Supplementary Materials for "Tractable and Scalable Schatten Quasi-Norm Approximations for Rank Minimization"

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In this supplementary material, we give the detailed proofs of some lemmas, properties and theorems, as well as some additional experimental results on synthetic data and four recommendation system data sets.

A More Notations

$\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and the set of all $m\times n$ matrices with real entries is denoted by $\mathbb{R}^{m\times n}$. Given matrices X and $Y\in\mathbb{R}^{m\times n}$, the inner product is defined by $\langle X,Y\rangle := \mathrm{Tr}(X^TY)$, where $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix. $\|X\|_2$ is the spectral norm and is equal to the maximum singular value of X. I denotes an identity matrix.

For any vector $x\in\mathbb{R}^n$, its ℓ_p quasi-norm for $0<p<1$ is defined as

$$\|x\|_p=\left(\sum_i |x_i|^p\right)^{1/p}.$$

In addition, the ℓ_1-norm and the ℓ_2-norm of x are $\|x\|_1=\sum_i|x_i|$ and $\|x\|_2=\sqrt{\sum_i x_i^2}$, respectively.

For any matrix $X\in\mathbb{R}^{m\times n}$, we assume the singular values of X are ordered as $\sigma_1(X)\geq\sigma_2(X)\geq\cdots\geq\sigma_r(X)>\sigma_{r+1}(X)=\cdots=\sigma_{\min(m,n)}(X)=0$, where $r=\mathrm{rank}(X)$. By writing $X=U\Sigma V^T$ in its standard singular value decomposition (SVD), we can extend $X=U\Sigma V^T$ to the following definitions.

The Schatten-p quasi-norm ($0<p<1$) of a matrix $X\in\mathbb{R}^{m\times n}$ is defined as follows:

$$\|X\|_{S_p}=\left(\sum_{i=1}^{\min(m,n)}(\sigma_i(X))^p\right)^{1/p}.$$

The trace norm (also called the nuclear norm or the Schatten-1 norm) of X is defined as

$$\|X\|_{\mathrm{tr}}=\sum_{i=1}^{\min(m,n)}\sigma_i(X).$$

The Frobenius norm (also called the Schatten-2 norm) of X is defined as

$$\|X\|_F=\sqrt{\mathrm{Tr}\left(X^TX\right)}=\sqrt{\sum_{i=1}^{\min(m,n)}\sigma_i^2(X)}.$$

B Proof of Theorem 1

In the following, we will first prove that the bi-trace norm $\|\cdot\|_{\text{Bi-tr}}$ is a quasi-norm.

Proof. By the definition of the bi-trace norm, for any a, a_1, $a_2\in\mathbb{R}$ and $a=a_1a_2$, we have

$$\begin{aligned}\|aX\|_{\text{Bi-tr}}&=\min_{aX=(a_1U)(a_2V^T)}\|a_1U\|_{\mathrm{tr}}\|a_2V\|_{\mathrm{tr}}\\&=\min_{X=UV^T}|a|\,\|U\|_{\mathrm{tr}}\|V\|_{\mathrm{tr}}\\&=|a|\min_{X=UV^T}\|U\|_{\mathrm{tr}}\|V\|_{\mathrm{tr}}\\&=|a|\,\|X\|_{\text{Bi-tr}}.\end{aligned}$$

By Lemma 1 of the main paper, i.e., $\|X\|_{\mathrm{tr}}=\min_{X=UV^T}\|U\|_F\|V\|_F$, and Lemma 6 in [1], there exist both matrices $\widehat{U}=U_{(d)}\Sigma_{(d)}^{1/2}$ and $\widehat{V}=V_{(d)}\Sigma_{(d)}^{1/2}$ (which are constructed in the same way as $U_\star$ and $V_\star$ in Section 5.1) such that $\|X\|_{\mathrm{tr}}=\|\widehat{U}\|_F\|\widehat{V}\|_F$ with the SVD of X, i.e., $X=U\Sigma V^T$. By the fact that $\|X\|_{\mathrm{tr}}\leq\sqrt{\mathrm{rank}(X)}\|X\|_F$, we have

$$\begin{aligned}\|X\|_{\text{Bi-tr}}&=\min_{X=UV^T}\|U\|_{\mathrm{tr}}\|V\|_{\mathrm{tr}}\\&\leq\|\widehat{U}\|_{\mathrm{tr}}\|\widehat{V}\|_{\mathrm{tr}}\\&\leq\sqrt{\mathrm{rank}(X)}\sqrt{\mathrm{rank}(X)}\|\widehat{U}\|_F\|\widehat{V}\|_F\\&\leq\mathrm{rank}(X)\|X\|_{\mathrm{tr}}.\end{aligned}$$

If $X\neq 0$, then $\mathrm{rank}(X)\geq 1$. On the other hand, we also have

$$\|X\|_{\mathrm{tr}}\leq\|X\|_{\text{Bi-tr}}.$$

By the above properties, there exists a constant $\alpha \geq 1$ such that the following holds for all $X, Y \in \mathbb{R}^{m\times n}$

$$\begin{aligned}\|X+Y\|_{\text{Bi-tr}} &\leq \alpha\|X+Y\|_{\text{tr}}\\ &\leq \alpha(\|X\|_{\text{tr}}+\|Y\|_{\text{tr}})\\ &\leq \alpha(\|X\|_{\text{Bi-tr}}+\|Y\|_{\text{Bi-tr}}).\end{aligned}$$

$\forall X \in \mathbb{R}^{m\times n}$ and $X = UV^T$, we have

$$\|X\|_{\text{Bi-tr}} = \min_{X=UV^T} \|U\|_{\text{tr}}\|V\|_{\text{tr}} \geq 0.$$

Moreover, if $\|X\|_{\text{Bi-tr}} = 0$, we have $\|X\|_{\text{tr}} \leq \|X\|_{\text{Bi-tr}} = 0$, i.e., $\|X\|_{\text{tr}} = 0$. Hence, $X = 0$. In short, the bi-trace norm $\|\cdot\|_{\text{Bi-tr}}$ is a quasi-norm. □

Before giving a complete proof for Theorem 1, we first present and prove the following lemmas.

Lemma 2 (Jensen's inequality)**.** *Assume that the function $g: \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous concave function on $[0, +\infty)$. Then, for any $t_i \geq 0$, $\sum_i t_i = 1$, and any $x_i \in \mathbb{R}^+$ for $i = 1, \ldots, n$, we have*

$$g\left(\sum_{i=1}^{n} t_i x_i\right) \geq \sum_{i=1}^{n} t_i g(x_i). \tag{16}$$

Lemma 3. *Suppose $Z \in \mathbb{R}^{m\times n}$ is of rank $r \leq \min(m, n)$, and denote its SVD by $Z = L\Sigma_Z R^T$, where $L \in \mathbb{R}^{m\times r}$, $R \in \mathbb{R}^{n\times r}$ and $\Sigma_Z \in \mathbb{R}^{r\times r}$. For any unitary matrix A satisfying $AA^T = A^T A = I_{r\times r}$, then $(A\Sigma_Z A^T)_{k,k} \geq 0$ for any $k = 1, \ldots, r$, and*

$$\text{Tr}^{1/2}(A\Sigma_Z A^T) \geq \text{Tr}^{1/2}(\Sigma_Z) = \|Z\|_{S_{1/2}}^{1/2},$$

where $\text{Tr}^{1/2}(B) = \sum_i B_{ii}^{1/2}$.

Proof. For any $k \in \{1, \ldots, r\}$, we have

$$(A\Sigma_Z A^T)_{k,k} = \sum_i a_{ki}^2 \sigma_i \geq 0,$$

where $\sigma_i \geq 0$ is the i-th singular value of Z. Then

$$\text{Tr}^{1/2}(A\Sigma_Z A^T) = \sum_k (\sum_i a_{ki}^2 \sigma_i)^{1/2}. \tag{17}$$

Let $t_i \geq 0$, $\sum_i t_i = 1$, and $x_i \geq 0$. Since the function $g(x) = x^{1/2}$ is concave on $\mathbb{R}^+$, and by Lemma 2, we have

$$g\left(\sum_i t_i x_i\right) \geq \sum_i t_i g(x_i).$$

Due to the fact that $\sum_i a_{ki}^2 = 1$ for any $k \in \{1, \ldots, r\}$, we have

$$\left(\sum_i a_{ki}^2 \sigma_i\right)^{1/2} \geq \sum_i a_{ki}^2 \sigma_i^{1/2}.$$

According to the above inequality and the fact that $\sum_k a_{ki}^2 = 1$ for any $i \in \{1, \ldots, r\}$, (17) can be written as follows:

$$\begin{aligned}\text{Tr}^{1/2}(A\Sigma_Z A^T) &= \sum_k (\sum_i a_{ki}^2 \sigma_i)^{1/2}\\ &\geq \sum_k \sum_i a_{ki}^2 \sigma_i^{1/2}\\ &= \sum_i \sigma_i^{1/2}\\ &= \text{Tr}^{1/2}(\Sigma_Z) = \|Z\|_{S_{1/2}}^{1/2}.\end{aligned}$$

This completes the proof. □

Proof of Theorem 1:

Proof. Assume that $U=L_U\Sigma_U R_U^T$ and $V=L_V\Sigma_V R_V^T$ are the thin SVDs of U and V, respectively, where $L_U\in\mathbb{R}^{m\times d}$, $L_V\in\mathbb{R}^{n\times d}$, and $R_U,\Sigma_U,R_V,\Sigma_V\in\mathbb{R}^{d\times d}$. Without loss of generality, we set $X=L_X\Sigma_X R_X^T$, where the columns of $L_X\in\mathbb{R}^{m\times d}$ and $R_X\in\mathbb{R}^{n\times d}$ are the left and right singular vectors associated with the top d singular values of X with rank at most r $(r\leq d)$, and $\Sigma_X=\textrm{diag}([\sigma_1(X),\cdots,\sigma_r(X),0,\cdots,0])\in\mathbb{R}^{d\times d}$.

By $X=UV^T$, which means that $L_X\Sigma_X R_X^T=L_U\Sigma_U R_U^T R_V\Sigma_V L_V^T$, then $\exists O_1,\widehat{O}_1\in\mathbb{R}^{d\times d}$ satisfy $L_X=L_U O_1$ and $L_U=L_X\widehat{O}_1$. Since $O_1=L_U^T L_X$ and $\widehat{O}_1=L_X^T L_U$, then $O_1^T=\widehat{O}_1$. Indeed, since $L_X=L_U O_1=L_X\widehat{O}_1 O_1$, we immediately have $\widehat{O}_1 O_1=O_1^T O_1=I_{d\times d}$. In addition, we obviously have $O_1\widehat{O}_1=O_1 O_1^T=I_{d\times d}$. Similarly, $\exists O_2\in\mathbb{R}^{d\times d}$ satisfying $O_2 O_2^T=O_2^T O_2=I_{d\times d}$ such that $R_X=L_V O_2$. Therefore, we have

$$\Sigma_X O_2^T=\Sigma_X R_X^T L_V=L_X^T L_U\Sigma_U R_U^T R_V\Sigma_V=O_1^T\Sigma_U R_U^T R_V\Sigma_V.$$

Let $O_3=O_2 O_1^T\in\mathbb{R}^{d\times d}$, then we have $O_3 O_3^T=O_3^T O_3=I_{d\times d}$, i.e., $\sum_i (O_3)_{ij}^2=\sum_j (O_3)_{ij}^2=1$ for $\forall i,j\in\{1,2,\ldots,d\}$, where $a_{i,j}$ denotes the element of the matrix A in the i-th row and the j-th column. Furthermore, let $O_4=R_U^T R_V$, we have $\sum_i (O_4)_{ij}^2\leq 1$ and $\sum_j (O_4)_{ij}^2\leq 1$ for $\forall i,j\in\{1,2,\ldots,d\}$.

By the above analysis, then we have $O_2\Sigma_X O_2^T=O_2 O_1^T\Sigma_U O_4\Sigma_V=O_3\Sigma_U O_4\Sigma_V$. Let ϱ_i and τ_j denote the i-th and the j-th diagonal elements of Σ_V and Σ_U, respectively. By Lemma 3, we can derive that

$$\begin{aligned}
\|X\|_{S_{1/2}} &\leq \left(\textrm{Tr}^{1/2}(O_2\Sigma_X O_2^T)\right)^2=\left(\textrm{Tr}^{1/2}(O_2 O_1^T\Sigma_U O_4\Sigma_V)\right)^2=\left(\textrm{Tr}^{1/2}(O_3\Sigma_U O_4\Sigma_V)\right)^2\\
&=\left(\sum_{i=1}^{d}\sqrt{\sum_{j=1}^{d}\tau_j (O_3)_{ij}(O_4)_{ji}\varrho_i}\right)^2=\left(\sum_{i=1}^{d}\sqrt{\varrho_i\sum_{j=1}^{d}\tau_j (O_3)_{ij}(O_4)_{ji}}\right)^2\\
&\stackrel{a}{\leq}\sum_{i=1}^{d}\varrho_i\sum_{i=1}^{d}\sum_{j=1}^{d}(\tau_j (O_3)_{ij}(O_4)_{ji})\\
&\stackrel{b}{\leq}\sum_{i=1}^{d}\varrho_i\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{((O_3)_{ij}^2\tau_j+(O_4)_{ji}^2\tau_j)}{2}\\
&\stackrel{c}{\leq}\sum_{i=1}^{d}\varrho_i\sum_{j=1}^{d}\tau_j\\
&=\|U\|_{\textrm{tr}}\|V\|_{\textrm{tr}}\leq\left(\frac{\|U\|_{\textrm{tr}}+\|V\|_{\textrm{tr}}}{2}\right)^2,
\end{aligned}$$

where the inequality $\stackrel{a}{\leq}$ holds due to the Cauchy−Schwartz inequality, the inequality $\stackrel{b}{\leq}$ follows from the basic inequality $xy\leq\frac{x^2+y^2}{2}$ for any real numbers x and y, and the inequality $\stackrel{c}{\leq}$ relies on the fact that $\sum_i (O_3)_{ij}^2=1$ and $\sum_i (O_4)_{ji}^2\leq 1$. Thus, we obtain

$$\|X\|_{S_{1/2}}\leq\|U\|_{\textrm{tr}}\|V\|_{\textrm{tr}}\leq\left(\frac{\|U\|_{\textrm{tr}}+\|V\|_{\textrm{tr}}}{2}\right)^2 \textrm{ and } \|X\|_{S_{1/2}}\leq\|U\|_{\textrm{tr}}\|V\|_{\textrm{tr}}\leq\frac{\|U\|_{\textrm{tr}}^2+\|V\|_{\textrm{tr}}^2}{2}.$$

On the other hand, set $U_\star=L_X\Sigma_X^{1/2}$ and $V_\star=R_X\Sigma_X^{1/2}$, then we have $X=U_\star V_\star^T$ and

$$\begin{aligned}
\|X\|_{S_{1/2}}&=[\textrm{Tr}^{1/2}(\Sigma_X)]^2=\|L_X\Sigma_X^{1/2}\|_{\textrm{tr}}\|R_X\Sigma_X^{1/2}\|_{\textrm{tr}}=\|U_\star\|_{\textrm{tr}}\|V_\star\|_{\textrm{tr}}=\frac{\|L_X\Sigma_X^{1/2}\|_{\textrm{tr}}^2+\|R_X\Sigma_X^{1/2}\|_{\textrm{tr}}^2}{2}\\
&=\frac{\|U_\star\|_{\textrm{tr}}^2+\|V_\star\|_{\textrm{tr}}^2}{2}=\left(\frac{\|L_X\Sigma_X^{1/2}\|_{\textrm{tr}}+\|R_X\Sigma_X^{1/2}\|_{\textrm{tr}}}{2}\right)^2=\left(\frac{\|U_\star\|_{\textrm{tr}}+\|V_\star\|_{\textrm{tr}}}{2}\right)^2.
\end{aligned}$$

Therefore, under the constraint $X=UV^T$, we have

$$\|X\|_{S_{1/2}}=\min_{X=UV^T}\left(\frac{\|U\|_{\textrm{tr}}+\|V\|_{\textrm{tr}}}{2}\right)^2=\min_{X=UV^T}\frac{\|U\|_{\textrm{tr}}^2+\|V\|_{\textrm{tr}}^2}{2}=\min_{X=UV^T}\|U\|_{\textrm{tr}}\|V\|_{\textrm{tr}}=\|X\|_{\textrm{Bi-tr}}.$$

This completes the proof. ◻

C Proof of Theorem 3

Before giving the proof of Theorem 3, we first prove the boundedness of multipliers and some variables of Algorithm 1. To prove the boundedness, we first give the following lemmas.

Lemma 4 ([2])**.** *Let $\mathcal{H}$ be a real Hilbert space endowed with an inner product $\langle\cdot,\cdot\rangle$ and a corresponding norm $\|\cdot\|$, and $y \in \partial\|x\|$, where $\partial f(x)$ denotes the subgradient of $f(x)$. Then $\|y\|^{*}=1$ if $x\neq 0$, and $\|y\|^{*}\leq 1$ if $x=0$, where $\|\cdot\|^{*}$ is the dual norm of $\|\cdot\|$. For instance, the dual norm of the trace norm is the spectral norm, $\|\cdot\|_2$, i.e., the largest singular value.*

Lemma 5 ([3])**.** *Assume that ∇g is Lipschitz continuous on $\mathrm{dom}(g):=\{X|g(X)<\infty\}$ satisfying the following condition: $\|\nabla g(X)-\nabla g(Y)\|_F\leq L_g\|X-Y\|_F$, $\forall X,Y\in\mathrm{dom}(g)$, with a Lipschitz constant L_g. Then*

$$g(X)\leq g(Y)+\langle\nabla g(Y),\, X-Y\rangle+\frac{L_g}{2}\|X-Y\|_F^2,\;\;\forall X,Y\in\mathrm{dom}(g).$$

Lemma 6. *Let $\lambda_{k+1}=\lambda_k+\beta_k(\mathcal{A}(U_{k+1}V_{k+1}^T)-b-e_{k+1})$, then the sequences $\{U_k,V_k\}$, $\{e_k\}$ and $\{\lambda_k\}$ produced by Algorithm 1 are all bounded.*

Proof. By the first-order optimality condition of the Lagrangian function $\mathcal{L}(U,V,e,\lambda,\beta)$ with respect to e, we have

$$0\in\partial_e\mathcal{L}(U_{k+1},V_{k+1},e,\lambda_k,\beta_k),$$

which equivalently states that

$$\beta_k\big(\mathcal{A}(U_{k+1}V_{k+1}^T)-e_{k+1}-b\big)+\lambda_k\in\frac{1}{\mu}\partial\|e_{k+1}\|_1.$$

Recalling $\lambda_{k+1}=\lambda_k+\beta_k(\mathcal{A}(U_{k+1}V_{k+1}^T)-b-e_{k+1})$, we have $\lambda_{k+1}\in\frac{1}{\mu}\partial\|e_{k+1}\|_1$. By Lemma 4, we have

$$\|\lambda_{k+1}\|_\infty\leq\frac{1}{\mu},$$

where $\|\cdot\|_\infty$ is the dual norm of $\|\cdot\|_1$. Thus, the sequence $\{\lambda_k\}$ is bounded.

By the definition of the linearization function $\widehat{\varphi}_k(U,U_k)$ in (8), we have $\widehat{\varphi}_k(U_k,U_k)=\varphi_k(U_k)$. Since U_{k+1} is the optimal solution of (9), and by Lemma 5, then we can derive that

$$\begin{aligned}&\mathcal{L}(U_{k+1},V_k,e_k,\lambda_k,\beta_k)\\=&\frac{1}{2}\|U_{k+1}\|_{\mathrm{tr}}+\frac{\beta_k}{2}\varphi_k(U_{k+1})+c\\\leq&\frac{1}{2}\|U_{k+1}\|_{\mathrm{tr}}+\frac{\beta_k}{2}\widehat{\varphi}_k(U_{k+1},U_k)+c\\\leq&\frac{1}{2}\|U_k\|_{\mathrm{tr}}+\frac{\beta_k}{2}\varphi_k(U_k)+c=\mathcal{L}(U_k,V_k,e_k,\lambda_k,\beta_k),\end{aligned}$$

where c is a constant independent of both U_k and U_{k+1}. Similarly, we have

$$\mathcal{L}(U_{k+1},V_{k+1},e_k,\lambda_k,\beta_k)\leq\mathcal{L}(U_{k+1},V_k,e_k,\lambda_k,\beta_k).$$

Furthermore, by the iteration procedure of Algorithm 1, we obtain

$$\begin{aligned}&\mathcal{L}(U_{k+1},V_{k+1},e_{k+1},\lambda_k,\beta_k)\\\leq&\mathcal{L}(U_{k+1},V_{k+1},e_k,\lambda_k,\beta_k)\leq\mathcal{L}(U_k,V_k,e_k,\lambda_k,\beta_k)\\=&\mathcal{L}(U_k,V_k,e_k,\lambda_{k-1},\beta_{k-1})+\alpha_k\|\lambda_k-\lambda_{k-1}\|_2^2,\end{aligned}$$

where $\alpha_k=\frac{\beta_k+\beta_{k-1}}{2\beta_{k-1}^2}$.

Since

$$\sum_{k=1}^{\infty}\alpha_k=\frac{\rho(\rho+1)}{2\beta_0(\rho-1)}<+\infty,$$

and recall the boundedness of $\{\lambda_k\}$, we have that $\{\mathcal{L}(U_k, V_k, e_k, \lambda_{k-1}, \beta_{k-1})\}$ is upper-bounded.

Note that $\lambda_k = \lambda_{k-1} + \beta_{k-1}(\mathcal{A}(U_k V_k^T) - b - e_k)$. Then we have

$$\frac{1}{2}(\|U_k\|_{\text{tr}} + \|V_k\|_{\text{tr}}) + \frac{1}{\mu}\|e_k\|_1 = \mathcal{L}(U_k, V_k, e_k, \lambda_{k-1}, \beta_{k-1}) - \frac{\|\lambda_k\|_2^2 - \|\lambda_{k-1}\|_2^2}{2\beta_{k-1}},$$

which is also upper-bounded. Thus the sequences $\{e_k\}$, $\{U_k\}$ and $\{V_k\}$ are all bounded. □

Proof of Theorem 3:

Proof. **(I)** By $\mathcal{A}(U_{k+1}V_{k+1}^T) - e_{k+1} - b = (\lambda_{k+1} - \lambda_k)/\beta_k$, the boundedness of $\{\lambda_k\}$, and $\lim_{k\to\infty} \beta_k = \infty$, we have

$$\lim_{k\to\infty} \|\mathcal{A}(U_{k+1}V_{k+1}^T) - e_{k+1} - b\|_2 = 0.$$

Hence, $\{(U_k,\, V_k,\, e_k)\}$ approaches to a feasible solution.

In the following, we will prove that the sequences $\{U_k\}$, $\{V_k\}$ and $\{e_k\}$ are Cauchy sequences.

By the boundedness of $\{\lambda_k\}$, $\{e_k\}$, $\{U_k\}$ and $\{V_k\}$, then both $\nabla\varphi_k(U_k)$ and t_k^{φ} are bounded. Furthermore, $\exists P_{k+1} \in \partial\|U_{k+1}\|_{\text{tr}}$ satisfies the following first-order optimality condition of (9)

$$\frac{1}{2}P_{k+1} + \beta_k t_k^{\varphi}\left[U_{k+1} - U_k + \frac{1}{t_k^{\varphi}}\nabla\varphi_k(U_k)\right] = 0. \tag{18}$$

By Lemma 4, we have $\|P_{k+1}\|_2 \leq 1$, which implies that $\{P_{k+1}\}$ is bounded.

$$\nabla\varphi_k(U_k) = \mathcal{A}^*[\mathcal{A}(U_kV_k^T) - e_k - b + \lambda_k/\beta_k]V_k = \frac{\mathcal{A}^*((\rho+1)\lambda_k - \rho\lambda_{k-1})V_k}{\beta_k}. \tag{19}$$

Substituting (19) into (18), it is easy to see that

$$\|U_{k+1} - U_k\|_F = \frac{\|\frac{1}{2}P_{k+1} + \beta_k\nabla\varphi_k(U_k)\|_F}{\beta_k t_k^{\varphi}} = \frac{\|P_{k+1} + 2\mathcal{A}^*((\rho+1)\lambda_k - \rho\lambda_{k-1})V_k\|_F}{2\beta_k t_k^{\varphi}}.$$

Consequently, if $m > n$,

$$\begin{aligned}
\|U_n - U_m\|_F &\leq \|U_n - U_{n+1}\|_F + \|U_{n+1} - U_{n+2}\|_F + \ldots + \|U_{m-1} - U_m\|_F \\
&= \frac{\|P_{n+1} + 2\mathcal{A}^*((\rho+1)\lambda_n - \rho\lambda_{n-1})V_n\|_F}{2\beta_n t_n^{\varphi}} + \frac{\|P_{n+2} + 2\mathcal{A}^*((\rho+1)\lambda_{n+1} - \rho\lambda_n)V_{n+1}\|_F}{2\beta_{n+1} t_{n+1}^{\varphi}} + \ldots + \frac{\|P_m + 2\mathcal{A}^*((\rho+1)\lambda_{m-1} - \rho\lambda_{m-2})V_{m-1}\|_F}{2\beta_{m-1} t_{m-1}^{\varphi}} \\
&\leq \delta_C\left(\frac{1}{\beta_n} + \frac{1}{\beta_{n+1}} + \ldots + \frac{1}{\beta_{m-1}}\right) = \frac{\delta_C}{\beta_n}\left(1 + \frac{1}{\rho} + \ldots + \frac{1}{\rho^{m-n-1}}\right) < \frac{\rho\delta_C}{(\rho-1)\beta_n},
\end{aligned}$$

where $\delta_C = \max\{\frac{\|P_{n+1} + 2\mathcal{A}^*((\rho+1)\lambda_n - \rho\lambda_{n-1})V_n\|_F}{2t_n^{\varphi}}, \frac{\|P_{n+2} + 2\mathcal{A}^*((\rho+1)\lambda_{n+1} - \rho\lambda_n)V_{n+1}\|_F}{2t_{n+1}^{\varphi}}, \ldots, \frac{\|P_m + 2\mathcal{A}^*((\rho+1)\lambda_{m-1} - \rho\lambda_{m-2})V_{m-1}\|_F}{2t_{m-1}^{\varphi}}\}$. Since $\frac{\rho\delta_C}{(\rho-1)\beta_n} \to 0$, it follows that indeed $\{U_k\}$ is a Cauchy sequence.

Similarly, $\{V_k\}$ and $\{e_k\}$ are also Cauchy sequences.

(II) Let (U_*, V_*, e_*) be a stationary point of (6), then the Karush-Kuhn-Tucker (KKT) conditions for (6) are formulated as follows:

$$\begin{aligned}
0 &\in \partial\|U_*\|_{\text{tr}} + 2\mathcal{A}^*(\lambda_*)V_*, \\
0 &\in \partial\|V_*\|_{\text{tr}} + 2(\mathcal{A}^*(\lambda_*))^T U_*, \\
0 &\in \frac{1}{\mu}\partial\|e_*\|_1 - \lambda_*, \\
e_* &= \mathcal{A}(U_*V_*^T) - b,
\end{aligned}$$

where λ_* is the associated Lagrangian multiplier. The first-order optimality condition of each subproblem at the $(k+1)$-th iteration is given by

$$\begin{aligned}
&0 \in \partial\|U_{k+1}\|_{\rm tr} + 2\beta_k t_k^{\varphi}\left[U_{k+1} - U_k + \frac{1}{t_k^{\varphi}}\nabla\varphi_k(U_k)\right],\\
&0 \in \partial\|V_{k+1}\|_{\rm tr} + 2\beta_k t_k^{\psi}\left[V_{k+1} - V_k + \frac{1}{t_k^{\psi}}\nabla\psi_k(V_k)\right],\\
&0 \in \frac{1}{\mu}\partial\|e_{k+1}\|_1 - \beta_k\left[\mathcal{A}(U_{k+1}V_{k+1}^T) - e_{k+1} - b + \lambda_k/\beta_k\right].
\end{aligned} \tag{20}$$

Since $\{U_k\}$, $\{V_k\}$ and $\{e_k\}$ are Cauchy sequences, then $\|U_{k+1}-U_k\|_F \to 0$, $\|V_{k+1}-V_k\|_F \to 0$ and $\|e_{k+1}-e_k\|_2 \to 0$. Let U_∞, V_∞ and e_∞ be their limit points, respectively, and λ_∞ be the associated Lagrangian multiplier. By the assumption $\|\lambda_{k+1} - \lambda_k\|_2 \to 0$, and (U_k, V_k, e_k) approaches a feasible solution, then $\beta_k\nabla\varphi_k(U_k) = \beta_k\mathcal{A}^*[\mathcal{A}(U_kV_k^T)-b-e_k+\lambda_k/\beta_k]V_k \to \mathcal{A}^*(\lambda_\infty)V_\infty$ and $\beta_k\nabla\psi_k(V_k) = \beta_k\{\mathcal{A}^*[\mathcal{A}(U_{k+1}V_k^T)-b-e_k+\lambda_k/\beta_k]\}^TU_{k+1} \to [\mathcal{A}^*(\lambda_\infty)]^TU_\infty$. Therefore, with $k\to\infty$, the following holds

$$\begin{aligned}
&0 \in \partial\|U_\infty\|_{\rm tr} + 2\mathcal{A}^*(\lambda_\infty)V_\infty,\\
&0 \in \partial\|V_\infty\|_{\rm tr} + 2[\mathcal{A}^*(\lambda_\infty)]^TU_\infty,\\
&0 \in \frac{1}{\mu}\partial\|e_\infty\|_1 - \lambda_\infty,\\
&e_\infty = \mathcal{A}(U_\infty V_\infty^T) - b.
\end{aligned}$$

Hence, the accumulation point $(U_\infty, U_\infty, e_\infty)$ of the sequence $\{(U_k, U_k, e_k)\}$ generated by Algorithm 1 satisfies the KKT conditions for the problem (6). □

D Proof of Theorem 4

To solve the bi-trace quasi-norm regularized problem (4) with the squared loss $\|\cdot\|_2^2$, the proposed algorithm is based on the proximal alternating linearized minimization (PALM) method for solving the following non-convex problem:

$$\min_{x,y} Q(x,y) := F(x) + G(y) + H(x,y), \tag{21}$$

where $F(x)$ and $G(y)$ are proper lower semi-continuous functions, and $H(x,y)$ is a smooth function with Lipschitz continuous gradients on any bounded set.

In Section 4.2 of the main paper, we stated that our PALM algorithm alternates between two blocks of variables, U and V. We establish the global convergence of our PALM algorithm by transforming the problem (4) into a standard form (21), and show that the transformed problem satisfies the condition needed to establish the convergence. First, the minimization problem (4) can be expressed in the form of (21) by setting

$$\begin{cases}
F(U) := \frac{1}{2}\|U\|_{\rm tr};\\
G(V) := \frac{1}{2}\|V\|_{\rm tr};\\
H(U,V) := \frac{1}{2\mu}\|\mathcal{A}(UV^T) - b\|_2^2.
\end{cases}$$

The conditions for global convergence of the PALM algorithm proposed in [4] are shown in the following lemma.

Lemma 7. *Let $\{(x_k, y_k)\}$ be a sequence generated by the PALM algorithm proposed in [4]. This sequence converges to a critical point of (21), if the following conditions hold:*

1. *$Q(x,y)$ is a Kurdyka-Lojasiewicz (KL) function;*
2. *$\nabla H(x,y)$ has Lipschitz constant on any bounded set;*
3. *$\{(x_k, y_k)\}$ is a bounded sequence.*

As stated in Lemma 7, the first condition requires that the objective function satisfies the KL property (For more details, see [4]). It is known that any proper closed semi-algebraic function is a KL function as such a function satisfies the KL property for all points in dom$f$ with $\phi(s) = cs^{1-\theta}$ for some $\theta \in [0, 1)$ and some $c > 0$. Therefore, we first give the following definitions of semi-algebraic sets and functions, and then prove that the proposed problem (4) with the squared loss $\|\cdot\|_2^2$ is also semi-algebraic.

Definition 3 ([4])**.** *A subset $S \subset \mathbb{R}^n$ is a real semi-algebraic set if there exists a finite number of real polynomial functions g_{ij}, $h_{ij} : \mathbb{R}^n \to \mathbb{R}$ such that*

$$S = \bigcup_j \bigcap_i \{u \in \mathbb{R}^n : g_{ij}(u) = 0,\; h_{ij}(u) < 0\}.$$

Moreover, a function $g(u)$ is called semi-algebraic if its graph $\{(u, t) \in \mathbb{R}^{n+1} : g(u) = t\}$ is a semi-algebraic set.

Semi-algebraic sets are stable under the operations of finite union, finite intersections, complementation and Cartesian product. The following are the semi-algebraic functions or the property of semi-algebraic functions used below:

- Real polynomial functions.
- Finite sums and product of semi-algebraic functions.
- Composition of semi-algebraic functions.

Lemma 8. *Each term in the proposed problem (4) with the squared loss $\|\cdot\|_2^2$ is a semi-algebraic function, and thus the function (4) is also semi-algebraic.*

Proof. It is easy to notice that the sets $\mathcal{U} = \{U \in \mathbb{R}^{m\times d} : \|U\|_\infty \leq D_1\}$ and $\mathcal{V} = \{V \in \mathbb{R}^{n\times d} : \|V\|_\infty \leq D_2\}$ are both semi-algebraic sets, where D_1 and D_2 denote two pre-defined upper-bounds for all entries of U and V, respectively.

For both terms $F(U) = \frac{1}{2}\|U\|_{\mathrm{tr}}$ and $G(V) = \frac{1}{2}\|V\|_{\mathrm{tr}}$: According to [4], we can know that the ℓ_1-norm is a semi-algebraic function. Since the trace norm is equivalent to the ℓ_1-norm on singular values of the associated matrix, it is natural that the trace norm is also semi-algebraic.

For the third term $H(U, V) = \frac{1}{2\mu}\|\mathcal{A}(UV^T) - b\|_2^2$, it is a real polynomial function, and thus is a semi-algebraic function [4]. Therefore, the proposed problem (4) with the squared loss $\|\cdot\|_2^2$ is semi-algebraic due to the fact that a finite sum of semi-algebraic functions is also semi-algebraic. □

For the second condition in Lemma 7, $H(U, V) = \frac{1}{2\mu}\|\mathcal{A}(UV^T) - b\|_2^2$ is a smooth polynomial function, and $\nabla H(U, V) = (\frac{1}{\mu}\mathcal{A}^*[\mathcal{A}(UV^T) - b]V,\ \frac{1}{\mu}\{\mathcal{A}^*[\mathcal{A}(UV^T) - b]\}^T U)$. It is natural that $\nabla H(U, V)$ has Lipschitz constant on any bounded set [5].

For the final condition in Lemma 7, $U_k \in \mathcal{U}$ and $V_k \in \mathcal{V}$ for any $k = 1, 2, \ldots$, which implies the sequence $\{(U_k, V_k)\}$ is bounded.

In short, we can know that three similar conditions as in Lemma 7 hold for our PALM algorithm. In other words, our PALM algorithm shares the same convergence property as in Lemma 7.

E Proof of Theorem 6

In order to prove Theorem 6, we first introduce the following Lemma (i.e., the Lemma 11 in [6] or the Theorem 1 in [7]).

Lemma 9. *Let $A, B \in \mathbb{R}^{n_1\times n_2}$, for any $p \in (0, 1]$, then we have*

$$\sum_{i=1}^{n} |\sigma_i^p(A) - \sigma_i^p(B)| \leq \sum_{i=1}^{n} \sigma_i^p(A - B),$$

where $n = \min(n_1, n_2)$.

Proof of Theorem 6:

Proof. ($\Longrightarrow$) By the definitions of $U_\star$ and $V_\star$ and Theorem 1 of the main paper, we have

$$X_0 = U_\star V_\star^T, \;\; \|X_0\|_{S_{1/2}}^{1/2} = \frac{\|U_\star\|_{\text{tr}} + \|V_\star\|_{\text{tr}}}{2}, \tag{22}$$

and $\text{rank}(U_\star) = \text{rank}(X_0) \leq r$ and $\text{rank}(V_\star) = \text{rank}(X_0) \leq r$. $\forall Z = U_\star W_2^T + W_1 V_\star^T + W_1 W_2^T$, we then obtain

$$\begin{aligned} X_0 + Z &= U_\star V_\star^T + U_\star W_2^T + W_1 V_\star^T + W_1 W_2^T \\ &= (U_\star + W_1)(V_\star + W_2)^T. \end{aligned} \tag{23}$$

Recall that $\mathcal{A}(U_\star V_\star^T) = \mathcal{A}(X_0) = b$. Then for all $Z \in \mathcal{N}(\mathcal{A}) \setminus \{\mathbf{0}\}$, we get

$$\mathcal{A}\left((U_\star + W_1)(V_\star + W_2)^T\right) = \mathcal{A}(X_0 + Z) = b.$$

Thus all feasible solutions to (3) can be represented as $X_0 + Z$ with $Z \in \mathcal{N}(\mathcal{A})$. To prove that $X_0 = U_\star V_\star^T$ is uniquely recovered by (3), we need to show that any feasible solution $X_1 = U_1 V_1^T$ ($X_1 \neq X_0$) to (3) satisfies $\|X_1\|_{S_{1/2}}^{1/2} > \|X_0\|_{S_{1/2}}^{1/2}$, where $U_1 \in \mathbb{R}^{m\times d}$ and $V_1 \in \mathbb{R}^{n\times d}$. Let $Z_1 = X_1 - X_0$, then $Z_1 \in \mathcal{N}(\mathcal{A}) \setminus \{\mathbf{0}\}$. Applying Theorem 1 of the main paper and by (23), the following holds for some W_1 and W_2:

$$\|X_1\|_{S_{1/2}}^{1/2} = \|X_0 + Z_1\|_{S_{1/2}}^{1/2} = (\|U_\star + W_1\|_{\text{tr}} + \|V_\star + W_2\|_{\text{tr}})/2, \tag{24}$$

where $U_\star + W_1 = U_{X_1}\Sigma_{X_1}^{1/2}$, $V_\star + W_2 = V_{X_1}\Sigma_{X_1}^{1/2}$, and $U_{X_1}\Sigma_{X_1}V_{X_1}^T$ is the same SVD form of X_1 as that of X_0. According to (22), (24), Theorem 1 of the main paper, and Lemma 9 with $p = 1$, we have

$$\begin{aligned} \|X_1\|_{S_{1/2}}^{1/2} =& \|X_0 + Z_1\|_{S_{1/2}}^{1/2} = \frac{1}{2}\left(\|U_\star + W_1\|_{\text{tr}} + \|V_\star + W_2\|_{\text{tr}}\right) \\ =& \frac{1}{2}\left(\sum_{i=1}^{d}\sigma_i(U_\star + W_1) + \sum_{i=1}^{d}\sigma_i(V_\star + W_2)\right) \\ \geq& \frac{1}{2}\left(\sum_{i=1}^{d}|\sigma_i(U_\star) - \sigma_i(W_1)| + \sum_{i=1}^{d}|\sigma_i(V_\star) - \sigma_i(W_2)|\right) \\ \geq& \frac{1}{2}\left[\sum_{i=1}^{r}\sigma_i(U_\star) - \sum_{i=1}^{r}\sigma_i(W_1) + \sum_{i=r+1}^{d}\sigma_i(W_1) + \sum_{i=1}^{r}\sigma_i(V_\star) - \sum_{i=1}^{r}\sigma_i(W_2) + \sum_{i=r+1}^{d}\sigma_i(W_2)\right] \\ >& \frac{1}{2}\left(\sum_{i=1}^{r}\sigma_i(U_\star) + \sum_{i=1}^{r}\sigma_i(V_\star)\right) \\ =& \frac{1}{2}\left(\|U_\star\|_{\text{tr}} + \|V_\star\|_{\text{tr}}\right) = \|X_0\|_{S_{1/2}}^{1/2}, \end{aligned}$$

which confirms that $X_0 = U_\star V_\star^T$ is uniquely recovered by (3).

($\Longleftarrow$) Conversely if (14) does not hold for some W_1 and W_2, i.e.,

$$\sum_{i=1}^{r}(\sigma_i(W_1) + \sigma_i(W_2)) \geq \sum_{i=r+1}^{d}(\sigma_i(W_1) + \sigma_i(W_2)), \tag{25}$$

then we can find W_1 and W_2 such that $(W_1)_r = -U_\star$ and $(W_2)_r = -V_\star$, where $(W_1)_r$ and $(W_2)_r$ denote the matrices induced by setting all but largest r singular values of W_1 and W_2 to 0, respectively. Using Theorem 1 of the main paper, then we can derive that

$$
\begin{aligned}
&\|X_0+Z\|_{S_{1/2}}^{1/2}\\
\leq&\frac{1}{2}\left(\|U_\star+W_1\|_{\text{tr}}+\|V_\star+W_2\|_{\text{tr}}\right)\\
=&\frac{1}{2}\left(\sum_{i=r+1}^{d}\sigma_i(W_1)+\sum_{i=r+1}^{d}\sigma_i(W_2)\right)\\
\leq&\frac{1}{2}\left(\sum_{i=1}^{r}\sigma_i(W_1)+\sum_{i=1}^{r}\sigma_i(W_2)\right)\\
=&\frac{1}{2}\left(\sum_{i=1}^{r}\sigma_i(U_\star)+\sum_{i=1}^{r}\sigma_i(V_\star)\right)\\
=&\|X_0\|_{S_{1/2}}^{1/2},
\end{aligned}
$$

i.e., $\|X_0+Z\|_{S_{1/2}}^{1/2}\leq\|X_0\|_{S_{1/2}}^{1/2}$. This shows that $X_0=U_\star V_\star^T$ is not the unique minimizer. □

F Proof of Theorem 7

Proof. With the squared loss, $\|\cdot\|_2^2$, (4) can be reformulated as follows:

$$\min_{U,V}\left\{\frac{\|U\|_{\text{tr}}+\|V\|_{\text{tr}}}{2}+\frac{1}{2\mu}\|\mathcal{A}(UV^T)-b\|_2^2\right\}. \tag{26}$$

Given $\widehat{V}$, the first-order optimality condition for the problem (26) with respect to U is given by

$$\frac{1}{\mu}\mathcal{A}^*(b-\mathcal{A}(\widehat{U}\widehat{V}^T))\widehat{V}\in\frac{1}{2}\partial\|\widehat{U}\|_{\text{tr}}. \tag{27}$$

According to Lemma 4 and (27), we can know that

$$\frac{1}{\mu}\|\mathcal{A}^*(b-\mathcal{A}(\widehat{U}\widehat{V}^T))\widehat{V}\|_2\leq\frac{1}{2},$$

where $\|A\|_2$ is the spectral norm of a matrix A and the dual norm of the trace norm. Recall that $\mathcal{A}^*(b-\mathcal{A}(\widehat{U}\widehat{V}^T))\widehat{V}\in\mathbb{R}^{m\times d}$ and $\text{rank}(\mathcal{A}^*(b-\mathcal{A}(\widehat{U}\widehat{V}^T))\widehat{V})\leq d$, then we obtain

$$\|\mathcal{A}^*(b-\mathcal{A}(\widehat{U}\widehat{V}^T))\widehat{V}\|_F\leq\sqrt{d}\|\mathcal{A}^*(b-\mathcal{A}(\widehat{U}\widehat{V}^T))\widehat{V}\|_2\leq\frac{\mu\sqrt{d}}{2}. \tag{28}$$

Let $\hat{X}=\widehat{U}\widehat{V}^T$. By the RSC assumption and (28), we have

$$
\begin{aligned}
\frac{\|X_0-\hat{X}\|_F}{\sqrt{mn}}&\leq\frac{\|\mathcal{A}(X_0-\hat{X})\|_2}{\kappa(\mathcal{A})\sqrt{lmn}}\\
&\leq\frac{\|\mathcal{A}(X_0)-b\|_2}{\kappa(\mathcal{A})\sqrt{lmn}}+\frac{\|b-\mathcal{A}(\hat{X})\|_2}{\kappa(\mathcal{A})\sqrt{lmn}}\\
&=\frac{\|e\|_2}{\kappa(\mathcal{A})\sqrt{lmn}}+\frac{\|\mathcal{A}^*(b-\mathcal{A}(\hat{X}))\hat{V}\|_F}{C_1\kappa(\mathcal{A})\sqrt{lmn}}\\
&\leq\frac{\epsilon}{\kappa(\mathcal{A})\sqrt{lmn}}+\frac{\mu\sqrt{d}}{2C_1\kappa(\mathcal{A})\sqrt{lmn}}.
\end{aligned}
$$

□

Lower bound on C_1

Next we discuss the lower boundedness of C_1, that is, it is lower bounded by a positive constant. By the characterization of the subdifferentials of the trace norm, we can know that

$$\partial\|X\|_{\mathrm{tr}} = \{Y \mid \langle Y,\, X\rangle = \|X\|_{\mathrm{tr}},\, \|Y\|_2 \le 1\}\,. \tag{29}$$

Let $\Xi = \mathcal{A}^*(b - \mathcal{A}(\widehat{U}\widehat{V}^T))\widehat{V}$, and by (27), we have that $\Xi \in \frac{\mu}{2}\partial\|\widehat{U}\|_{\mathrm{tr}}$. By (29), we obtain

$$\left\langle \frac{2}{\mu}\Xi,\, \widehat{U} \right\rangle = \|\widehat{U}\|_{\mathrm{tr}}.$$

Note that $\|X\|_{\mathrm{tr}} \ge \|X\|_F$ and $\langle X, Y\rangle \le \|X\|_F\|Y\|_F$ for any same-sized matrices X and Y. Then

$$\frac{2}{\mu}\|\Xi\|_F\|\widehat{U}\|_F \ge \left\langle \frac{2}{\mu}\Xi,\, \widehat{U} \right\rangle = \|\widehat{U}\|_{\mathrm{tr}} \ge \|\widehat{U}\|_F.$$

Recall that $\|\widehat{U}\|_F > 0$ and $\mu \neq 0$, thus we obtain

$$\|\mathcal{A}^*(b - \mathcal{A}(\widehat{U}\widehat{V}^T))\widehat{V}\|_F = \|\Xi\|_F \ge \frac{\mu}{2}.$$

$\widehat{U}$ is the optimal solution of the problem (26) with given $\widehat{V}$, then

$$\frac{1}{2\mu}\|\mathcal{A}(\widehat{U}\widehat{V}^T) - b\|_2^2 < \frac{1}{2\mu}\|\mathcal{A}(\widehat{U}\widehat{V}^T) - b\|_2^2 + \frac{1}{2}\|\widehat{U}\|_{\mathrm{tr}} \le \frac{1}{2\mu}\|b\|_2^2 = \nu,$$

where $\nu > 0$ is a constant. Hence,

$$C_1 = \frac{\|\mathcal{A}^*(b - \mathcal{A}(\hat{X}))\hat{V}\|_F}{\|b - \mathcal{A}(\hat{X})\|_2} > \frac{\sqrt{\mu}}{2\sqrt{2\nu}}.$$

G Proof of Theorem 8

According to Theorem 4 of the main paper, we can know that $(\widehat{U}, \widehat{V})$ is a critical point of the problem (15). To prove Theorem 8, we first give the following lemma [8].

Lemma 10. *Let $\mathcal{L}(X) = \frac{1}{\sqrt{mn}}\|X - \widehat{X}\|_F$ and $\hat{\mathcal{L}}(X) = \frac{1}{\sqrt{|\Omega|}}\|\mathcal{P}_\Omega(X - \widehat{X})\|_F$ be the actual and empirical loss function respectively, where $X,\ \widehat{X} \in \mathbb{R}^{m\times n}$ $(m \ge n)$. Furthermore, assume entry-wise constraint $\max_{i,j}|X_{ij}| \le \delta$. Then for all rank-r matrices X, with probability greater than $1 - 2\exp(-m)$, there exists a fixed constant C such that*

$$\sup_{X\in S_r} |\hat{\mathcal{L}}(X) - \mathcal{L}(X)| \le C\delta\left(\frac{mr\log(m)}{|\Omega|}\right)^{1/4},$$

where $S_r = \{X \in \mathbb{R}^{m\times n} : rank(X) \le r, \|X\|_F \le \sqrt{mn}\delta\}$.

Suppose that $M = \max_{i,j}(X_{ij} - \widehat{X}_{ij})^2 \le (2\delta)^2$ and $\epsilon = 9\delta$ as in [8]. According to Theorem 2 in [8], thus we have

$$\begin{aligned}
&\sup_{X\in S_r} |\hat{\mathcal{L}}(X) - \mathcal{L}(X)| \\
\le&\frac{2\epsilon}{\sqrt{|\Omega|}} + \left(\frac{M^2}{2}\frac{2mr\log(9\delta m/\epsilon)}{|\Omega|}\right)^{1/4} \\
\le&\frac{18\beta_1}{\sqrt{|\Omega|}} + 2\delta\left(\frac{mr\log(m)}{|\Omega|}\right)^{1/4} \\
=&\left(2 + \frac{18}{(|\Omega| mr\log(m))^{1/4}}\right)\delta\left(\frac{mr\log(m)}{|\Omega|}\right)^{1/4}.
\end{aligned}$$

Therefore, the constant C can be set to $2 + \frac{18}{(|\Omega| mr\log(m))^{1/4}}$.

Proof of Theorem 8:

Proof.

$$\begin{aligned}
&\frac{\|D-\widehat{U}\widehat{V}^T\|_F}{\sqrt{mn}}\\
\leq&\left|\frac{\|D-\widehat{U}\widehat{V}^T\|_F}{\sqrt{mn}}-\frac{\|\mathcal{P}_{\Omega}(D-\widehat{U}\widehat{V}^T)\widehat{V}\|_F}{C_3\sqrt{|\Omega|}}\right|+\frac{\|\mathcal{P}_{\Omega}(D-\widehat{U}\widehat{V}^T)\widehat{V}\|_F}{C_3\sqrt{|\Omega|}}\\
=&\left|\frac{\|D-\widehat{U}\widehat{V}^T\|_F}{\sqrt{mn}}-\frac{\|\mathcal{P}_{\Omega}(D-\widehat{U}\widehat{V}^T)\|_F}{\sqrt{|\Omega|}}\right|+\frac{\|\mathcal{P}_{\Omega}(D-\widehat{U}\widehat{V}^T)\widehat{V}\|_F}{C_3\sqrt{|\Omega|}}.
\end{aligned}$$

Let $\tau(\Omega):=\left|\frac{1}{\sqrt{mn}}\|D-\widehat{U}\widehat{V}^T\|_F-\frac{1}{\sqrt{|\Omega|}}\|\mathcal{P}_{\Omega}(D-\widehat{U}\widehat{V}^T)\|_F\right|$, then we need to bound $\tau(\Omega)$. It is clear that $\text{rank}(\widehat{U}\widehat{V}^T)\leq d$, and thus $\widehat{U}\widehat{V}^T\in S_d$. According to Lemma 10, then with probability greater than $1-2\exp(-m)$, then there exists a fixed constant $C_2=2+\frac{18}{(|\Omega|mr\log(m))^{1/4}}$ such that

$$\begin{aligned}
\sup_{\hat{U}\hat{V}^T\in S_d}\tau(\Omega)=&\left|\frac{\|\widehat{U}\widehat{V}^T-D\|_F}{\sqrt{mn}}-\frac{\|\mathcal{P}_{\Omega}(\widehat{U}\widehat{V}^T)-\mathcal{P}_{\Omega}(D)\|_F}{\sqrt{|\Omega|}}\right|\\
\leq&C_2\delta\left(\frac{md\log(m)}{|\Omega|}\right)^{\frac{1}{4}}.
\end{aligned}\tag{30}$$

We also need to bound $\|\mathcal{P}_{\Omega}(\widehat{U}\widehat{V}^T-D)\widehat{V}\|_F$. Given $\widehat{V}$, the optimization problem with respect to U is formulated as follows:

$$\min_{U}\frac{\|U\|_{\text{tr}}}{2}+\frac{1}{2\mu}\|\mathcal{P}_{\Omega}(U\widehat{V}^T)-\mathcal{P}_{\Omega}(D)\|_F^2.\tag{31}$$

Since $(\widehat{U},\widehat{V})$ is a KKT point of the problem (15), the first-order optimality condition for the problem (31) is given by

$$\mathcal{P}_{\Omega}(D-\widehat{U}\widehat{V}^T)\widehat{V}\in\frac{\mu}{2}\partial\|\widehat{U}\|_{\text{tr}}.\tag{32}$$

Using Lemma 4, we obtain

$$\|\mathcal{P}_{\Omega}(\widehat{U}\widehat{V}^T-D)\widehat{V}\|_2\leq\frac{\mu}{2},$$

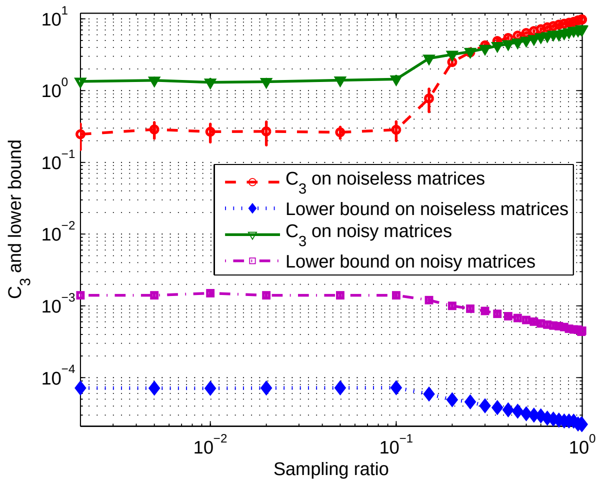
where $\|X\|_2$ is the spectral norm of a matrix X. Recall that $\text{rank}(\mathcal{P}_{\Omega}(\widehat{U}\widehat{V}^T-D)\widehat{V})\leq d$, we have

$$\|\mathcal{P}_{\Omega}(\widehat{U}\widehat{V}^T-D)\widehat{V}\|_F\leq\sqrt{d}\|\mathcal{P}_{\Omega}(\widehat{U}\widehat{V}^T-D)\widehat{V}\|_2\leq\frac{\sqrt{d}\mu}{2}.\tag{33}$$

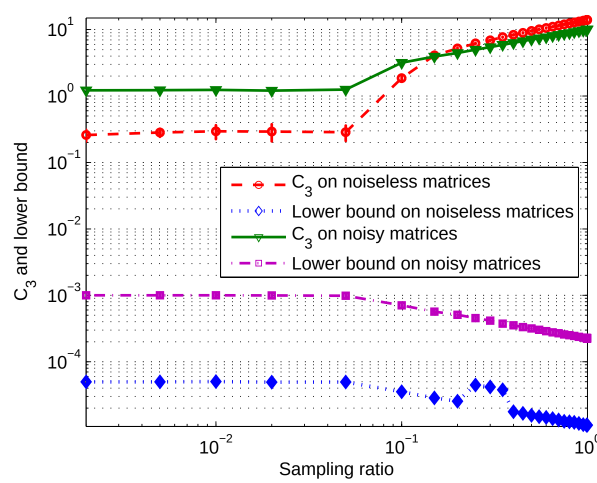
By (30) and (33), we have

$$\begin{aligned}
\frac{\|X_0-\widehat{U}\widehat{V}^T\|_F}{\sqrt{mn}}\leq&\frac{\|E\|_F}{\sqrt{mn}}+\frac{\|D-\widehat{U}\widehat{V}^T\|_F}{\sqrt{mn}}\\
\leq&\frac{\|E\|_F}{\sqrt{mn}}+\tau(\Omega)+\frac{\|\mathcal{P}_{\Omega}(D-\widehat{U}\widehat{V}^T)\widehat{V}\|_F}{C_3\sqrt{|\Omega|}}\\
\leq&\frac{\|E\|_F}{\sqrt{mn}}+C_2\delta\left(\frac{md\log(m)}{|\Omega|}\right)^{\frac{1}{4}}+\frac{\sqrt{d}\mu}{2C_3\sqrt{|\Omega|}}.
\end{aligned}$$

This completes the proof. □



(a) Matrices of size 100×100



(b) Matrices of size 200×200

Figure 1: Average results (mean and std.) of C_3 vs. sampling ratio. For a fixed sampling ratio, we can observe that although C_3 is not a constant, the value of C_3 is stable with respect to the projection operator $\mathcal{P}_\Omega$ (Best viewed zoomed in).

Lower bound on C_3

Finally, we also discuss the lower boundedness of C_3, that is, it is lower bounded by a positive constant. Let $Q=\mathcal{P}_\Omega(D-\widehat{U}\widehat{V}^T)\widehat{V}$, and by (32), we have that $Q\in\frac{\mu}{2}\partial\|\widehat{U}\|_{\text{tr}}$. By (29), we obtain

$$\left\langle\frac{2}{\mu}Q,\,\widehat{U}\right\rangle=\|\widehat{U}\|_{\text{tr}}.$$

Note that $\|A\|_{\text{tr}}\geq\|A\|_F$ and $\langle A,B\rangle\leq\|A\|_F\|B\|_F$ for any matrices A and B of the same size.

$$\frac{2}{\mu}\|Q\|_F\|\widehat{U}\|_F\geq\left\langle\frac{2}{\mu}Q,\,\widehat{U}\right\rangle=\|\widehat{U}\|_{\text{tr}}\geq\|\widehat{U}\|_F.$$

Recall that $\|\widehat{U}\|_F>0$ and $\mu\neq 0$, thus we obtain

$$\|\mathcal{P}_\Omega(D-\widehat{U}\widehat{V}^T)\widehat{V}\|_F=\|Q\|_F\geq\frac{\mu}{2}.$$

Since $\widehat{U}$ is the optimal solution of the problem (31) with given $\widehat{V}$, then

$$\frac{1}{2\mu}\|\mathcal{P}_\Omega(D-\widehat{U}\widehat{V}^T)\|_F^2<\frac{1}{2\mu}\|\mathcal{P}_\Omega(D-\widehat{U}\widehat{V}^T)\|_F^2+\frac{1}{2}\|\widehat{U}\|_{\text{tr}}\leq\frac{1}{2\mu}\|\mathcal{P}_\Omega(D)\|_F^2=\gamma,$$

where $\gamma>0$ is a constant. Hence,

$$C_3=\frac{\|\mathcal{P}_\Omega(D-\widehat{U}\widehat{V}^T)\widehat{V}\|_F}{\|\mathcal{P}_\Omega(D-\widehat{U}\widehat{V}^T)\|_F}>\frac{\sqrt{\mu}}{2\sqrt{2\gamma}}.$$

In fact, the value of C_3 is much greater than its lower bound, $\frac{\sqrt{\mu}}{2\sqrt{2\gamma}}$, as shown in Figure 1, where the ordinate is the average results over 100 independent runs, and the abscissa denotes the sampling ratio, which is chosen from $\{0.002,0.005,0.01,0.05,0.1,\ldots,0.95,0.99,0.995,0.999\}$. Moreover, the regularization parameter μ is set to 5 and 100 for noisy matrices ($nf=0.1$) and noiseless matrices, respectively.

H Complexity Analysis

For MC and RPCA problems, the running time of our PALM and LADM algorithms is mainly consumed in performing some matrix multiplications. The time complexity of some multiplications operators is $O(mnd)$. In addition, the time complexity of performing SVD on matrices of the same sizes as U_k and V_k is $O(md^2+nd^2)$. In short, the total time complexity of our PALM and LADM algorithms is $O(nmd)$ $(d \ll m, n)$. Moreover, it is known that the parallel matrix multiplication on multicore architectures can be efficiently implemented. Thus, in practice our PALM and LADM algorithms are fast and scales well to handle large-scale problems.

I More Experimental Results

For the MC problem, e.g., synthetic matrix completion and collaborative filtering, we propose an efficient proximal alternating linearized minimization (PALM) algorithm to solve (15), and then extend it to solve the Tri-tr quasi-norm regularized matrix completion problem.

In the following, we present our bi-trace quasi-norm minimization models for the RPCA problems (e.g., the text separation task):

$$\min_{U,V} \frac{1}{2}(\|U\|_{\text{tr}}+\|V\|_{\text{tr}})+\frac{1}{\mu}\|\mathcal{P}_{\Omega}(D-UV^T)\|_1, \tag{34}$$

and

$$\min_{U,V} \frac{1}{2}(\|U\|_{\text{tr}}+\|V\|_{\text{tr}})+\frac{1}{\mu}\|\mathcal{P}_{\Omega}(D-UV^T)\|_{1/2}^{1/2}, \tag{35}$$

where $\mathcal{P}_\Omega$ denotes the linear projection operator, i.e., $\mathcal{P}_\Omega(D)_{ij}=D_{ij}$ if $(i,j)\in\Omega$, and $\mathcal{P}_\Omega(D)_{ij}=0$ otherwise. Similar to (34), the Tri-tr quasi-norm penalty can also be used to the RPCA problem.

To efficiently solve the RPCA problems (34) and (35), we also need to introduce an auxiliary variable E (as the same role as e), and can assume, without loss of generality, that the unknown entries of D are simply set as zeros, i.e., $D_{\Omega^C}=0$, and E_{Ω^C} may be any values such that $\mathcal{P}_{\Omega^C}(D)=\mathcal{P}_{\Omega^C}(UV^T)+\mathcal{P}_{\Omega^C}(E)$. Therefore, the RPCA problems (34) and (35) are reformulated as follows:

$$\min_{U,V,E} \frac{1}{2}(\|U\|_{\text{tr}}+\|V\|_{\text{tr}})+\frac{1}{\mu}\|\mathcal{P}_{\Omega}(E)\|_1, \text{ s.t., } UV^T+E=D, \tag{36}$$

$$\min_{U,V,E} \frac{1}{2}(\|U\|_{\text{tr}}+\|V\|_{\text{tr}})+\frac{1}{\mu}\|\mathcal{P}_{\Omega}(E)\|_{1/2}^{1/2}, \text{ s.t., } UV^T+E=D. \tag{37}$$

In fact, we can apply directly Algorithm 1 with the soft-thresholding operator [9] to solve (36). In contrast, for solving (37) we can update E via solving the following problem with Lagrange multiplier Y_k,

$$\min_{E} \frac{1}{\mu}\|\mathcal{P}_{\Omega}(E)\|_{1/2}^{1/2}+\frac{\beta_k}{2}\|E-M_{k+1}\|_F^2, \tag{38}$$

where $M_{k+1}=D-U_{k+1}V_{k+1}^T-Y_k/\beta_k$. In general, the ℓ_p $(0<p<1)$ quasi-norm leads to a non-convex, non-smooth, and non-Lipschitz optimization problem [10]. Fortunately, we introduce the following half-thresholding operator in [11, 12] to solve (38).

Lemma 11. *Let $y=(y_1,y_2,\ldots,y_n)^T$, and $x^*=(x_1^*,x_2^*,\ldots,x_n^*)^T$ be an $\ell_{1/2}$ quasi-norm solution of the following minimization*

$$\min_x \|y-x\|_2^2+\lambda\|x\|_{1/2}^{1/2},$$

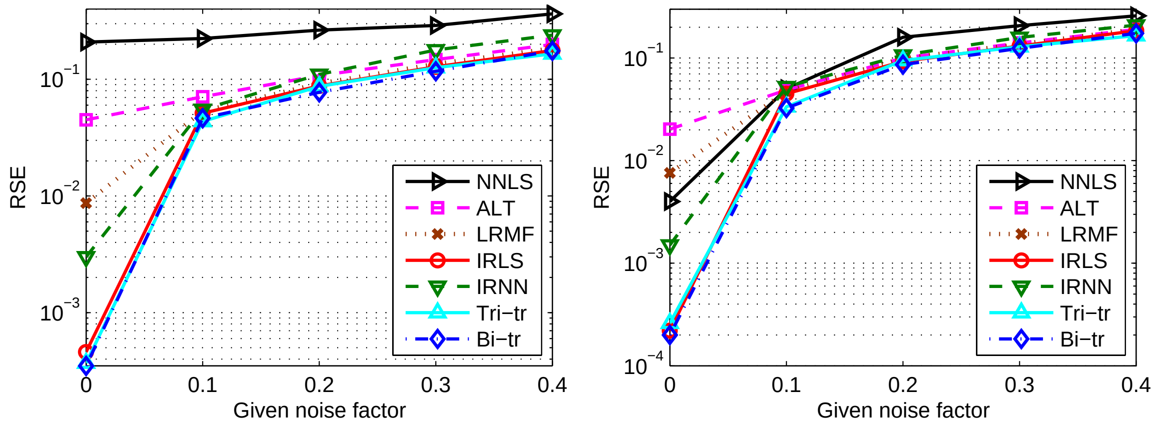
then the solution x^* can be given by $x^*=H_\lambda(y)$, where the half-thresholding operator $H_\lambda(\cdot)$ is defined as

$$H_\lambda(y_i)=\begin{cases} \frac{2}{3}y_i[1+\cos(\frac{2\pi}{3}-\frac{2\phi_\lambda(y_i)}{3})], & |y_i|>\frac{\sqrt[3]{54}}{4}\lambda^{\frac{2}{3}},\\ 0, & \text{otherwise,}\end{cases}$$

where $\phi_\lambda(y_i)=\arccos(\frac{\lambda}{8}(|y_i|/3)^{-3/2})$.

By Lemma 11, the closed-form solution of (38) is given by

$$(E_{k+1})_{i,j}=\begin{cases} H_{2/(\mu\beta_k)}((M_{k+1})_{i,j}), & (i,j)\in\Omega,\\ (M_{k+1})_{i,j}, & \text{otherwise.}\end{cases}$$



(a) Noisy matrices of size 100×100 (b) Noisy matrices of size 200×200

Figure 2: The recovery RSE results of NNLS, ALT, LRMF, IRLS, IRNN, and our Tri-tr and Bi-tr methods on noisy random matrices with different noise levels.

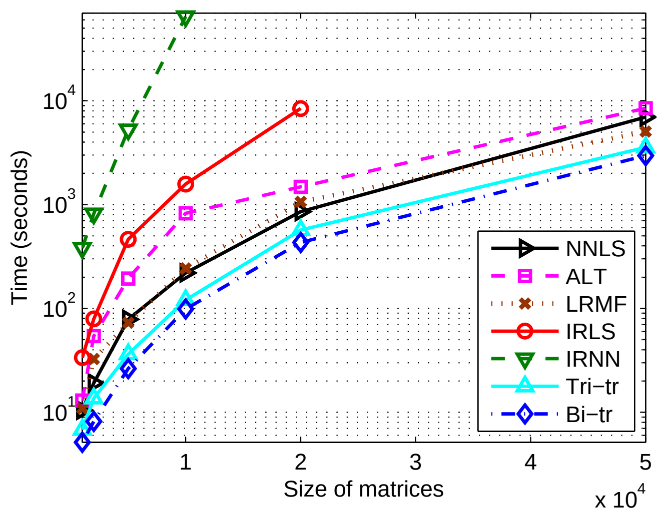


Figure 3: The running time of NNLS, ALT, LRMF, IRLS, IRNN, and our Tri-tr and Bi-tr methods as the size of noisy random matrices increases.

I.1 Implementation Details for Comparison

We present the implementation detail for all other algorithms in our comparison. The code for NNLS is downloaded from `http://www.math.nus.edu.sg/~mattohkc/NNLS.html`. For ALT, the given rank is set to $d = \lfloor 1.25r \rfloor$ for synthetic data and 50 for four recommendation system datasets, and the maximum number of iterations is set to 100 and 50, respectively, and its code is downloaded from `http://www.cs.utexas.edu/~cjhsieh/`. For LRMF, the code is downloaded from `http://ttic.uchicago.edu/~ssameer/#code`, and the IRLS code is downloaded from `http://www.math.ucla.edu/~wotaoyin/papers/improved_matrix_lq.html`. The given rank of LRMF and IRLS is set to the same value as our algorithms, e.g., $d = \lfloor 1.25r \rfloor$ for synthetic data. In addition, the code for IRNN is downloaded from `https://sites.google.com/site/canyilu/`. For LMaFit, the code is downloaded from `http://lmafit.blogs.rice.edu/`, and the $\mathrm{S}_p + \ell_p$ code is downloaded from `https://sites.google.com/site/feipingnie/publications`. Note that the regularization parameter μ is generally set to $\sqrt{\max(m, n)}$ as suggested in [13]. All the experiments were conducted on an Intel Xeon E7-4830V2 2.20GHz CPU with 64G RAM.

Table 1: Characteristics of the recommendation datasets.

Dataset	# row	# column	# rating
MovieLens1M	6,040	3,906	1,000,209
MovieLens10M	71,567	10,681	10,000,054
MovieLens20M	138,493	27,278	20,000,263
Netflix	480,189	17,770	100,480,507

Table 2: Regularization parameter settings for different algorithms.

Datasets	NNLS	ALT	LRMF	LMaFit	IRLS	Ours
	μ	λ	λ	λ	λ	μ
MovieLens1M	1.70	50	5	–	1e-6	100
MovieLens10M	4.80	100	5	–	1e-6	100
MovieLens20M	6.63	150	5	–	1e-6	100
Netflix	16.76	150	5	–	1e-6	100

I.2 Synthetic Data

In order to evaluate the robustness of our methods against noise, we generated the noisy input by the following procedure [14]:

$$b = \mathcal{A}(X_0 + nf * \Theta),$$

where the elements of the noise matrix Θ are i.i.d. standard Gaussian random variables, and nf is the given noise factor.

We also conduct some experiments on noisy matrices of size 100×100 or 200×200 with different noise factors, and report the RSE results of all algorithms with 20% SR in Figure 2. It is clear that ALT and LRMF have very similar performance, and usually outperform NNLS in terms of RSE. Moreover, the recovery performance of both our methods are similar to that of IRLS and IRNN, and they consistently perform better than the other methods. Moreover, we also present the running time of all those methods with 20% SR as the size of noisy random matrices increases, as shown in Figure 3. We can observe that the running time of IRNN and IRLS increases dramatically when the size of matrices increases, and they could not yield experimental results within 48 hours when the size of matrices is $50,000 \times 50,000$. On the contrary, both our methods are much faster than the other methods. This further justifies that both our methods have very good scalability and can address large-scale problems. As NNLS uses the PROPACK package [15] to compute a partial SVD in each iteration, it usually runs slightly faster than ALT.

I.3 Real-World Recommendation System Data

In this part, we present the detailed descriptions for four real-world recommendation system data sets and the detailed regularization parameter settings for different algorithms, as shown in Table 1 and Table 2, respectively. For IRNN, the regularization parameter λ is dynamically decreased by $\lambda_k = 0.7\lambda_{k-1}$, where $\lambda_0 = 10\|\mathcal{P}_\Omega(D)\|_\infty$. We also report the running time of all these algorithms on the four data sets, as shown in Figure 4, from which it is clear that both our methods are much faster than the other methods, except LMaFit. Compared with LMaFit, both our methods remain competitive in speed, but achieve much lower RMSE than LMaFit on all the data sets (as shown in Figure 2 in the main paper).

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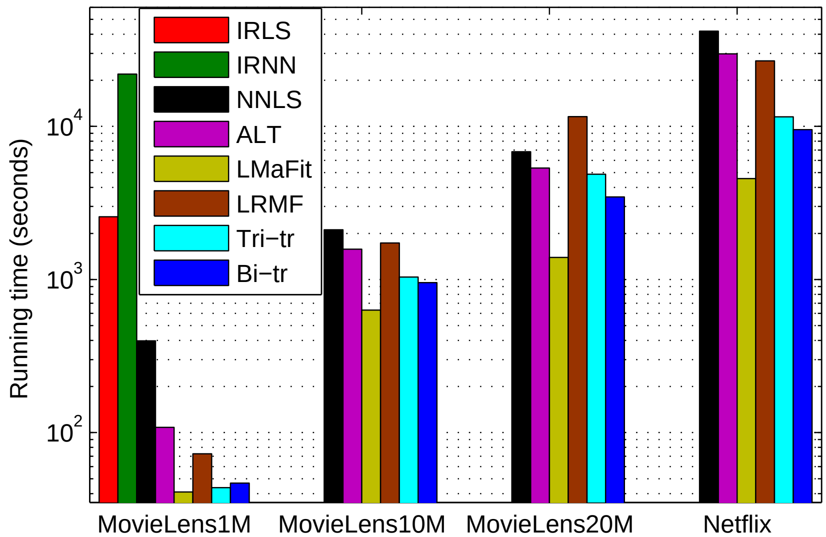

Figure 4: Running time (seconds) for comparison on the four data sets (Best viewed in color).

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