sigma}}_{\mathcal{S}^{n'}}^{\boldsymbol{\vartheta},n'}\|_2 = \Lambda_{\max}(\widehat{\mathbf{\Sigma}}_{\mathcal{S}^{n'}}^{\boldsymbol{\vartheta},n'})\\ &\leq \sigma_\epsilon^2 \Lambda_{\max}(\mathbf{M}_{\mathcal{S}^{n'}}^{n'-1}) + (\lambda^{n'})^2 \Lambda_{\max}(\widehat{\mathrm{Cov}}(\boldsymbol{z}_{\mathcal{S}^{n'}}^{n'}))\Lambda_{\max}^2(\mathbf{M}_{\mathcal{S}^{n'}}^{n'-1})\\ &\leq C_8 \frac{\sigma_\epsilon^2 \log p}{n'},\end{aligned}$$

for some constant C_8. Thus, for the posterior covariance matrix, we have

$$\begin{aligned}&\|\mathbf{\Sigma}_{\bar{\mathcal{S}}}^{\boldsymbol{\vartheta},n}\|_2\\ &= \Lambda_{\min}^{-1}\left[(\mathbf{\Sigma}_{\bar{\mathcal{S}}}^{\boldsymbol{\vartheta},N'-1})^{-1} + [(\widehat{\mathbf{\Sigma}}_{\mathcal{S}^{N'}}^{\boldsymbol{\vartheta},N'})^{-1}]_{\bar{\mathcal{S}}} + \cdots + [(\widehat{\mathbf{\Sigma}}_{\mathcal{S}^{n}}^{\boldsymbol{\vartheta},n})^{-1}]_{\bar{\mathcal{S}}}\right]\\ &\leq \frac{1}{\Lambda_{\min}\left[[(\widehat{\mathbf{\Sigma}}_{\mathcal{S}^{N'}}^{\boldsymbol{\vartheta},N'})^{-1}]_{\bar{\mathcal{S}}}\right] + \cdots \Lambda_{\min}\left[(\widehat{\mathbf{\Sigma}}_{\mathcal{S}^{n}}^{\boldsymbol{\vartheta},n})^{-1}\right]_{\bar{\mathcal{S}}}}\\ &= \frac{1}{\Lambda_{\max}^{-1}(\widehat{\mathbf{\Sigma}}_{\mathcal{S}^{N'}}^{\boldsymbol{\vartheta},N'}) + \cdots \Lambda_{\max}^{-1}(\widehat{\mathbf{\Sigma}}_{\mathcal{S}^{n}}^{\boldsymbol{\vartheta},n})}\\ &\leq \frac{2C_8 \sigma_\epsilon^2 \log p}{(N'+n)(n-N'+1)}\\ &\leq \frac{C_9 \sigma_\epsilon^2 \log p}{n^2},\end{aligned} \tag{22}$$

for some constant C_9. If we let

$$\Delta_{\bar{\mathcal{S}}}(N') = \|(\mathbf{\Sigma}_{\bar{\mathcal{S}}}^{\boldsymbol{\vartheta},N'-1})^{-1}\|_2 \|\boldsymbol{\delta}_{\bar{\mathcal{S}}}^{N'-1}\|_2,$$

then combining (18),(21), and (22) gives us the following bound on $\boldsymbol{\delta}_{\bar{\mathcal{S}}}^n$

$$\begin{aligned}\|\boldsymbol{\delta}_{\bar{\mathcal{S}}}^n\|_2 &\leq \frac{C_9 \sigma_\epsilon^2 \log p}{n^2}\left(\Delta_{\bar{\mathcal{S}}}(N') + \sum_{n'=N'}^{n} \frac{C_4 C_7 \sqrt{s^* n'}}{\sigma_\epsilon \sqrt{\log p}}\right)\\ &\leq \frac{C_{10}\sigma_\epsilon \sqrt{s^* \log p}}{\sqrt{n}} + \frac{C_9 \sigma_\epsilon^2 \log p \Delta_{\bar{\mathcal{S}}}(N')}{n^2},\end{aligned} \tag{23}$$

for some constant C_{10}. After dropping off the higher order term, (23) is equivalent to

$$\|\boldsymbol{\vartheta}_{\bar{\mathcal{S}}}^n - \boldsymbol{\vartheta}_{\bar{\mathcal{S}}}\|_2^2 \leq \frac{C_2 \sigma_\epsilon^2 s^* \log p}{n}$$

and thus completes the proof. □