A Proofs

The proof of Proposition 1 relies on the following variational form of Kullback–Leibler divergence, which is given in Theorem 5.2.1 of Robert Gray’s textbook *Entropy and Information Theory* [10].

**Fact 1.** Fix two probability measures $P$ and $Q$ defined on a common measurable space $(\Omega, \mathcal{F})$. Suppose that $P$ is absolutely continuous with respect to $Q$. Then

$$D(P||Q) = \sup_X \{ E_P[X] - \log E_Q[e^X] \},$$

where the supremum is taken over all random variables $X$ such that the expectation of $X$ under $P$ is well defined, and $e^X$ is integrable under $Q$.

**Proof of Proposition 1.**

$$I(T; \phi) = \sum_{i=1}^{n} P(T = i) D(P(\phi = \cdot | T = i) \| P(\phi = \cdot))$$

$$\geq \sum_{i=1}^{n} P(T = i) D(P(\phi_i = \cdot | T = i) \| P(\phi_i = \cdot))$$

Applying Fact 1 with $P = P(\phi_i = \cdot | T = i), Q = P(\phi_i = \cdot), and X = \lambda(\phi_i - \mu_i), we have

$$D(P(\phi_i = \cdot | T = i) \| P(\phi_i = \cdot)) \geq \sup_{\lambda} \lambda \Delta_i - \frac{\lambda^2 \sigma^2}{2}$$

where $\Delta_i \equiv E[\phi_i | T = i] - \mu_i$. Taking the derivative with respect to $\lambda$, we find that the optimizer is $\lambda = \Delta_i / \sigma^2$.

This gives

$$2\sigma^2 I(T; \phi) \geq \sum_{i=1}^{n} P(T = i) \Delta_i^2 = E[\Delta_i^2].$$

By the Tower property of conditional expectation and Jensen’s inequality

$$E[\phi_T - \mu_T] = E[\Delta_T] \leq \sqrt{E[\Delta_T^2]} \leq \sigma \sqrt{2I(T; \phi)}.$$

**Proof of Proposition 2.** Set

$$M_k = \max_{i \leq k} \phi_i = \sigma \max_{i \leq k} \frac{\phi_i}{\sigma}.$$

Basic facts about the maximum of independent standard Gaussian random variables then imply

$$\sigma^{-1} E[M_k] \geq c \sqrt{2k} \quad \& \quad \sigma^{-1} E[M_k] - \sqrt{2k} \xrightarrow{k \to \infty} 0,$$

where $c$ is a numerical constant that does not depend on $k$ or $\sigma$. Now we have,

$$E[\phi_{T_B}] = E[M_{\lfloor B \rfloor}].$$

The result then follows by observing that for $B \geq 2$

$$B \leq 2 \lfloor B \rfloor$$

and

$$\sqrt{B} - \sqrt{\lfloor B \rfloor} \xrightarrow{B \to \infty} 0.$$
Proof of Proposition 3. Following the same analysis as in the sub-Gaussian setting, we have

\[ D\left( \mathbf{P}(\phi_i = \cdot | T = i) \| \mathbf{P}(\tilde{\phi}_i = \cdot) \right) \geq \sup_{\lambda < 1/b} \lambda \Delta_i - \lambda^2 \sigma^2 / 2. \]

The RHS is greater than the value from setting \( \lambda = 1/b \). Therefore, we have

\[ D\left( \mathbf{P}(\phi_i = \cdot | T = i) \| \mathbf{P}(\phi_i = \cdot) \right) \geq \frac{\Delta_i}{b} - \frac{\sigma^2}{2b^2}. \]

Summing over \( \mathbf{P}(T = i) \) gives

\[ \mathbf{E}[\phi_T - \mu] \leq bI(T; \phi) + \frac{\sigma^2}{2b}. \]

When \( b > 1, \lambda = 1/\sqrt{b} < 1/b \) is also a feasible point. Putting in this value of \( \lambda \) into the calculations above gives the second bound

\[ \mathbf{E}[\phi_T - \mu_T] \leq \sqrt{b}I(T; \phi) + \frac{\sigma^2}{2\sqrt{b}}. \]

Proof of Corollary 1. Let \( Z_i = (\phi_i - \mu_i)^2 \). Then, \( Z_i \in [-C^2, C^2] \) and hence is \( C^2 \)-sub-Gaussian. This implies that

\[ \mathbf{E}[Z_T - \mu_T] \leq C\sqrt{2I(T; Z)} \leq C\sqrt{2I(T; \phi)} \]

where the last step follows from the data-processing inequality for mutual-information, and the fact that \( Z_i \) is a deterministic function of \( \phi_i \).

Proof of Proposition 4.

\[
D\left( \mathbf{P}(\phi_T = \cdot) \| \mathbf{P}(\tilde{\phi}_T = \cdot) \right) \leq D\left( \mathbf{P}(\phi_T = \cdot , T = i) \| \mathbf{P}(\tilde{\phi}_T = \cdot , T = i) \right) \\
= \sum_{T=1}^{m} \mathbf{P}(T = i) D\left( \mathbf{P}(\phi_T = \cdot | T = i) \| \mathbf{P}(\tilde{\phi}_T = \cdot | T = i) \right) \\
= \sum_{T=1}^{m} \mathbf{P}(T = i) D\left( \mathbf{P}(\phi_i = \cdot | T = i) \| \mathbf{P}(\phi_i = \cdot) \right) \\
\leq \sum_{T=1}^{m} \mathbf{P}(T = i) D\left( \mathbf{P}(\phi = \cdot | T = i) \| \mathbf{P}(\phi = \cdot) \right) \\
= I(T; \phi),
\]

where both inequalities follow from the data-processing inequality for KL divergence.

Proof of Proposition 5. Since \( \phi_i \sim \text{Uniform}(0,1) \), \( Z_{\epsilon,i} = 1(\phi_i < \epsilon) \) is a Bernoulli random variable with parameter \( \epsilon \) and \( \mathbf{E}[Z_i] = \epsilon \). We use the fact that a probability \( p \) Bernoulli random variable is sub-Gaussian with parameter \( [4] \)

\[
\sigma = \sqrt{\frac{1 - 2p}{2 \log((1 - p)/p)}} \leq \sqrt{\frac{1}{2 \log(1/2p)}}.
\]

Combining this with Proposition 1, we have the desired result

\[
\mathbf{E}[Z_T - \mathbf{E}[\mu_T]] \leq \mathbb{P}(\exists_T < \epsilon) - \epsilon \leq \sqrt{I(T; Z_{\epsilon}) / \log(1/2\epsilon)}.
\]

The second inequality follows by the data-processing inequality.
Proof of Proposition 6. Note that
\[ \max_{x \in \mathcal{X}} \mu(x) \geq \mathbb{E}[\mu(X^*)] \]
and
\[ \mathbb{E}[\max_{x \in \mathcal{X}} f_\theta(x)] = \mathbb{E}[f_\theta(X^*)] \]
Therefore,
\[ \mathbb{E}[\max_{x \in \mathcal{X}} f_\theta(x)] - \max_{x \in \mathcal{X}} \mu(x) \leq \mathbb{E}[f_\theta(X^*)] - \mathbb{E}[\mu(X^*)] \leq \sigma \sqrt{2I(X^*; \theta)} = \sigma \sqrt{2H(X^*)} \]
\[ \square \]

Proof of Lemma 1. Since, conditional on \( H_k, T_{k+1} \) is independent of \( \phi \), the data-processing inequality for mutual information implies,
\[ I(T_{k+1}; \phi) \leq I(H_k; \phi). \]
Now we have,
\[ I(H_k; \phi) = \sum_{i=1}^k I((T_i, Y_{T_i}); \phi|H_{i-1}). \]
We complete the proof by simplifying the expression for \( I((T_i, Y_{T_i}); \phi|H_{i-1}) \). Let \( \phi_{(-i)} = (\phi_j : j \neq i) \). Then,
\[ I((T_i, Y_{T_i}); \phi|H_{i-1}) = I(T_i; \phi|H_{i-1}) + I(Y_{T_i}; \phi|H_{i-1}, T_i) = I(Y_{T_i}; \phi|H_{i-1}, T_i) = I(Y_{T_i}; \phi_{T_i}|H_{i-1}, T_i) + I(Y_{T_i}; \phi_{(-T_i)}|H_{i-1}, T_i, \phi_{T_i}) \]
where the final equality follows because, conditioned on \( \phi_{T_i}, Y_{T_i} \) is independent of \( \phi_{(-T_i)} \).
\[ \square \]

Proof of Lemma 2.
\[ I(X; Y) = -\frac{1}{2} \log \left( 1 - \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \right) = -\frac{1}{2} \log \left( 1 - \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \right) = \frac{1}{2} \log \left( 1 + \frac{\sigma_1^2}{\sigma_2^2} \right). \]
\[ \square \]

Proof of Proposition 7. In order to apply our general result, we need to convert the bound on the differences in the expectations of a random variable into a bound on probability. Let
\[ Z_i = 1(\phi_i - \mu_i > \tau) \quad i \in \mathbb{N}, \]
so that \( Z_i \) is a bernoulli random variable with expectation
\[ e_i = \mathbb{E}[Z_i] \leq \exp\left(-\frac{n\tau^2}{2}\right) \]
It is known [4] that a bernoulli random variable with parameter \( p < \frac{1}{2} \) is \( \sigma \)-sub-Gaussian with
\[ \sigma = \sqrt{\frac{1-2p}{\log((1-p)/p)}} \leq \sqrt{\frac{1}{2 \log(1/2p)}} \]
and applying this with \( p = \mathbb{E}[Z_i] \) shows that \( Z_i \) is sub-Gaussian with effective standard deviation less than \( \sqrt{\frac{c}{n\tau^2}} \) where \( c \) is a universal numerical constant. This shows that
\[ \mathbb{E}[Z_{T_{k+1}} - e_{T_{k+1}}] \quad \leq \quad \frac{\sqrt{c}}{\tau} \sqrt{\frac{2I(T_{k+1}; \phi)}{n}} \]
\[ \quad \leq \quad \frac{\sqrt{c}}{\tau} \sqrt{\frac{2 \sum_{i=1}^k I(Y_{T_i}; \phi_{T_i}|Y_{T_1},...Y_{T_{i-1}})}{n}} \]