A Proofs

Proof of Lemma 8. We continue the proof.
Case 1.2, $|x_2 - d| > |x_2 - b|$**:**
Case 1.2.1, $|c - x_1| \le |x_2 - d|$**:** We have that $|x_2 - b| < |x_1 - b| \le R_\epsilon |x_1 - c|$. Thus,

$$\begin{aligned}
\frac{A}{B} &\ge \frac{|a-b|\,|x_1-x_2|}{BR_\epsilon\,|a-x_1|\,|x_1-c|} \\
&= \frac{|a-b|}{R_\epsilon\,|a-x_1|}\cdot\frac{|d-x_2|}{|c-d|} \ge \frac{|d-x_2|}{R_\epsilon\,|c-d|} \\
&\ge \frac{|d-x_2|}{R_\epsilon(|d-x_2|+|x_2-x_1|+|x_1-c|)} \\
&\ge \frac{|d-x_2|}{R_\epsilon(|d-x_2|+(1+R_\epsilon)\,|x_1-c|)} \\
&= \frac{1}{R_\epsilon\left(1+(1+R_\epsilon)\frac{|x_1-c|}{|d-x_2|}\right)} \ge \frac{1}{R_\epsilon(2+R_\epsilon)} \\
&\ge \frac{1}{4R_\epsilon(1+2R_\epsilon)}\,.
\end{aligned}$$

Case 1.2.2, $|c - x_1| > |x_2 - d|$**:** As before, we bound A and B separately:

$$\begin{aligned}
B &\le \frac{|c-d|\,|x_1-x_2|}{|c-x_1|\,|x_2-b|} \\
&\le \frac{|x_1-x_2|}{|x_2-b|}\cdot\frac{|c-x_1|+R_\epsilon\,|c-x_1|+|x_2-d|}{|c-x_1|} \\
&\le (2+R_\epsilon)\frac{|x_1-x_2|}{|x_2-b|}\,,
\end{aligned}$$

and

$$A = \frac{|b-a|\,|x_1-x_2|}{|a-x_1|\,|x_2-b|} \ge \frac{|x_1-x_2|}{|x_2-b|}\,.$$

Putting these together,

$$\frac{A}{B} \ge \frac{1}{2+R_\epsilon} \ge \frac{1}{4R_\epsilon(1+2R_\epsilon)}\,,$$

where the second inequality holds because $R_\epsilon(1+8R_\epsilon) \ge 2/3$.
Case 2, $|x_1 - b| > |x_1 - a|$ **and** $|x_2 - b| < |x_2 - a|$**:** In this case, x_1 and x_2 are on opposite sides of the point $(a+b)/2$. Let M be a positive constant. We will choose $M = 4$ later.
Case 2.1, $|c - d| \le M\,|c - x_1|$**:** We bound

$$B \le \frac{M\,|x_1-x_2|}{|x_2-d|} \le \frac{MR_\epsilon\,|x_1-x_2|}{|x_2-b|}$$

and conclude

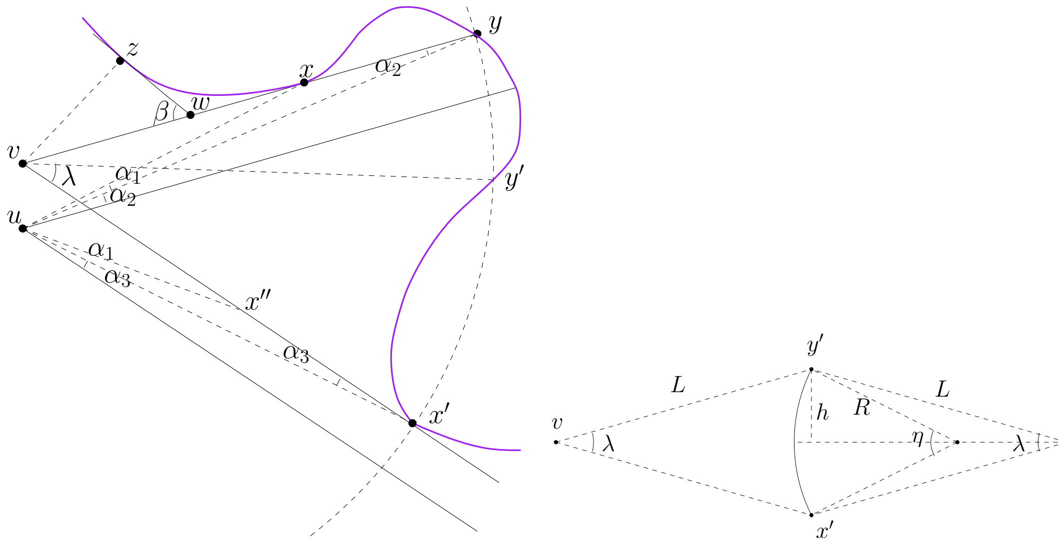
$$\frac{A}{B} \ge \frac{|a-b|}{MR_\epsilon\,|a-x_1|} \ge \frac{1}{MR_\epsilon} \ge \frac{1}{4R_\epsilon(1+2R_\epsilon)}\,.$$

Case 2.2, $|c - d| > M\,|c - x_1|$**:**
Case 2.2.1, $|c - d| \le M\,|a - b|$**:** We have that

$$\frac{A}{B} \ge \frac{1}{MR_\epsilon^2} \ge \frac{1}{4R_\epsilon(1+2R_\epsilon)}\,.$$

Case 2.2.2, $|c - d| > M\,|a - b|$**:** Let x_0 be a point on the line segment $[x_1, x_2]$. Let β_1 be the angle between line segments $[c, x_1]$ and $[x_1, x_0]$. We write

$$|c-x_0|^2 = |x_1-x_0|^2 + |x_1-c|^2 - 2\,|x_1-c|\cdot|x_1-x_0|\cos\beta_1$$



(a) The geometry used in Lemma 11.

(b) The geometry used in Lemma 11.

Figure 4: If two points are close to each other, they have similar views.

$$
\begin{aligned}
&\leq \frac{1}{M^2}|c-d|^2 + \frac{1}{M^2}|c-d|^2 + \frac{2}{M^2}|c-d|^2 \\
&= \frac{4}{M^2}|c-d|^2 \ .
\end{aligned}
$$

By the triangle inequality,

$$
|d-x_0| \geq |c-d| - |c-x_0| \geq \left(1-\frac{2}{M}\right)|c-d| \ .
$$

Let β_2 be the angle between line segments $[d,x_2]$ and $[x_2,x_0]$. Let $w = 1-2/M$. We write

$$
\begin{aligned}
w^2|c-d|^2 \leq |d-x_0|^2 & \\
&= |x_2-x_0|^2 + |x_2-d|^2 - 2|x_2-d|\cdot|x_2-x_0|\cos\beta_2 \\
&\leq \frac{1}{M^2}|c-d|^2 + |x_2-d|^2 + \frac{2}{M}|d-x_2|\cdot|c-d| \ .
\end{aligned}
$$

Thus,

$$
|x_2-d|^2 + \frac{2}{M}|d-x_2|\cdot|c-d| + (\frac{4}{M} - \frac{3}{M^2} - 1)|c-d|^2 \geq 0 \, ,
$$

which is a quadratic inequality in $|x_2-d|$. Thus it holds that

$$
|x_2-d| \geq \left(-\frac{1}{M} + \left|\frac{2}{M}-1\right|\right)|c-d| \ .
$$

If we choose $M=4$, then $|x_2-d| \geq 0.25\,|c-d|$ and

$$
B \leq \frac{4\,|x_1-x_2|}{|x_1-c|} \leq \frac{4R_\epsilon\,|x_1-x_2|}{|x_1-a|} \ ,
$$

yielding

$$
\frac{A}{B} \geq \frac{|a-b|}{4R_\epsilon\,|b-x_2|} \geq \frac{1}{4R_\epsilon} \geq \frac{1}{4R_\epsilon(1+2R_\epsilon)} \ .
$$

Finally, observe that Case 3 follows by symmetry from Case 1. □

Proof of Lemma 11. We say a line segment L is not fully visible from a point x if there exists a point on the line segment that is not visible from x. We denote this event by $L \notin \text{VIEW}(x)$. Let L be a line segment chosen by Hit-and-Run from u. So, as the next point in the Markov chain, Hit-and-Run chooses a point uniformly at random from L. We know that

$$P_u(\{x : x \notin \text{VIEW}(v)\}) \leq P_u(\{L : L \notin \text{VIEW}(v)\}) ,$$

So it suffices to show

$$P_u(\{L : L \notin \text{VIEW}(v)\}) \leq \max\left(\frac{4}{\pi}, \frac{\kappa}{\sin(\pi/8)}\right)\frac{\epsilon'}{\epsilon} . \tag{7}$$

To sample the line segment L, first we sample a random two dimensional plane containing u and v, and then sample the line segment inside this plane. To prove (7), we show that in any two dimensional plane containing u and v, the ratio of invisible to visible region is bounded by $\max\left(\frac{4}{\pi}, \frac{\kappa}{\sin(\pi/8)}\right)\frac{\epsilon'}{\epsilon}$.

Consider the geometry shown in Figure 4(a). Let $\mathcal{H}$ be the intersection of $\partial\Sigma$ and a two dimensional plane containing u and v. For a line ℓ and points q and u, we write $[q, \ell, u]$ to denote that u and a small neighborhood of q on $\mathcal{H}$ are on the opposite sides of ℓ. For example, in Figure 4(a), we have that $[x, \ell(v, x), u]$. Define a subset

$$Q = \{q \in \mathcal{H} \,:\, \ell(v, q) \text{ is tangent to } \mathcal{H} \text{ at } q \text{ and } [q, \ell(v, q), u]\} .$$

Any line $\ell(v, q)$ such that $[q, \ell(v, q), u]$ creates some space that is visible to u and invisible to v. If Q is empty, then the entire $\mathcal{H}$ is in the view of v and $P_u(\{x : x \notin \text{VIEW}(v)\}) = 0$.Otherwise, let x be a member of Q. Let $y \in \mathcal{H}$ be the closest point to x such that $[v, y]$ is tangent to $\mathcal{H}$ at x. Let α_1 be the angle between $[x, u]$ and $[u, y]$, and let α_2 be the angle between $[y, v]$ and $[y, u]$. Because $|u - v| \leq |v - z| \leq |v - x|$, $\alpha_1 + \alpha_2 \leq \pi/2$. Further, if the lengths of $|u - v|$ and $|v - x|$ are fixed, $\alpha_1 + \alpha_2$ is maximized when $[v, u]$ is orthogonal to $[u, x]$. If x is the only member of Q, then maximum invisible angle is α_1, which can be bounded as follows:

$$\sin\alpha_1 \leq \sin(\alpha_1 + \alpha_2) \leq \frac{|u - v|}{|v - x|} \leq \frac{|u - v|}{\epsilon} .$$

Otherwise, assume Q has more members. The same upper bound holds for members that are also on the line $\ell(v, x)$. So next we consider members of Q that are not on the line $\ell(v, x)$. Assume Q has only one such member and let x' be that tangent point (see Figure 4(a). The same argument can be repeated if Q has more such members). We consider two cases. **Case 1:** $|v - x'| \geq |v - y|$. Let α_3 be the angle between $[v, x']$ and $[u, x']$. If $\alpha_3 \leq \alpha_2$, then

$$\sin(\alpha_1 + \alpha_3) \leq \sin(\alpha_1 + \alpha_2) \leq \frac{|u - v|}{\epsilon} .$$

Otherwise, $\alpha_3 > \alpha_2$. Consider point x'' such that the angle between $[u, x']$ and $[u, x'']$ is α_1. We show that $|v - x''| \geq |v - x|$ by contradiction. Assume $|v - x| > |v - x''|$. Thus, $|x' - x''| > |y - x|$ and $|u - x| > |u - x''|$. By law of sines, $|x - y| / \sin\alpha_1 = |u - x| / \sin\alpha_2$ and $|x'' - x'| / \sin\alpha_1 = |u - x''| / \sin\alpha_3$. Because $|u - x| > |u - x''|$ and $\alpha_3 > \alpha_2$, we have that $|u - x| / \sin\alpha_2 > |u - x''| / \sin\alpha_3$, and thus $|x'' - x'| / \sin\alpha_1 < |x - y| / \sin\alpha_1$. This implies $|x'' - x'| < |x - y|$, a contradiction. Thus,

$$\sin(\alpha_1 + \alpha_3) \leq \frac{|u - v|}{|v - x''|} \leq \frac{|u - v|}{|v - x|} \leq \frac{|u - v|}{\epsilon} .$$

Next we consider the second case. **Case 2:** $|v - x'| < |v - y|$. Consider the arc on $\mathcal{H}$ from y to x'. Let y' be the last point on this arc such that $|v - y'| = |v - y|$. Let η be the change of angle between the tangent of $\mathcal{H}$ at y' and the tangent of $\mathcal{H}$ at x' (tangents are defined in clockwise direction), and let λ be the angle between $[v, y']$ and $[v, x']$. Angle η is minimized when the tangent at y' is orthogonal to $[v, y']$. Thus $\eta \geq \pi/2 - \lambda$. If $\lambda < \pi/4$, then $\eta \geq \pi/4$. Angle λ is smallest when the arc from y' to x' changes with maximum curvature $\kappa/\mathcal{R}_{\mathcal{H}}$, i.e. it is a segment of a circle with radius $\mathcal{R}_{\mathcal{H}}/\kappa$. Figure 4(b) shows this case, where $R = \mathcal{R}_{\mathcal{H}}/\kappa$ and $L = |v - y'|$. We have that

$$\frac{\sin(\lambda/2)}{\sin(\eta/2)} \geq \frac{h/L}{h/R} = \frac{R}{L} = \frac{\mathcal{R}_{\mathcal{H}}}{\kappa\,|v - y|}$$

Thus,

$$\frac{\lambda}{2} \geq \sin(\lambda/2) \geq \frac{\mathcal{R}_{\mathcal{H}}}{\kappa\,|v - y|}\sin(\eta/2) = \frac{\sin(\pi/8)\mathcal{R}_{\mathcal{H}}}{\kappa\,|v - y|} \geq \frac{\sin(\pi/8)}{\kappa} ,$$

where the last step follows by $|v - y| \le \mathcal{R}_{\mathcal{H}}$. Thus

$$\lambda \ge \lambda_0 \stackrel{\text{def}}{=} \min\left(\frac{\pi}{4}, \frac{\sin(\pi/8)}{\kappa}\right) .$$

So for every $|u - v| / \epsilon$ invisible region, we have at least λ_0 visible region. Thus,

$$P_u(\{x : x \notin \text{VIEW}(v)\}) \le \frac{|u - v|}{\lambda_0 \epsilon} = \max\left(\frac{4}{\pi}, \frac{\kappa}{\sin(\pi/8)}\right) \frac{|u - v|}{\epsilon} .$$

□

Before proving Lemma 12, we show a useful inequality. Consider points $u, v, w \in \Sigma$ that see each other. Let $\mathcal{C}$ be the convex hull of

$$\{a(u, v), b(u, v), a(u, w), b(u, w), a(v, w), b(v, w)\} .$$

Let i and j be distinct members of $\{u, v, w\}$. We use $a'(i, j)$ and $b'(i, j)$ to denote the endpoints of $\ell_{\mathcal{C}}(i, j)$ that are closer to i and j, respectively. Because $|a'(i, j) - b'(i, j)|$ is convex combination of two line segments that are inside Σ,

$$|a'(i, j) - b'(i, j)| \le D_\Sigma . \tag{8}$$

Also $[a(i, j), b(i, j)] \subset [a'(i, j), b'(i, j)]$, and thus $|a(i, j) - i| \le |a'(i, j) - i|$ and $|b(i, j) - j| \le |b'(i, j) - j|$. We can write

$$\begin{aligned}
\ell_{\mathcal{C}}(i, j) &= \frac{|i - j| \cdot |a'(i, j) - b'(i, j)|}{|a'(i, j) - i| \cdot |j - b'(i, j)|} \\
&\le \frac{|i - j| \cdot |a(i, j) - b(i, j)|}{|a(i, j) - i| \cdot |j - b(i, j)|} \cdot \frac{|a'(i, j) - b'(i, j)|}{|a(i, j) - b(i, j)|} \\
&\le D_\Sigma \frac{\ell_\Sigma(i, j)}{d(i, \partial\Sigma)} ,
\end{aligned} \tag{9}$$

where the last inequality holds because $|a(i, j) - b(i, j)| \ge d(i, \partial\Sigma)$.

Proof of Lemma 12. Let $A \subset \Sigma$ be a measurable subset of Σ. We prove that

$$P_u(A) - P_v(A) \le 1 - \frac{\epsilon}{8e^4 D_\Sigma} .$$

We partition A into five subsets, and estimate the probability of each of them separately:

$$\begin{aligned}
A_1 &= \{x \in A \,:\, |x - u| < F(u)\} , \\
A_2 &= \left\{x \in A \,:\, \left|(x - u)^\top (u - v)\right| > \frac{1}{\sqrt{n}} |x - u| \cdot |u - v| \right\} , \\
A_3 &= \Bigg\{x \in A \,:\, |x - u| < \frac{1}{6} |u - a(u, x)| , \\
&\qquad\qquad \text{or } |x - u| < \frac{1}{6} |u - a(x, u)| \Bigg\} , \\
A_4 &= \{x \in A \,:\, x \in \text{VIEW}(u),\, x \notin \text{VIEW}(v)\} , \\
S &= A \setminus A_1 \setminus A_2 \setminus A_3 \setminus A_4 .
\end{aligned}$$

The definition of $F(u)$ immediately yields $P_u(A_1) \le 1/8$. Now consider A_2 and let C be the cap of the unit sphere centered at u in the direction of v, defined by $C = \{x : (u - v)^\top x \ge \frac{1}{\sqrt{n}} |u - v|\}$. If $x \sim P_u$, then $P(x \in A_2)$ is bounded above by the probability that a uniform random line through u intersects C, which has probability equal to the ratio between the surface of C and the surface of the half-sphere. A standard computation to show

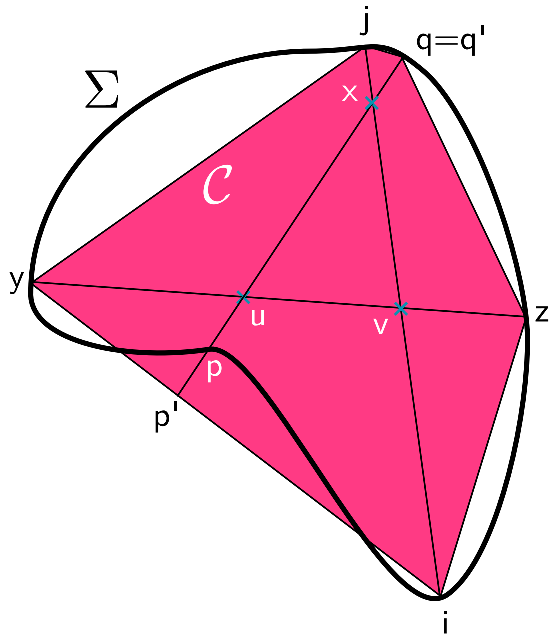


Figure 5: Illustration for Lemma 12 proof

that this ratio is less than 1/6, and hence $P_u(A_2) \leq 1/6$. The probability that $x \in A_3$ is at most 1/6, since x is chosen from a segment of a chord of length at most $|\ell(u,x)|/6$. Finally, to bound $P(A_4)$, we apply Lemma 11:

$$P_u\left(x \in A \,:\, x \in \text{VIEW}(u),\, x \notin \text{VIEW}(v)\right) \leq \max\left(\frac{4}{\pi}, \frac{\kappa}{\sin(\pi/8)}\right)\frac{\epsilon'}{\epsilon} \leq \frac{1}{6}\,.$$

The combined probability of A_1, A_2, A_3, and A_4 is at most $1/8 + 1/6 + 1/6 + 1/6 < 3/4$.

We now turn to bounding $P_u(S)$ and show that $P_u(S) \leq 2e^4(D_\Sigma/\epsilon)P_v(S)$. Because points in S are visible from both u and v, by (2)

$$P_v(S) = \frac{2}{n\pi_n}\int_S \frac{1}{\ell_\Sigma(v,x)\,|x-v|^{n-1}}\,.$$

Now, any $x \in S$ must respect the following

$$|x-u| \geq F(u) \geq \frac{\sqrt{n}}{2}\,|u-v|\,, \tag{10}$$

$$\left|(x-u)^\top(u-v)\right| \leq \frac{1}{\sqrt{n}}\,|x-u|\cdot|u-v|\,, \tag{11}$$

$$|x-u| \geq \frac{1}{6}\,|u-a(u,x)|\,, \text{ and} \tag{12}$$

$$|x-u| \geq \frac{1}{6}\,|u-a(x,u)|\,. \tag{13}$$

As illustrated in Figure 5, we define the points $y = a(u,v)$, $z = a(v,u)$, $p = a(u,x)$, $q = a(x,u)$, $i = a(v,x)$ and $j = a(x,v)$ with convex hull $\mathcal{C}$. Also let p' and q' be the endpoints of $\ell_{\mathcal{C}}(u,x)$. If $p' = p$ and $q' = q$, we proceed with the argument in the proof of Lemma 9 of Lovász (1999) to get the desired result. Otherwise, assume $q' = q$ and p' is the intersection of the lines $\ell(u,p)$ and $\ell(y,i)$. (See Figure 5. A similar argument holds when $q \neq q'$.) From (12) and (13), we get that

$$2\,|x-u| > \frac{1}{6}\,|p-q|\,.$$

We have that $|p-q| \geq \epsilon$, and by (8), $|p'-q'| \leq D_\Sigma$. Thus $|p'-q'| \leq (D_\Sigma/\epsilon)\,|p-q|$. Thus,

$$\frac{1}{6}\,|p'-q'| \leq \frac{2D_\Sigma}{\epsilon}\,|x-u|\,. \tag{14}$$

To relate $P_v(S)$ to $P_u(S)$, we need to bound $|x-v|$ and $\ell(x,v)$ in terms of $|x-u|$ and $\ell(x,u)$:

$$\begin{aligned}|x-v|^2 &= |x-u|^2 + |u-v|^2 + 2(x-u)^\top(u-v) \\ &\le |x-u|^2 + |u-v|^2 + \frac{2}{\sqrt{n}}|x-u|\cdot|u-v| && \ldots\text{By (11)} \\ &\le |x-u|^2 + \frac{4}{n}|x-u|^2 + \frac{4}{n}|x-u|^2 && \ldots\text{By (10)} \\ &= \left(1+\frac{8}{n}\right)|x-u|^2 \ .\end{aligned}$$

Thus,

$$|x-v| \le \left(1+\frac{4}{n}\right)|x-u| \ . \tag{15}$$

First we use convexity of $\mathcal{C}$ to bound $\ell_{\mathcal{C}}(x,v)$ in terms of $\ell_{\mathcal{C}}(x,u)$, and then we use (9) to bound $\ell_\Sigma(x,v)$ and $\ell_\Sigma(x,u)$ in terms of $\ell_{\mathcal{C}}(x,v)$ and $\ell_{\mathcal{C}}(x,u)$. By Menelaus' Theorem (wrt triangle uvx and transversal line $[y,i]$),

$$\frac{|x-i|}{|v-i|} = \frac{|u-y|}{|v-y|}\cdot\frac{|x-p'|}{|u-p'|} \ .$$

We have that

$$\frac{|u-y|}{|v-y|} = 1 - \frac{|v-u|}{|v-y|} > 1 - d_{\mathcal{C}}(u,v)\,,$$

and thus

$$\begin{aligned}\frac{|x-v|}{|v-i|} &= \frac{|x-i|}{|v-i|} - 1 \\ &\ge (1-d_{\mathcal{C}}(u,v))\frac{|x-p'|}{|u-p'|} - 1 \\ &= \frac{|x-u|}{|u-p'|}\left(1 - d_{\mathcal{C}}(u,v)\frac{|x-p'|}{|x-u|}\right) \\ &> \frac{|x-u|}{|u-p'|}\left(1 - d_{\mathcal{C}}(u,v)\frac{|p'-q'|}{|x-u|}\right) \\ &> \frac{|x-u|}{|u-p'|}\left(1 - \frac{12D_\Sigma\epsilon}{24D_\Sigma\epsilon}\right) \\ &> \frac{1}{2}\frac{|x-u|}{|u-p'|}\,,\end{aligned}$$

where we have used (14) and $d_{\mathcal{C}}(u,v) = d_\Sigma(u,v) < \epsilon/(24D_\Sigma)$ (the condition in the statement of the lemma); we conclude that

$$|v-i| < 2\frac{|x-v|}{|x-u|}|u-p'| \ . \tag{16}$$

Next we prove a similar inequality for $|v-j|$. It is easy to check that

$$\frac{|z-v|}{|u-z|} = 1 - \frac{|u-v|}{|u-z|} > 1 - d_{\mathcal{C}}(u,v)\,,$$

and combining with Menelaus' Theorem

$$\frac{|v-j|}{|x-j|} = \frac{|q'-u|}{|x-q'|}\cdot\frac{|z-v|}{|u-z|}$$

we can show

$$\frac{|x-v|}{|x-j|} = \frac{|v-j|}{|x-j|} - 1$$

$$
\begin{aligned}
&\geq (1 - d_{\mathcal{C}}(u,v))\frac{|q'-u|}{|x-q'|} - 1 \\
&= \frac{|x-u|}{|x-q'|}\left(1 - d_{\mathcal{C}}(u,v)\frac{|q'-u|}{|x-u|}\right) \\
&> \frac{|x-u|}{|x-q'|}\left(1 - d_{\mathcal{C}}(u,v)\frac{|p'-q'|}{|x-u|}\right) \\
&> \frac{|x-u|}{|x-q'|}\left(1 - \frac{12D_\Sigma\epsilon}{24D_\Sigma\epsilon}\right) \\
&> \frac{1}{2}\frac{|x-u|}{|x-q'|} ,
\end{aligned}
$$

where we have used (14) and $d_{\mathcal{C}}(u,v) = d_\Sigma(u,v) < \epsilon/(24D_\Sigma)$. Thus,

$$
|x-j| < 2\frac{|x-v|}{|x-u|}|x-q'| \ ,
$$

and combining this with the trivial observation that $|x-v| \leq 2\frac{|x-v|}{|x-u|}|x-u|$, and Equation 16 yields

$$
\ell_{\mathcal{C}}(x,v) = |v-i| + |v-x| + |x-j| \leq 2\frac{|x-v|}{|x-u|}\ell_{\mathcal{C}}(x,u) \ .
$$

Thus,

$$
\begin{aligned}
\ell_\Sigma(x,v) &= \ell_{\mathcal{C}}(x,v) \\
&\leq 2\frac{|x-v|}{|x-u|}\ell_{\mathcal{C}}(x,u) \\
&\leq \frac{2D_\Sigma}{\epsilon}\frac{|x-v|}{|x-u|}\ell_\Sigma(x,u) \ .
\end{aligned} \tag{17}
$$

Where the last step holds by (9). Now we are ready to lower bound $P_v(S)$ in terms of $P_u(S)$.

$$
\begin{aligned}
P_v(S) &= \frac{2}{n\pi_n}\int_S \frac{dx}{\ell_\Sigma(x,v)\,|x-v|^{n-1}} \\
&\geq \frac{\epsilon}{n\pi_n D_\Sigma}\int_S \frac{|x-u|\,dx}{\ell_\Sigma(x,u)\,|x-v|^{n}} && \ldots\text{By } (17) \\
&\geq \frac{\epsilon}{n\pi_n D_\Sigma}\left(1+\frac{4}{n}\right)^{-n}\int_S \frac{dx}{\ell_\Sigma(x,u)\,|x-u|^{n-1}} && \ldots\text{By } (15) \\
&\geq \frac{\epsilon}{2e^4 D_\Sigma}P_u(S) \ .
\end{aligned}
$$

Finally,

$$
\begin{aligned}
P_u(A) - P_v(A) &\leq P_u(A) - P_v(S) \\
&\leq P_u(A) - \frac{\epsilon}{2e^4D_\Sigma}P_u(S) \\
&\leq P_u(A) - \frac{\epsilon}{2e^4D_\Sigma}\left(P_u(A) - \frac{3}{4}\right) \\
&= \frac{3\epsilon}{8e^4D_\Sigma} + \left(1 - \frac{\epsilon}{2e^4D_\Sigma}\right)P_u(A) \\
&\overset{(a)}{\leq} \frac{3\epsilon}{8e^4D_\Sigma} + 1 - \frac{4\epsilon}{8e^4D_\Sigma} \\
&= 1 - \frac{\epsilon}{8e^4D_\Sigma} \ .
\end{aligned}
$$

In the step (a), we used the fact that $D_\Sigma \geq \epsilon$ and $P_u(A) \leq 1$. $\square$

Proof of Lemma 13. Let $\{S_1, S_2\}$ be a partitioning of Σ. Define

$$\Sigma_1 = \{x \in S_1 \ : \ P_x(S_2) \le \delta\} \ ,$$
$$\Sigma_2 = \{x \in S_2 \ : \ P_x(S_1) \le \delta\} \ ,$$
$$\Sigma_3 = \Sigma \setminus \Sigma_1 \setminus \Sigma_2 \ .$$

Case 1: $\text{VOL}(\Sigma_1) \le \text{VOL}(S_1)/2$. We have that

$$\int_{S_1} P_x(S_2) dx \ge \int_{S_1 \setminus \Sigma_1} P_x(S_2) dx \ge \delta \text{VOL}(S_1 \setminus \Sigma_1) \ge \frac{\delta}{2} \text{VOL}(S_1) \ .$$

Thus,

$$\frac{1}{\min\{\text{VOL}(S_1), \text{VOL}(S_2)\}} \int_{S_1} P_x(S_2) dx \ge \frac{\delta}{2} \ .$$

Case 2: $\text{VOL}(\Sigma_1) > \text{VOL}(S_1)/2$ and $\text{VOL}(\Sigma_2) > \text{VOL}(S_2)/2$. Similar to the argument in the previous case,

$$\int_{S_1} P_x(S_2) \ge \delta\, \text{VOL}(S_1 \setminus \Sigma_1) \, ,$$

and

$$\int_{S_1} P_x(S_2) = \int_{S_2} P_x(S_1) \ge \delta\, \text{VOL}(S_2 \setminus \Sigma_2) \ .$$

Thus,

$$\int_{S_1} P_x(S_2) \ge \frac{\delta}{2} \text{VOL}(\Sigma \setminus \Sigma_1 \setminus \Sigma_2) = \frac{\delta}{2} \text{VOL}(\Sigma_3) \ .$$

Let $\Omega_i = g^{-1}(\Sigma_i)$ for $i = 1, 2, 3$. Define

$$(u(x), v(x)) = \operatorname*{argmin}_{u \in \Omega_1, v \in \Omega_2, \{u,v,x\} \text{ are collinear}} d_\Omega(u, v) \, , \qquad h(x) = (1/3) \min(1, d_\Omega(u(x), v(x))) \ .$$

By definition, $h(x)$ satisfies condition of Theorem 7. Let $\epsilon = \frac{r}{2n}$ and notice that $\text{VOL}(\Omega^\epsilon) \ge \text{VOL}(\Omega)/2$. We have that

$$\begin{aligned} \int_{S_1} P_x(S_2) &\ge \frac{\delta}{2} \text{VOL}(\Omega_3) \\ &\ge \frac{\delta}{2} \mathbf{E}_\Omega(h(x)) \min(\text{VOL}(\Omega_1), \text{VOL}(\Omega_2)) \\ &= \frac{\delta}{4} \mathbf{E}_{\Omega^\epsilon}(h(x)) \min(\text{VOL}(\Sigma_1), \text{VOL}(\Sigma_2)) \ . \end{aligned}$$

Let $x \in \Omega^\epsilon$. We consider two cases. <u>In the first case</u>, $|u(x) - v(x)| \ge \epsilon/10$. Thus,

$$d_\Omega(u(x), v(x)) \ge \frac{4}{D_\Omega} |u(x) - v(x)| \ge \frac{2}{5nD_\Omega} \ .$$

<u>In the second case</u>, $|u(x) - v(x)| < \epsilon/10$, then $|u(x) - x| \le \epsilon/10$ and $|v(x) - x| \le \epsilon/10$. Thus, $u, v \in \Omega^{\epsilon'}$ for $\epsilon' = 9\epsilon/10$. Thus by Assumption 2, $g(u), g(v) \in \Sigma^{\epsilon''}$ for $\epsilon'' = 9\epsilon/(10L_\Omega)$. By Lemma 10,

$$d_\Omega(u(x), v(x)) \ge \frac{\widetilde{d}_\Sigma(g(u(x)), g(v(x)))}{4L_\Sigma^2 L_\Omega^2 R_{\epsilon'}(1 + 2R_{\epsilon'})} \ .$$

Next we lower bound $\widetilde{d}_\Sigma(g(u), g(v))$. If $g(u)$ and $g(v)$ cannot see each other, then $\widetilde{d}_\Sigma(g(u), g(v)) \ge 8\epsilon''/D_\Sigma$. Next we assume that $g(u)$ and $g(v)$ see each other. Because $g(u) \in \Sigma_1$ and $g(v) \in \Sigma_2$,

$$d_{tv}(P_{g(u)} - P_{g(v)}) \ge 1 - P_{g(u)}(S_2) - P_{g(v)}(S_1) \ge 1 - 2\delta = 1 - \frac{\epsilon''}{8e^4 D_\Sigma} \ .$$

Lemma 12, applied to $g(u), g(v) \in \Sigma^{\epsilon''}$, gives us that

$$d_\Sigma(g(u), g(v)) \ge \frac{\epsilon''}{24D_\Sigma} \qquad \text{or} \qquad |g(u) - g(v)| \ge \frac{2}{\sqrt{n}} \min\left(\frac{2F(g(u))}{\sqrt{n}}, G\epsilon''\right) \ .$$

By (6), $F(g(u)) \ge \epsilon''/16$. We get the desired lower bound by taking a minimum over all cases. $\square$