Limits on Sparse Support Recovery via Linear Sketching with Random Expander Matrices

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Abstract

Linear sketching is a powerful tool for the problem of sparse signal recovery, having numerous applications such as compressive sensing, data stream computing, graph sketching, and routing. Motivated by applications where the *positions* of the non-zero entries in a sparse vector are of primary interest, we consider the problem of *support recovery* from a linear sketch taking the form $\mathbf{Y} = \mathbf{X}\beta + \mathbf{Z}$. We focus on a widely-used expander-based construction in the columns of the measurement matrix $\mathbf{X} \in \mathbb{R}^{n \times p}$ are random permutations of a sparse binary vector containing $d \ll n$ ones and $n - d$ zeros. We provide a sharp characterization of the number of measurements required for an information-theoretically optimal decoder, thus permitting a precise comparison to the i.i.d. Gaussian construction. Our findings reveal both positive and negative results, showing that the performance nearly matches the Gaussian construction at moderate-to-high noise levels, while being worse by an arbitrarily large factor at low noise levels.

1 Introduction

In recent years, there has been a tremendous amount of research in linear sketching and sparse signal recovery, in which the goal is to recover a sparse vector $\beta \in \mathbb{R}^p$ based on n noisy observations of the form

$$\mathbf{Y} = \mathbf{X}\beta + \mathbf{Z}, \qquad (1)$$

where $\mathbf{X} \in \mathbb{R}^{n \times p}$ is a known measurement matrix, and $\mathbf{Z} \in \mathbb{R}^n$ is additive noise. This simple model itself comes with a large number of applications, such as compressive sensing, data stream computing, graph sketching, routing, and wireless communication. Perhaps the most common performance criterion is to characterize a distance between β and an estimate $\hat{\beta}$, such as $\|\beta - \hat{\beta}\|_1$ or $\|\beta - \hat{\beta}\|_2$ [1–3].

In many applications, the primary interest is not in the values of β, but rather in the locations of the non-zero entries: In cognitive radio settings [4], one may seek to determine which channels are in use and which are available; in routing and data stream problems [5], one may be interested in determining a sparse set of IP addresses that have been accessed (or more generally, accessed an unusually large number of times, *cf.*, the *heavy hitters* problem [5,6]). This motivates the study of the *support recovery* problem, where the goal is to recover the non-zero indices $S := \{i : \beta_i \neq 0\}$. As well as being of direct interest, this can lead to other estimation guarantees, such ℓ_2 error [7].

The measurement matrix $\mathbf{X}$ plays a key role in determining both the required number of measurements for successful recovery, and the ability to perform the estimation efficiently. For example, i.i.d. Gaussian matrices tend to provide the best theoretical guarantees [1], but can pose significant limitations in terms of storage and computation. On the other hand, constructions based on structured matrices (Hadamard, Fourier, etc.) permit efficient operations such as matrix multiplications without needing explicit storage, but often at the cost of requiring more measurements.

Constructions based on *expanders* [8–11] have proved to be a promising approach for achieving the best of both worlds. Such constructions are highly sparse and thus permit efficient storage and computation. Moreover, they have been shown to achieve order-optimal results in terms of the ℓ_1-ℓ_1 error guarantee, expressing the difference $\|\hat{\beta} - \beta\|_1$ as a function of the noise norm $\|\mathbf{Z}\|_1$ [8]. However, to our knowledge, precise

characterizations of how these compare to the popular i.i.d. Gaussian construction have remained elusive. While the latter are less practical due to their storage and computation requirements, they generally come with the best known theoretical guarantees.

In this paper, we study the performance of expander-based measurements in the context of support recovery, and provide several results that allow for rigorous comparisons to the i.i.d. Gaussian construction. As formalized below, we focus on the most widely-used class of random matrices for constructing expanders, where each column is a random permutation of a sparse binary vector. Such constructions are particularly common in computer science applications. We study both upper and lower bounds on the number of measurements that hold regardless of the computational complexity. A key finding is that when the noise level is not too low, the sample complexity is nearly identical to that of Gaussian constructions, even up to the constant factors. In contrast, we show that Gaussian constructions may require significantly fewer measurements at low noise levels.

Information-theoretic studies of support recovery can be found in [7, 12–17]. The sharp thresholds for the linear model with i.i.d. Gaussian matrices that provide the baseline comparison for the present paper were first given in [15], though our analysis follows a more general approach for possibly non-linear models performed in [17]. The analysis therein heavily relies on the measurement matrix being i.i.d. (not necessarily Gaussian). A key contribution of the present work is the development of *change-of-measure* techniques in which the error probability under a non-i.i.d. distribution is bounded in terms of that of an i.i.d. distribution. We also address a second difficulty, namely, characterizing the concentration of relevant sums of *non-independent* random variables, as opposed to the i.i.d. sums arising in [17]. Although we do not consider it here, our techniques can directly be applied to an analogous group testing problem [18] to show the asymptotic optimality of our non-i.i.d. construction in the case that the number of defective items is $\Theta(1)$.

Notation We use bold symbols for collections of n scalars or vectors; vectors of other sizes may still have the regular font. We write β_S (respectively, $\mathbf{X}_S$ to denote the subvector of β (respectively, submatrix of $\mathbf{X}$) containing the entries (respectively, columns) indexed by S. We write $\beta_i := \beta_{\{i\}}$ and $\mathbf{X}_i := \mathbf{X}_{\{i\}}$, and let X_{ij} denote element (i, j) of $\mathbf{X}$. The complement with respect to $\{1, \ldots, p\}$ is denoted by $(\cdot)^c$. For a given joint distribution P_{XY}, the corresponding marginal distributions are denoted by P_X, P_Y, $P_{Y|X}$, and so on. We use standard notations for the entropy and mutual information (e.g., $H(X)$, $I(X;Y|Z)$), and we define the binary entropy function $H_2(\epsilon) := -\epsilon \log \epsilon - (1 - \epsilon) \log(1 - \epsilon)$. We make use of the standard asymptotic notations $O(\cdot)$, $o(\cdot)$, $\Theta(\cdot)$, $\Omega(\cdot)$ and $\omega(\cdot)$. We define the function $[\cdot]^+ = \max\{0, \cdot\}$, and write the floor function as $\lfloor \cdot \rfloor$. The function log has base e.

1.1 Definition and Probabilistic Constructions of Expander Matrices

In the following, we formally define the notion of an expander matrix $\mathbf{X} \in \{0, 1\}^{n \times p}$. For a given subset $S \subseteq \{1, \ldots, p\}$, we let $\mathcal{N}_{\mathbf{X}}(S) := \{i \in \{1, \ldots, n\} : X_{ij} = 1 \text{ for some } j \in S\}$; interpreting $\mathbf{X}$ as representing a bipartite graph from $\{1, \ldots, p\}$ to $\{1, \ldots, n\}$, we can view $\mathcal{N}_{\mathbf{X}}(S)$ as being the neighborhood set of S.

Definition 1. A matrix $\mathbf{X} \in \{0, 1\}^{n \times p}$ containing exactly d ones per column is a *lossless (k, d, ϵ)-expander matrix* if, for any $S \subseteq \{1, \ldots, p\}$ with $|S| \leq k$, we have $|\mathcal{N}_{\mathbf{X}}(S)| \geq (1 - \epsilon)d|S|$.

One can think of this definition as stating that the bipartite graph representing $\mathbf{X}$ is "well-connected" in the sense that every "small" subset of the left-vertices has a number of neighbors within a factor of $1 - \epsilon$ of the highest number possible.

There has been an extensive amount of work on both existence proofs for expander matrices with given parameters (p, n, k, d, ϵ) via the probabilistic method, and also explicit constructions; see [8, 19] and the references therein. In this paper, we consider a standard probabilistic construction in which each column $\mathbf{X}_i$ of $\mathbf{X}$ is a uniformly random permutation of a fixed vector containing d ones and $n - d$ zeros [8, 20]. We define the ratio $\rho := \frac{d}{n}$, and since we are interested in sparse matrices, we limit our attention to the case that $\rho \leq \frac{1}{2}$.

The following result, proved in Appendix A, is obtained by starting with a general bound of Berinde [8] and then bounding it in a manner that is more specific to the scaling regimes considered in this paper.

Lemma 1. *For the preceding construction with $k = \Theta(1)$, $\frac{d}{n} = \Theta(1)$ and $\epsilon = \Theta(1)$, the random matrix $\mathbf{X}$ is a (k, d, ϵ)-expander matrix with probability approaching one as $p \to \infty$ provided that*

$$\epsilon \log \frac{n}{dk} - H_2(\epsilon) > \frac{\log p}{d}(1 + \eta) \qquad (2)$$

for some $\eta > 0$.

This result will be used in our numerical evaluations in Section 3 to check when the random matrices that we consider indeed form expander matrices. Note that our results will be concerned with the scaling $n = \Theta(k \log p) = \Theta(\log p)$; since we are also considering $\frac{d}{n} = \Theta(1)$, this means that $d = \Theta(\log p)$.

We briefly mention that under the scaling laws in Lemma 1, the storage of $\mathbf{X}$ requires roughly $npH_2(\rho)$ bits; for small values of ρ, this is far below the np real numbers required to store an i.i.d. Gaussian matrix.

1.2 Support Recovery: Problem Statement

The support set S is assumed to be equiprobable on $\mathcal{S}$, defined to contain the $\binom{p}{k}$ subsets of $\{1,\ldots,p\}$ with cardinality k. Given S, the entries of β_{S^c} are deterministically set to zero, and the remaining entries are generated according to some distribution $\beta_S \sim P_{\beta_S}$. We assume that these non-zero entries follow the same distribution for all of the $\binom{p}{k}$ possible realizations of S, and that this distribution is permutation-invariant.

The measurement matrix $\mathbf{X}$ is assumed to have the expander-based distribution given in the previous subsection. We write $\mathbf{X}_i$ to denote the random i-th column; generically denoting one such column by $\mathbf{X}_0$, the corresponding distribution can be written as

$$P_{\mathbf{X}_0}(\mathbf{x}_0) = \frac{1}{\mu_n}\prod_{i=1}^{n} P_X(x_{0,i})\mathbb{1}\{N_1(\mathbf{x}_0) = d\}, \qquad (3)$$

where $P_X \sim \text{Bernoulli}\big(\frac{d}{n}\big)$, $N_1(\mathbf{x}_0)$ is the number of ones in $\mathbf{x}_0$, and μ_n is a normalizing constant. Since all binary vectors with d ones are equiprobable under this distribution, it simply amounts to a random permutation, as desired. We let $P_{\mathbf{X}_0}^{\ell}$ denote the ℓ-fold product of $P_{\mathbf{X}_0}$, so that

$$\mathbf{X} \sim P_{\mathbf{X}_0}^p. \qquad (4)$$

The i-th row of $\mathbf{X}$ is denoted by $X^{(i)}$.

Given S, $\mathbf{X}$, and β, each entry of the observation vector $\mathbf{Y}$ is generated in a conditionally independent manner, with the i-th entry $Y^{(i)}$ distributed according to

$$Y^{(i)} = \langle X^{(i)}, \beta\rangle + Z^{(i)}, \qquad (5)$$

where $Z^{(i)} \sim N(0,\sigma^2)$ is additive Gaussian noise. The corresponding conditional probability density function (PDF) is denoted by $P_{Y|X_S\beta_S}$, and we let $P_{Y|X_S\beta_S}^n(\cdot|\cdot,b_s)$ denote the n-fold product of $P_{Y|X_S\beta_S}(\cdot|\cdot,b_s)$. Note that here and subsequently, $P_{Y|X_S\beta_S}$ denotes the Gaussian conditional PDF, whereas $P_{\mathbf{X}_0}$ is a probability mass function (PMF). We allow β_S to be discrete or continuous, meaning P_{β_S} may be a PMF or PDF.

Given $\mathbf{X}$ and $\mathbf{Y}$, a *decoder* forms an estimate $\hat{S}$ of S. We assume that the decoder knows the system model (including k, P_{β_S} and $P_{Y|X_S\beta_S}$); that is, it knows the relevant probability distributions, but not the specific realizations. The error probability is given by

$$P_{\mathrm{e}} := \mathbb{P}[\hat{S} \neq S], \qquad (6)$$

and is taken over the realizations of S, β, $\mathbf{X}$, and $\mathbf{Y}$ (the decoder is assumed to be deterministic).

1.3 Joint Distributions and Properties

It will prove convenient to work with random variables that are implicitly conditioned on $S = s = \{1,\ldots,k\}$. We write P_{β_s} and $P_{Y|X_s\beta_s}$ in place of P_{β_S} and $P_{Y|X_S\beta_S}$ to emphasize that $S = s$, and we define

$$P_{\beta_s\mathbf{X}_s\mathbf{Y}}(b_s,\mathbf{x}_s,\mathbf{y}) := P_{\beta_s}(b_s)P_{\mathbf{X}_0}^k(\mathbf{x}_s)P_{Y|X_s\beta_s}^n(\mathbf{y}|\mathbf{x}_s,b_s). \qquad (7)$$

It will also prove useful to also introduce the following counterpart for matrices that are i.i.d. on $P_X \sim \text{Bernoulli}\big(\frac{d}{n}\big)$; letting $P_X^{n\times p}$ denote the corresponding i.i.d. matrix distribution, we define

$$\widetilde{P}_{\beta_s\mathbf{X}_s\mathbf{Y}}(b_s,\mathbf{x}_s,\mathbf{y}) := P_{\beta_s}(b_s)P_X^{n\times k}(\mathbf{x}_s)P_{Y|X_s\beta_s}^n(\mathbf{y}|\mathbf{x}_s,b_s). \qquad (8)$$

It is easy to show (e.g., see Appendix D) that any given symbol in a vector with distribution $P_{\mathbf{X}_0}$ (*cf.*, (3)) has distribution P_X. It follows that the marginal distribution corresponding to a single measurement is the same in both (7) and (8), and is given by

$$P_{\beta_s X_s Y}(b_s, x_s, y) := P_{\beta_s}(b_s)P_X^k(x_s)P_{Y|X_s\beta_s}(y|x_s,b_s). \qquad (9)$$

In accordance with the above definitions, we make use of the following random variables:

$$(\beta_s, X_s, Y) \sim P_{\beta_s X_s Y} \qquad (10)$$

$$(\beta_s, \mathbf{X}_s, \mathbf{Y}) \sim P_{\beta_s \mathbf{X}_s \mathbf{Y}}. \qquad (11)$$

A key idea that we will use in our analysis is a *change of measure* from the expander-based column distribution $P_{\mathbf{X}_0}$ to the i.i.d. Bernoulli distribution P_X^n. Formally, we have the following.

Lemma 2. *For any choices of n and d, we have for all sequences $\mathbf{x}_0 \in \{0,1\}^n$ that*

$$P_{\mathbf{X}_0}(\mathbf{x}_0) \leq (n+1)P_X^n(\mathbf{x}_0). \qquad (12)$$

Proof. This result will follow upon proving that $\mu_n \geq \frac{1}{n+1}$ in (3). By definition, μ_n is the probability that $N_1(\widetilde{\mathbf{X}}_0) = d$ under $\widetilde{\mathbf{X}}_0 \sim \prod_{i=1}^{n} P_X(x_{0,i})$. Since $N_1(\mathbf{x}_0)$ can only take one of $n + 1$ values, the inequality $\mu_n \geq \frac{1}{n+1}$ follows by noting that $N_1(\widetilde{\mathbf{X}}_0) \sim \text{Binomial}\big(n, \frac{d}{n}\big)$, and hence its most probable value is d. $\square$

Changes of measure of this type hold in greater generality beyond permutations of binary vectors [21, Ch. 2], and also beyond discrete measures (e.g., one can similarly change measure from uniform on a sphere to i.i.d. Gaussian [22]). Analyses based on such arguments are common in information-theoretic studies of

channel coding, but we are not aware of any previous works applying them to support recovery problems.

As in [16, 17, 23], we consider partitions of the support set $s \in \mathcal{S}$ into two sets $s_{\text{dif}} \neq \emptyset$ and s_{eq}; these can be thought of as corresponding to $s \backslash \bar{s}$ and $s \cap \bar{s}$ for some incorrect support set $\bar{s}$. For fixed $s \in \mathcal{S}$ and a corresponding pair $(s_{\text{dif}}, s_{\text{eq}})$, we introduce the notation

$$P_{Y|X_{s_{\text{dif}}} X_{s_{\text{eq}}} \beta_s}(y|x_{s_{\text{dif}}}, x_{s_{\text{eq}}}, b_s) := P_{Y|X_s \beta_s}(y|x_s, b_s) \tag{13}$$

$$P_{Y|X_{s_{\text{eq}}} \beta_s}(y|x_{s_{\text{eq}}}, b_s) := \sum_{x_{s_{\text{dif}}}} P_X^{\ell}(x_{s_{\text{dif}}})$$
$$\times P_{Y|X_{s_{\text{dif}}} X_{s_{\text{eq}}} \beta_s}(y|x_{s_{\text{dif}}}, x_{s_{\text{eq}}}, b_s), \tag{14}$$

where $\ell := |s_{\text{dif}}|$. The key quantities in our bounds are the following conditional mutual informations computed with respect to (10), (13) and (14):

$$I_{s_{\text{dif}}, s_{\text{eq}}}(b_s) := I(X_{s_{\text{dif}}}; Y|X_{s_{\text{eq}}}, \beta_s = b_s) \tag{15}$$

$$= I\left(\sum_{i \in s_{\text{dif}}} b_i X_i; \sum_{i \in s_{\text{dif}}} b_i X_i + Z \,\Big|\, \beta_s = b_s\right), \tag{16}$$

where (16) follows from (5) by noting that conditioning on $X_{s_{\text{eq}}}$ amounts to subtracting it from the signal.

2 Main Results

We provide two main results; the first considers a discrete distribution on the non-zero entries β_S, and the second considers a continuous distribution.

2.1 Discrete Non-zero Entries

We first consider the distribution of β_s used in [15] and [17, Sec. IV-B], where β_s is a uniformly random permutation of a fixed vector $b_s = (b_1, \ldots, b_k)$. We assume that k, $(b_1, \ldots, b_k)$, ρ and σ^2 are fixed, i.e., they do not scale with p.

Theorem 1. *Under the preceding setup with fixed values of k, $b_s = (b_1, \ldots, b_k)$, and σ^2, and using the random measurement matrix (4) with $\rho := \frac{d}{n} = \Theta(1)$, there exists a decoder such that $P_{\text{e}} \to 0$ as $p \to \infty$, provided that*

$$n \geq \max_{(s_{\text{dif}}, s_{\text{eq}}) : s_{\text{dif}} \neq \emptyset} \frac{|s_{\text{dif}}| \log p}{I_{s_{\text{dif}}, s_{\text{eq}}}(b_s)} (1+\eta) \quad \text{(Achievability).} \tag{17}$$

Conversely, any decoder satisfies $P_{\text{e}} \to 1$ as $p \to \infty$ whenever

$$n \leq \max_{(s_{\text{dif}}, s_{\text{eq}}) : s_{\text{dif}} \neq \emptyset} \frac{|s_{\text{dif}}| \log p}{I_{s_{\text{dif}}, s_{\text{eq}}}(b_s)} (1 - \eta) \quad \text{(Converse)} \tag{18}$$

for some $\eta > 0$.

Proof. See Section 4. □

Remark 1. Under i.i.d. Gaussian measurement matrices with $N(0, \sigma_X^2)$ entries, analogous bounds are known to hold with $I_{s_{\text{dif}}, s_{\text{eq}}}(b_s) = \frac{1}{2} \log \left(1 + \frac{\sigma_X^2}{\sigma^2} \sum_{i \in s_{\text{dif}}} b_i^2\right)$ [15, 17]. In order to equalize the signal power in the two cases, one should set $\sigma_X^2 = \rho$, the non-centralized second moment arising in the expander-based distribution. In Section 2.2 we analytically compare these results in the limits of small and large noise levels, and in Section 3 we numerically compare the bounds for a wide range of finite noise levels.

We emphasize that the converse result not only gives conditions under which $P_{\text{e}} \not\to 0$, but also under which $P_{\text{e}} \to 1$. The latter is stronger, not only stating the "some" realizations of the random matrix $\mathbf{X}$ "sometimes" lead to errors, but instead that "almost all" realizations "almost always" lead to errors.

The mutual informations in our bounds can easily be computed for fixed $(s_{\text{dif}}, s_{\text{eq}})$ by evaluating the relevant integrals corresponding to differential entropies. While the optimization over $(s_{\text{dif}}, s_{\text{eq}})$ may be difficult for large k, this can be potentially simplified by further bounding the mutual information (e.g., see [24]). Moreover, a valid converse can be obtained without performing the full optimization, and we found numerically that the maximum is usually achieved by one of the extreme cases $|s_{\text{dif}}| = 1$ or $|s_{\text{dif}}| = k$.

2.2 Low-SNR and High-SNR Asymptotics

Theorem 1 provides an exact threshold on the number of measurements in terms of the vector b_s, noise level σ^2 and parameter $\rho := \frac{d}{n}$. The key quantities are the mutual informations $I_{s_{\text{dif}}, s_{\text{eq}}}(b_s)$, which we can study using the form given in (16). The threshold in (17)–(18) in fact has the same form as that for i.i.d. Gaussian matrices [15, 17], except that the mutual information is computed with respect to a different distribution P_X. Since we have assumed Gaussian noise, we obtain from a well-known saddlepoint property of the mutual information [25, Ex. 9.21] that the mutual informations are strictly higher for Gaussian measurements, and thus fewer measurements are required.

We proceed by showing that the corresponding gap is "small" at low signal-to-noise ratios (SNRs), and "large" at high SNRs; see Section 3 for numerical evaluations supporting this. Here the SNR is the ratio of the signal power to the noise power, given by SNR $:= \frac{k\rho}{\sigma^2}$.

First consider the low-SNR regime, where $(b_1, \ldots, b_k)$ and ρ are fixed but $\sigma^2 \ll 1$. Using the formula $I(X; X+Z) = \frac{\text{Var}[X]}{2\sigma^2} + O\left(\frac{1}{\sigma^4}\right)$ for the low-SNR asymptotics of a fixed random variable X (e.g., see [26,

Eq. (50)]; we have an extra factor of $\frac{1}{2}$ since we work in $\mathbb{R}$ rather than $\mathbb{C}$), we obtain

$$I_{s_{\mathrm{dif}},s_{\mathrm{eq}}}(b_s) = \frac{1}{2\sigma^2}\rho(1-\rho)\sum_{i \in s_{\mathrm{dif}}} b_i^2 + O\left(\frac{1}{\sigma^4}\right), \quad (19)$$

since the variance of Bernoulli(ρ) is $\rho(1-\rho)$. Similarly, when the measurement matrix is i.i.d. on $P_X \sim N(0,\rho)$ (see Remark 1), we obtain (19) with $\rho(1-\rho)$ replaced by ρ. Thus, *for the same average SNR, the expander distribution only requires a factor of $\frac{1}{1-\rho}$ more measurements asymptotically.* This is typically very close to one, since we want ρ small to ensure that $\mathbf{X}$ is sparse. In fact, this gap can be removed altogether by letting the non-zero entries of each column be ± 1 with equal probability, rather than always being one.

In contrast, the behaviors of the two ensembles differ significantly at high SNRs. The easiest way to see this is by trivially upper bounding (15) by the entropy: $I_{s_{\mathrm{dif}},s_{\mathrm{eq}}}(b_s) \leq H\left(\sum_{i \in s_{\mathrm{dif}}} b_i X_i\right)$, which is uniformly bounded with respect to the SNR since k is fixed.[1] In contrast, using the Gaussian mutual information formula $I(X; X+Z) = \frac{1}{2}\log\left(1 + \mathrm{SNR}\right)$, we see that with Gaussian measurements each term $I_{s_{\mathrm{dif}},s_{\mathrm{eq}}}(b_s)$ grows logarithmically in the SNR, and can be arbitrarily large.

It was shown in [27] that under Gaussian measurements and the polynomial-time LASSO decoder, the required number of measurements tends to $2k\log p$ in the limit of high SNR. Thus, unlike the optimal decoder, the performance saturates for the LASSO when Gaussian measurements are used. We found numerically that this saturation point may be above or below that of the expander ensemble with optimal decoding, depending on the choice of ρ. Specifically, the LASSO with Gaussian measurements may outperform the optimal decoder with expander-based measurements if ρ is small, whereas the opposite may be true if ρ is large. It remains an open question as to how well the LASSO performs with expander-based measurements.

2.3 Continuous Non-zero Entries

We now turn to the case that β_s is i.i.d. on $N(0,\sigma_\beta^2)$ for some variance σ_β^2. As noted in [15], one cannot expect to achieve $P_{\mathrm{e}} \to 0$ in this case, since the non-zero entries can be arbitrarily small. Nevertheless, we can characterize the required number of measurements to achieve $P_{\mathrm{e}} \to \delta > 0$, as the following result shows.

Theorem 2. *Under the preceding setup for the linear model with fixed values of k and σ^2, a distribution*

Figure 1: Asymptotic number of measurements for various $\rho = \frac{d}{n}$, and for the i.i.d. Gaussian construction.

P_{β_s} *i.i.d. on $N(0,\sigma_\beta^2)$, and the random measurement matrix in* (4) *with $\rho := \frac{d}{n} = \Theta(1)$, the optimal decoder (i.e. the decoder minimizing P_{e} in the absence of computational limitations) satisfies*

$$P_e = \mathbb{P}\left[n \leq \max_{(s_{\mathrm{dif}},s_{\mathrm{eq}}):\,s_{\mathrm{dif}}\neq\emptyset} \frac{|s_{\mathrm{dif}}|\log p}{I_{s_{\mathrm{dif}},s_{\mathrm{eq}}}(\beta_s)}\right] + o(1). \quad (20)$$

Proof. The proof is similar to that of Theorem 1; the differences are outlined in Appendix C. $\square$

Observe that the threshold in the probability in (20) coincides with that in (17)–(18). Thus, this result provides a natural counterpart to Theorem 1 for the case that the values of the non-zero entries are not fixed in advance. Similarly to Remark 1, an analogous bound holds in the case of i.i.d. Gaussian measurements upon a suitable modification of the mutual information terms $I_{s_{\mathrm{dif}},s_{\mathrm{eq}}}(\cdot)$.

3 Numerical Evaluations and Comparisons

In this section, we numerically compare the exact thresholds in Theorem 1 with those of i.i.d. Gaussian measurements with the same power (having the same form but with $I_{s_{\mathrm{dif}},s_{\mathrm{eq}}}(b_s) = \frac{1}{2}\log\left(1 + \frac{\rho}{\sigma^2}\sum_{i \in s_{\mathrm{dif}}} b_i^2\right)$ [15, 17]). Note that we focus on the large-p asymptotics, replacing the arbitrarily small constant η in (17)–(18) by zero. For simplicity, we set $k = 15$ and $b_s = (b_0,\ldots,b_0)$ (i.e., all of the non-zero entries are equal); we observed similar behavior when this was replaced by a vector with half $+b_0$'s and half $-b_0$'s, and when different values of k were used.

Figure 1 plots the asymptotic thresholds on the number of measurements for various values of ρ, as a func-

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[1]For example, an upper bound independent of the SNR is obtained by upper bounding the entropy by the logarithm of the number of possible values.
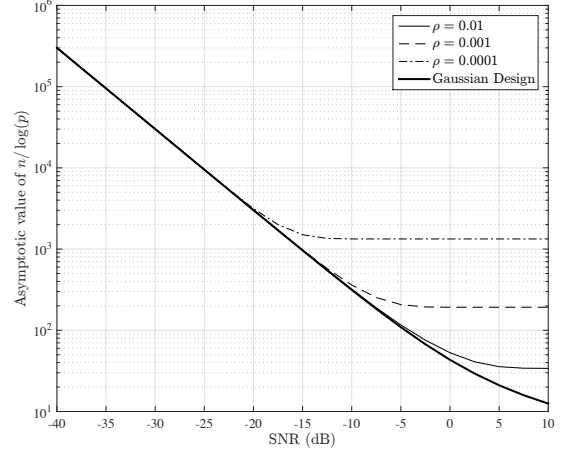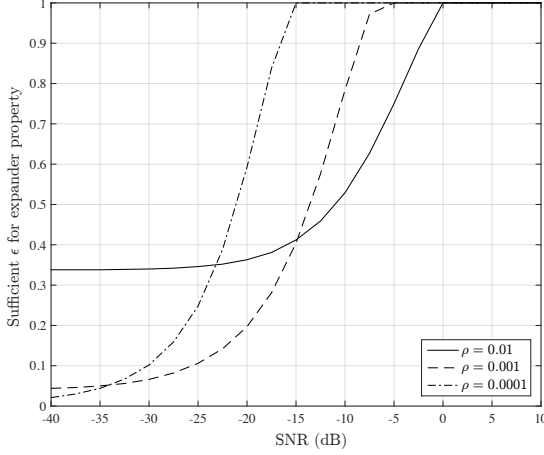
Figure 2: Infimum of ϵ such that (2) holds for some $\eta > 0$ when the number of measurements coincides with the threshold in Theorem 1.

tion of the per-sample SNR in dB:

$$\mathrm{SNR}_{\mathrm{dB}} := 10 \log \frac{k\rho}{\sigma^2}. \qquad (21)$$

Note that the Gaussian ensemble depends only on the SNR and not on ρ, and hence there is only one curve for this ensemble, despite there being three for the expander-based ensemble.

As predicted by our findings in Section 2.2, the gap to the performance of Gaussian measurements is insignificant at low SNRs, but grows at higher SNRs. In fact, each of the curves exhibits a rapid transition from the regime where the performance nearly matches the Gaussian curve to that where the number of measurements saturates. The main effect of increasing ρ is in lowering the final saturated value, whereas the behaviors at low SNR are all similar. In fact, although it is not visible at the scale shown, smaller values of ρ are slightly more favorable at low SNR, yielding a smaller multiplicative factor $\frac{1}{1-\rho}$ (see Section 2.2).

Next, we check in which cases the parameters yield expander matrices with high probability. Specifically, Figure 2 plots the infima of ϵ such that the sufficient condition (2) holds for some $\eta > 0$, under the values of the SNR and number of measurements shown in Figure 1. In all cases, the expander property holds with small-to-moderate values of ϵ when the SNR is not too high. In contrast, at high SNRs, the value of ϵ approaches one, meaning that the expander property fails to hold. Interestingly, there appears to be a close correspondence between the expansion property holding and the support recovery performance being similar to the Gaussian design. Note, however, that we are only plotting a sufficient condition in Figure 1, so smaller values of ϵ may be possible.

4 Proofs

Here we provide the main steps of the proof of Theorem 1, deferring several technical details to the appendices in the supplementary material.

4.1 Preliminary Definitions

Along with the definitions in Section 1.3, we introduce the joint distributions

$$P_{\mathbf{Y}|\mathbf{X}_{s_{\mathrm{dif}}}\mathbf{X}_{s_{\mathrm{eq}}}}(\mathbf{y}|\mathbf{x}_{s_{\mathrm{dif}}}, \mathbf{x}_{s_{\mathrm{eq}}}) := P_{\mathbf{Y}|\mathbf{X}_s}(\mathbf{y}|\mathbf{x}_s) \qquad (22)$$

$$P_{\mathbf{Y}|\mathbf{X}_{s_{\mathrm{eq}}}}(\mathbf{y}|\mathbf{x}_{s_{\mathrm{eq}}}) := \\ \sum_{\mathbf{x}_{s_{\mathrm{dif}}}} P_{\mathbf{X}_0}^{\ell}(\mathbf{x}_{s_{\mathrm{dif}}}) P_{\mathbf{Y}|\mathbf{X}_{s_{\mathrm{dif}}}\mathbf{X}_{s_{\mathrm{eq}}}}(\mathbf{y}|\mathbf{x}_{s_{\mathrm{dif}}}, \mathbf{x}_{s_{\mathrm{eq}}}) \qquad (23)$$

$$\widetilde{P}_{\mathbf{Y}|\mathbf{X}_{s_{\mathrm{eq}}}}(\mathbf{y}|\mathbf{x}_{s_{\mathrm{eq}}}) := \\ \sum_{\mathbf{x}_{s_{\mathrm{dif}}}} P_X^{n\times\ell}(\mathbf{x}_{s_{\mathrm{dif}}}) P_{\mathbf{Y}|\mathbf{X}_{s_{\mathrm{dif}}}\mathbf{X}_{s_{\mathrm{eq}}}}(\mathbf{y}|\mathbf{x}_{s_{\mathrm{dif}}}, \mathbf{x}_{s_{\mathrm{eq}}}), \qquad (24)$$

where $P_{\mathbf{Y}|\mathbf{X}_s}$ is the marginal distribution of (7). Moreover, we define

$$\tilde{\imath}(\mathbf{x}_{s_{\mathrm{dif}}}; \mathbf{y}|\mathbf{x}_{s_{\mathrm{eq}}}) := \log \frac{P_{\mathbf{Y}|\mathbf{X}_{s_{\mathrm{dif}}}\mathbf{X}_{s_{\mathrm{eq}}}}(\mathbf{y}|\mathbf{x}_{s_{\mathrm{dif}}}, \mathbf{x}_{s_{\mathrm{eq}}})}{\widetilde{P}_{\mathbf{Y}|\mathbf{X}_{s_{\mathrm{eq}}}}(\mathbf{y}|\mathbf{x}_{s_{\mathrm{eq}}})}, \qquad (25)$$

and similarly

$$\imath^n(\mathbf{x}_{s_{\mathrm{dif}}}; \mathbf{y}|\mathbf{x}_{s_{\mathrm{eq}}}, b_s) := \sum_{i=1}^n \imath(x_{s_{\mathrm{dif}}}^{(i)}; y^{(i)}|x_{s_{\mathrm{eq}}}^{(i)}, b_s) \qquad (26)$$

$$\imath(x_{s_{\mathrm{dif}}}; y|x_{s_{\mathrm{eq}}}, b_s) := \log \frac{P_{Y|X_{s_{\mathrm{dif}}}X_{s_{\mathrm{eq}}}\beta_s}(y|x_{s_{\mathrm{dif}}}, x_{s_{\mathrm{eq}}}, b_s)}{P_{Y|X_{s_{\mathrm{eq}}}\beta_s}(y|x_{s_{\mathrm{eq}}}, b_s)}. \qquad (27)$$

Quantities of this form are often referred to as *information densities* [17, 22]. We note, however, that the denominator in (25) contains the i.i.d.-based distribution $\widetilde{P}_{\mathbf{Y}|\mathbf{X}_{s_{\mathrm{eq}}}}$, as opposed to the true marginal distribution $P_{\mathbf{Y}|\mathbf{X}_{s_{\mathrm{eq}}}}$. The idea is that, when equipped with the change-of-measure result in Lemma 2, the former is easier to analyze, since it permits a reduction to a summation over the measurements as in (26). Such "modified" information densities have also appeared in the channel coding literature (e.g., see [28,29]) for handling non-i.i.d. random codebook constructions.

Observe that averaging (27) with respect to the random variables in (10) conditioned on $\beta_s = b_s$ yields the conditional mutual information in (15).

4.2 Non-asymptotic Bounds

We begin by providing non-asymptotic upper and lower bounds on the error probability in terms of tail probabilities of the above information densities.

Theorem 3. *For any constants $\delta_1 > 0$ and γ, there exists a decoder such that*

$$P_{\mathrm{e}} \le \mathbb{P}\left[\bigcup_{\substack{(s_{\mathrm{dif}}, s_{\mathrm{eq}}) \\ s_{\mathrm{dif}} \ne \emptyset}} \left\{ \imath^n(\mathbf{X}_{s_{\mathrm{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\mathrm{eq}}}, \beta_s) \le \log \binom{p-k}{|s_{\mathrm{dif}}|} \right. \right.$$
$$\left. \left. + 2 \log \left(\frac{k}{\delta_1} \binom{k}{|s_{\mathrm{dif}}|} (n+1)^{|s_{\mathrm{dif}}|} \right) + \gamma \right\} \right] + P_0(\gamma) + 2\delta_1, \tag{28}$$

where

$$P_0(\gamma) := \mathbb{P}\left[\log \frac{P_{\mathbf{Y}|\mathbf{X}_s, \beta_s}(\mathbf{Y}|\mathbf{X}_s, \beta_s)}{P_{\mathbf{Y}|\mathbf{X}_s}(\mathbf{Y}|\mathbf{X}_s)} > \gamma \right]. \tag{29}$$

Proof. The proof follows the approach of [17, Thm. 1], but uses the modified information density in (25), and performs additional steps involving change of measure from the random permutation distribution to an i.i.d. distribution. See Appendix B for details. □

Theorem 4. *Fix $\delta_1 > 0$, and let $(s_{\mathrm{dif}}(b_s), s_{\mathrm{eq}}(b_s))$ be an arbitrary partition of $s = \{1, \dots, k\}$ (with $s_{\mathrm{dif}} \ne \emptyset$) depending on $b_s \in \mathbb{R}^k$. For any decoder, we have*

$$P_{\mathrm{e}} \ge \mathbb{P}\left[\imath^n(\mathbf{X}_{s_{\mathrm{dif}}(\beta_s)}; \mathbf{Y} | \mathbf{X}_{s_{\mathrm{eq}}(\beta_s)}, \beta_s) \right.$$
$$\left. \le \log \binom{p-k+|s_{\mathrm{dif}}(\beta_s)|}{|s_{\mathrm{dif}}(\beta_s)|} + \log \delta_1 \right] - \delta_1. \tag{30}$$

Proof. The proof is identical to that of [17, Thm. 2]; while an i.i.d. measurement matrix distribution was used therein, an inspection of the proof reveals that having i.i.d. columns is enough. □

While the preceding theorems resemble their counterparts for i.i.d. measurement matrices [17], there are two notable differences. First, $\imath^n$ is no longer an i.i.d. summation in the present setting, and it is thus more difficult to characterize the corresponding deviations from the mean. Moreover, the change of measure in Lemma 2 plays a key role in the proof of Theorem 3, and leads to the additional factor $(n+1)^{|s_{\mathrm{dif}}|}$ in (28).

4.3 Simplifications of Theorems 3 and 4

Next, we reduce the preceding theorems to a form that is more amenable to inferring bounds on the required number of measurements. We start with the achievability part:

1. Observe that, conditioned on $\beta_s = b_s$, the mean of $\imath^n(\mathbf{X}_{s_{\mathrm{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\mathrm{eq}}}, \beta_s)$ is $n I_{s_{\mathrm{dif}}, s_{\mathrm{eq}}}(b_s)$ (*cf.*, (15)); this is due to to the fact the marginal distribution corresponding to any single measurement is given by (9).

2. Fix $\delta_2 \in (0, 1)$ and $\kappa > 0$, and suppose that for a fixed value b_s of β_s, we have for all $(s_{\mathrm{dif}}, s_{\mathrm{eq}})$ that

$$n \ge \frac{\log \binom{p-k}{|s_{\mathrm{dif}}|} + 2 \log \left(\frac{k}{\delta_1} \binom{k}{|s_{\mathrm{dif}}|} (n+1)^{|s_{\mathrm{dif}}|} \right) + \gamma}{I_{s_{\mathrm{dif}}, s_{\mathrm{eq}}}(b_s)(1 - \delta_2)} \tag{31}$$

$$\frac{\mathrm{Var}\left[\imath^n(\mathbf{X}_{s_{\mathrm{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\mathrm{eq}}}, \beta_s) \mid \beta_s = b_s \right]}{I_{s_{\mathrm{dif}}, s_{\mathrm{eq}}}(b_s)^2} \le n\kappa. \tag{32}$$

Combining these conditions with the previous step, the union bound, and Chebyshev's inequality, we obtain

$$\mathbb{P}\left[\bigcup_{\substack{(s_{\mathrm{dif}}, s_{\mathrm{eq}}) \\ s_{\mathrm{dif}} \ne \emptyset}} \left\{ \imath^n(\mathbf{X}_{s_{\mathrm{dif}}}; \mathbf{Y} | \mathbf{X}_{s_{\mathrm{eq}}}, \beta_s) \le \log \binom{p-k}{|s_{\mathrm{dif}}|} \right. \right.$$
$$\left. \left. + 2 \log \left(\frac{k}{\delta_1} \binom{k}{|s_{\mathrm{dif}}|} (n+1)^{|s_{\mathrm{dif}}|} \right) + \gamma \right\} \, \middle| \, \beta_s = b_s \right]$$
$$\le (2^k - 1) \frac{\kappa}{n\delta_2^2}, \tag{33}$$

since there are exactly $2^k - 1$ ways of splitting s into $(s_{\mathrm{dif}}, s_{\mathrm{eq}})$ with $s_{\mathrm{dif}} \ne \emptyset$.

3. Combining the above observations gives

$$P_{\mathrm{e}} \le \mathbb{P}\left[\beta_s \notin \mathcal{B}(\delta_1, \delta_2, \gamma) \right] + (2^k - 1) \frac{\kappa}{n\delta_2^2} + P_0(\gamma) + 2\delta_1, \tag{34}$$

where

$$\mathcal{B}(\delta_1, \delta_2, \gamma) := \left\{ b_s : (31) \text{ and } (32) \text{ hold for all} \right.$$
$$\left. (s_{\mathrm{dif}}, s_{\mathrm{eq}}) \text{ with } s_{\mathrm{dif}} \ne \emptyset \right\}. \tag{35}$$

The simplification of Theorem 4 is done using similar steps. Fix $\delta_2 > 0$, and suppose that, for a fixed value b_s of β_s, the pair $(s_{\mathrm{dif}}, s_{\mathrm{eq}}) = (s_{\mathrm{dif}}(b_s), s_{\mathrm{eq}}(b_s))$ is such that

$$n \le \frac{\log \binom{p-k+|s_{\mathrm{dif}}|}{|s_{\mathrm{dif}}|} - \log \delta_1}{I_{s_{\mathrm{dif}}, s_{\mathrm{eq}}}(b_s)(1 + \delta_2)} \tag{36}$$

and (32) holds. Combining these with Chebyshev's inequality, we find that the first probability in (30), with an added conditioning on $\beta_s = b_s$, is lower bounded by $1 - \frac{\kappa}{n\delta_2^2}$. Recalling that the partition $(s_{\mathrm{dif}}, s_{\mathrm{eq}})$ is an arbitrary function of β_s, we can ensure that (36) coincides with

$$n \le \max_{(s_{\mathrm{dif}}, s_{\mathrm{eq}}) : s_{\mathrm{dif}} \ne \emptyset} \frac{\log \binom{p-k+|s_{\mathrm{dif}}|}{|s_{\mathrm{dif}}|} - \log \delta_1}{I_{s_{\mathrm{dif}}, s_{\mathrm{eq}}}(b_s)(1 + \delta_2)} \tag{37}$$

by choosing each pair $(s_{\mathrm{dif}}, s_{\mathrm{eq}})$ as a function of b_s to achieve this maximum. Combining these observations, we obtain

$$P_{\mathrm{e}} \ge \mathbb{P}\left[\beta_s \in \mathcal{B}'(\delta_1, \delta_2) \right] \left(1 - \frac{\kappa}{n\delta_2^2} \right) - \delta_1, \tag{38}$$

where

$$\mathcal{B}'(\delta_1, \delta_2) := \big\{ b_s : (32) \text{ and } (36) \text{ hold for all} \\ (s_{\mathrm{dif}}, s_{\mathrm{eq}}) \text{ with } s_{\mathrm{dif}} \neq \emptyset \big\}. \quad (39)$$

4.4 Bounding the Variance

In order to apply (35) and (39), we must characterize the variance appearing in (32). Under the setting of i.i.d. measurement matrices studied in [17], the variance of $\imath^n(\mathbf{X}_{s_{\mathrm{dif}}}; \mathbf{Y}|\mathbf{X}_{s_{\mathrm{eq}}}, b_s)$ simplifies to n times the variance of a single-letter information density $\imath(X_{s_{\mathrm{dif}}}; Y|X_{s_{\mathrm{eq}}}, b_s)$. This is not the case in the present setting, as the columns of $\mathbf{X}_s$ are generated according to the non-i.i.d. distribution $P_{\mathbf{X}_0}$. However, we can still characterize each such variance accurately, as shown in the following.

Lemma 3. *For any fixed k, σ^2, $b_s = (b_1, \ldots, b_k)$ and $(s_{\mathrm{dif}}, s_{\mathrm{eq}})$, we have*

$$\mathrm{Var}\big[\imath^n(\mathbf{X}_{s_{\mathrm{dif}}}; \mathbf{Y}|\mathbf{X}_{s_{\mathrm{eq}}}, \beta_s) \,|\, \beta_s = b_s\big] \\ = nV_{s_{\mathrm{dif}}, s_{\mathrm{eq}}}(b_s) + O(1), \quad (40)$$

where

$$V_{s_{\mathrm{dif}}, s_{\mathrm{eq}}}(b_s) := \mathrm{Var}\big[\imath(X_{s_{\mathrm{dif}}}; Y|X_{s_{\mathrm{eq}}}, \beta_s) \,|\, \beta_s = b_s\big] \\ - \sum_{i \in s} \mathrm{Var}\big[\mathbb{E}[\imath(X_{s_{\mathrm{dif}}}; Y|X_{s_{\mathrm{eq}}}, \beta_s) \,|\, X_{s \setminus \{i\}}] \,|\, \beta_s = b_s\big]. \quad (41)$$

Proof. This follows by writing the variance as a sum of covariance terms, and then characterizing the joint distribution between two different columns of the measurement matrix $\mathbf{X}$. See Appendix D for details. $\square$

It follows immediately from Lemma 3 that (32) holds for all b_s within an arbitrary set $\mathcal{B}_0$ upon setting $\kappa = \sup_{(s_{\mathrm{dif}}, s_{\mathrm{eq}}), b_s \in \mathcal{B}_0} \frac{V_{s_{\mathrm{dif}}, s_{\mathrm{eq}}}(b_s)}{I_{s_{\mathrm{dif}}, s_{\mathrm{eq}}}(b_s)^2}(1 + o(1))$. Note however, that in order to ensure that κ is finite, one should avoid letting $V_{s_{\mathrm{dif}}, s_{\mathrm{eq}}}(b_s) = \infty$ or $I_{s_{\mathrm{dif}}, s_{\mathrm{eq}}}(b_s) = 0$; while this will not be an issue for Theorem 1, we will choose $\mathcal{B}_0$ accordingly in the proof of Theorem 2 in Appendix C.

4.5 Proof of Theorem 1

Theorem 1 follows without difficulty from the preceding two subsections. For the achievability part, we first observe from (29) that

$$\gamma = \log \frac{1}{\min_{b_s} P_{\beta_s}(b_s)} \implies P_0(\gamma) = 0, \quad (42)$$

which follows since $P_{\mathbf{Y}|\mathbf{X}_s}(\mathbf{y}|\mathbf{x}_s) = \sum_{b_s} P_{\beta_s}(b_s) P_{\mathbf{Y}|\mathbf{X}_s, \beta_s}(\mathbf{y}|\mathbf{x}_s, b_s)$. Since k and

$(b_1, \ldots, b_k)$ are fixed (not scaling with p) in the theorem statement, we have $\gamma = \Theta(1)$. Similarly, and since $\rho = \Theta(1)$ by assumption, we have $I_{s_{\mathrm{dif}}, s_{\mathrm{eq}}}(b_s) = \Theta(1)$ for all $(s_{\mathrm{dif}}, s_{\mathrm{eq}})$. Moreover, since P_{β_s} is simply a random permutation, the variance in (32) is the same for all b_s on its support, and behaves as $O(n)$ by Lemma 3. This implies that $\kappa = \Theta(1)$ in (32). Hence, and by letting δ_1 vanish slowly as a function of p, we see that the second, third and fourth terms in (34) vanish provided that $\delta_2 \to 0$ sufficiently slowly.

The first term in (34) also vanishes provided that n satisfies (31). By Stirling's approximation and the fact that $\delta_2 \to 0$, we can readily simplify this to

$$n \geq \frac{|s_{\mathrm{dif}}| \log p + 2|s_{\mathrm{dif}}| \log n}{I_{s_{\mathrm{dif}}, s_{\mathrm{eq}}}(b_s)}(1 + o(1)). \quad (43)$$

Considering without loss of generality the case that (17) holds with equality, we see that (43) is indeed satisfied for sufficiently large p, since $\log n = O(\log \log p) = o(\log p)$.

For the converse part, we may also assume without loss of generality that (18) holds with equality, as the decoder can always choose to ignore measurements. We obtain the desired result be applying similar (yet simpler) steps to those above starting from (38).

5 Conclusion

We have provided exact thresholds on the required number of measurements for support recovery via linear sketching, using random constructions known to produce expander matrices with high probability. A key tool in our analysis was a change-of-measure technique, which may prove useful in handling other random constructions. We obtained bounds nearly matching those of Gaussian measurements at low SNRs, while showing that the gap between the two is significant at high SNRs. An important extension of this work is handling the case that k scales with p, which appears to require more sophisticated concentration inequalities in place of Chebyshev's inequality. Unfortunately, Bernstein's inequality (used in [17]) does not appear to be suitable, since we need to deal with sums of *non-independent* random variables. Finally, our results motivate the study of practical decoding methods for obtaining the optimal thresholds, or characterizing how the thresholds change when one switches to practical decoders such as the LASSO.

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