SUPPLEMENTARY MATERIAL

A Calibration of the parameters

We only prove the calibration in the setting of linear regression with square loss (i.e., for Theorem 3 only and not for the general Theorem 1). It remains an open question whether the calibration of the parameters can be performed in the general setting of Section 3. We leave this question for future research. Furthermore, for the sake of clarity the adaption to $Y$ (which is only necessary for clipping) is not considered here. However, it can be achieved simultaneously by updating the clipping range based on the past observations $Y_s$, $s \leq t-1$ (see Gerchinovitz, 2013, Section 4.5).

The calibration algorithm (Algorithm 3) works as follows. We define large enough grids of parameters for each doubling session $j \geq 0$

$$
G_j = \{(d_0, \alpha, U, B) \in [1, \ldots, d] \times \mathbb{R}_+^d \text{ such that}
\begin{align*}
&d_0 \in \{0\} \cup \{2^k, k = 0, \ldots, \lfloor \log_2 d \rfloor \}, \\
&\alpha \in \{2^k, k = -2j + \lfloor \log_2 (Bd_0/Y^2) \rfloor, \ldots, \lfloor j + \log_2 d_0 \rfloor \}, \\
&U \in \{2^k, k = -2j, \ldots, 2j + \lfloor 2 \log_2 Y \rfloor \}, \\
&B \in \{2^k, k = -2j, \ldots, 2j + \lfloor 2 \log_2 Y \rfloor \} \}
\end{align*}
\right)
$$

(7)

For each set of parameters $p = (d_0, \alpha, U, B) \in G_j$, we perform a local version of SAEW to obtain an estimator $\hat{\theta}_{p,j}$ at time $t = 2^j - 1$. Then, the calibration algorithm uses the online aggregation procedure BOA of Wintenberger (2014) to make predictions from $t = 2^j$ to $2^{j+1} - 1$. Its predictions are based on online combinations of the (clipped) forecasts made by the $\hat{\theta}_{p,j}$.

**Theorem 4.** Let $Y > \max_{t=1, \ldots, T} |Y_t|$ almost surely. With probability $1 - \delta$, the excess risk of the estimator $\hat{f}_T$ produced by Algorithm 3 is of order

$$
O_T \left( \frac{Y^2}{T} \log \left( \frac{(\log d)(\log T + \log Y)}{\delta} \right) + \frac{d_0 X^2 \sigma^2}{\alpha^* T} \log \left( \frac{d \log T}{\delta} \right) \right),
$$

where $d_0 = \|\theta^*\|_0$ and $\alpha^* > 0$ is the largest value of $\alpha$ satisfying Inequality (SC).

The proof is postponed to Appendix B.7.

**Remark A.1.** Similarly to the restricted eigenvalue condition of the Lasso, we believe that the strong convexity condition for $\alpha^*$ might be necessary on subspaces of dimension lower than or equal to $d_0$ only. However, to do so, SAEW should be used with a subroutine that produces sparse $\hat{\theta}_{t-1}$. Up to our knowledge, such procedures do not exist for convex optimization in the $\ell_1$-ball. As stated previously, sparse procedures such as RDA of Xiao (2010) cannot be used as subroutines since they perform optimization in the $\ell_2$-ball and suffer a linear dependence on $d$. We leave this question for future work.

**Remark A.2.** For the sake of clarity, the above result is only stated asymptotically. However the bound also holds in finite time up to universal multiplicative constant (as done in the proof). Additional negligible terms of order $O(1/T^2)$ then appear in the bound. Furthermore, the finite time bound also achieves the best of the two regimes (slow rate vs fast rate) as in Theorem 3.

**Remark A.3.** Theorem 4 has been proven only for square linear regression. However, it also holds for any strongly-convex loss function, with locally bounded gradients (i.e., with LIST condition, see Wintenberger, 2014).

**Remark A.4.** To perform the calibration, we left the original framework of Section 2. First, because of the clipping, the estimators $\tilde{f}_{t-1}$ produced by Algorithm 3 are not linear any-more. Second, the meta-algorithm implies that we can observe the gradients of all subrou-

Algorithm 3: Calibration algorithm

**Parameters:** $Y > 0$, $\delta > 0$

**Initialization:** $t_0 = t = 1$ and $\bar{\theta}^{(0)} = 0$

For each $j = 0, 1, \ldots$

- Define the grid $G_j$ as in (7)
- For parameters $p = (d_0, \alpha, U, B) \in G_j$:
  - Define $\delta_j = \delta/(2(j+1)^2)$
  - Run SAEW with parameter $(d_0, \alpha, U, B, \delta_j)$ for $t = 0, \ldots, 2^j - 1$ and get the estimator $\hat{\theta}_{2^j - 1}$, denoted by $\bar{\theta}_{p,j}$.
  - Define the clipped predictor $f_{p,j}: x \mapsto [x^T \bar{\theta}_{p,j}]_Y$ where $[\cdot]_Y := \max \{ -Y, \min\{\cdot, Y\} \}$.
- For $t = 2^j, \ldots, 2^{j+1} - 1$,
  - predict $\tilde{f}_{t-1}(X_t)$ by performing BOA with experts $(f_{p,j})_{p \in G_j}$
  - output the estimator $\tilde{f}_t = \tilde{f}_j$
- Define the average estimator $\tilde{f}_{j+1} = 2^{-j} \sum_{t=2^j}^{2^{j+1}-1} \tilde{f}_t$.
times SAEW simultaneously. Tuning the parameters in the original setting is left for future work.

B Proofs

B.1 Lemma 5

We first state Lemma 5, a classical result in strong convexity, as it will be useful in the proofs. It relates the $\ell_2$-error of an estimator with its excess risk when the risk is strongly convex.

**Lemma 5.** If the risk is $2\alpha$-strongly convex, then
\[ \|\theta - \theta^*\|_2^2 \leq \alpha^{-1} \text{Risk}(\theta) \]
for all $\theta \in \mathbb{R}^d$.

**Proof.** Let $\theta \in \mathbb{R}^d$, by (SC) applied with $\theta_1 = \theta^*$ and $\theta_2 = \theta$, we get
\[ \|\theta - \theta^*\|_2^2 \leq \alpha^{-1} \mathbb{E}[\ell_i(\theta) - \ell_i(\theta^*)] \]
\[ + \alpha^{-1} \mathbb{E}[\nabla \ell_i(\theta^*)]^\top (\theta^* - \theta). \]

But, $\mathbb{E}[\nabla \ell_i(\theta^*)]^\top (\theta^* - \theta) \leq 0$. Otherwise, taking into account the convexity of the domain, the direction $\theta - \theta^*$ is a decreasing feasible direction, which contradicts the optimality of $\theta^*$.

B.2 Proof of Theorem 1

Let $(\delta_i)$ be a non-increasing sequence in $(0, 1)$ such that $\sum_{i=1}^{\infty} \delta_i \leq \delta$.

**Step 1.** Proof by induction that the subroutines always perform the optimization in the correct $\ell_1$-ball. We prove by induction on $i \geq 0$ that with probability at least $1 - \sum_{j=1}^{i} \delta_j$
\[ \|\theta^* - [\hat{\theta}_{i-1}]_{d_0}\|_1 \leq U^{2i/2}. \] (Hi)

$\mathcal{H}_0$ is satisfied by assumption since $\|\theta^*\|_1 \leq U$ and $[\hat{\theta}_{0-1}]_{d_0} = [\hat{\theta}_0]_{d_0} = 0$ (see SAEW for the definition of $[\hat{\theta}_0]$).

Let $i \geq 0$ and assume $(\mathcal{H}_i)$. The following Lemma (whose proof is postponed to Appendix B.3) states that the gradients are indeed upper-bounded by $B$ in sup-norm.

**Lemma 6.** Let $i \geq 0$. Under $(\mathcal{H}_i)$, for all $t \in [t_i, t_{i+1} - 1]$, $\|\nabla \ell_i(\hat{\theta}_{i-1})\|_\infty \leq B$ almost surely.

Therefore, from the regret bound (5), the subroutine $\mathcal{S}_i$ satisfies for all $t \in [t_i, t_{i+1} - 1]$
\[ \sum_{s=t_i}^{t} \ell_s(\hat{\theta}_{s-1}) - \ell_s(\theta^*) \]
\[ \leq U^{2i/2} \left( a \sum_{s=t_i}^{t} \|\nabla \ell_s(\hat{\theta}_{s-1})\|_\infty^2 + bB \right). \]

Bounding the cumulative risk with the regret thanks to Theorem 10 in Appendix C.2, it yields with probability at least $1 - \sum_{j=1}^{i} \delta_j$,
\[ \sum_{s=t_i}^{t} \mathbb{E}[\ell_s(\hat{\theta}_{s-1})] - \mathbb{E}[\ell_s(\theta^*)] \leq U^{2i/2} \text{Err}_t \] (8)
where $\text{Err}_t := a' \sqrt{\sum_{t_i}^{t} \|\nabla \ell_s(\hat{\theta}_{s-1})\|_\infty^2 + b'B}$ with
\[ a' := a + \sqrt{2 \log \left( 1 + \frac{1}{2} \log \left( \frac{t - t_i + 1}{2} \right) \right)} - \log \delta_{i+1}, \]
and
\[ b' := b + \frac{1}{2} + \log \left( 1 + \frac{1}{2} \log \left( \frac{t - t_i + 1}{2} \right) \right) - \log \delta_{i+1}. \] (9)

Thus, recalling that by definition (see SAEW)
\[ \hat{\theta}_t := (t - t_i + 1)^{-1} \sum_{s=t_i}^{t} \hat{\theta}_{s-1}, \]
and because the losses are i.i.d., Jensen’s inequality yields
\[ \text{Risk}(\hat{\theta}_t) = \mathbb{E}[\ell_{t+1}(\hat{\theta}_t)] - \mathbb{E}[\ell_{t+1}(\theta^*)] \]
\[ \leq \mathbb{E} \left[ (t - t_i + 1) \sum_{s=t_i}^{t} \mathbb{E}[\ell_s(\hat{\theta}_{s-1})] - \mathbb{E}[\ell_s(\theta^*)] \right] \]
\[ \leq U \text{Err}_t \]
\[ \leq \frac{U}{2^{2/2}(t - t_i + 1)}. \] (11)

Together with the strong convexity of the risk (Lemma 5), this entails
\[ \|\hat{\theta}_t - \theta^*\|_2^2 \leq \frac{U \text{Err}_t}{\alpha^{2/2}(t - t_i + 1)}. \] (12)

We thus control the $\ell_2$-error of $\hat{\theta}_t$. However, in order to control the $\ell_1$-error without paying a factor $d$, we need to truncate some coordinates of $\hat{\theta}_t$. By definition of $[\hat{\theta}_t]_{d_0}$ (see SAEW), we have
\[ [\hat{\theta}_t]_{d_0} \in \arg \min_{\theta \in \mathbb{R}^d \|\theta\|_0 \leq d_0} \{\|\theta - \theta\|_2\}. \] (13)

Now, (13) together with $\|\theta^*\|_0 \leq d_0$ (by assumption) yields
\[ \|\hat{\theta}_t - [\hat{\theta}_t]_{d_0}\|_2 \leq \|\hat{\theta}_t - \theta^*\|_2. \] (14)
Furthermore, because both $\|\theta^*\|_0 \leq d_0$ and $\|\hat{\theta}_1\|_0 \leq d_0$, we have
\[
\|\hat{\theta}_1\|_0 - \theta^* \|_0 \leq 2d_0.
\] (15)
Therefore, with probability at least $1 - \sum_{j=0}^{i+1} \delta_j$, we have
\[
\|\hat{\theta}_1 - \theta^* \|_0 \leq \sqrt{2d_0 \|\hat{\theta}_1 - \theta^* \|_2^2} \leq \sqrt{2d_0 (\|\hat{\theta}_1 - \theta^* \|_2 + \|\theta_1 - \theta^* \|_2)} \leq \sqrt{2d_0 \|\theta_1 - \theta^* \|_2^2} \leq 2\sqrt{2d_0 \|\theta_1 - \theta^* \|_2} \leq 2\sqrt{2d_0 \|\theta_1 - \theta^* \|_2} =: \varepsilon_t,
\] (16)
where the last equality holds by definition of $\varepsilon_t$ (see SAEW). Finally, $(H_{i+1})$ is fulfilled by definition of $\varepsilon_{t+1}$ (see SAEW), which satisfies $\varepsilon_{t+1} - \varepsilon_t \leq U^2-2^{-i+1}$. The induction is thus completed.

In the rest of the proof, we consider that $(H_i)$ are satisfied for all $i \geq 0$. This occurs with probability $1 - \sum_{j=1}^{\infty} \delta_j \geq 1 - \delta$ as stated by Step 1.

**Step 2. Fast rate for the excess risk of $\hat{\theta}_1$.** First, we prove that the excess risk of $\hat{\theta}_1$ is upper-bounded as
\[
\text{Risk}(\hat{\theta}_1) \leq \frac{d_0 B^2}{\alpha} \left( \frac{27a^2}{t} + \frac{211b^2}{t^2} \right) + 2\alpha U^2 \frac{d_0 t^2}{d_0 t^2},
\] (17)
for all $t \geq 1$, where $a' = a_{[2 \log_2 t]}$ and $b' = b_{[2 \log_2 t]}$.

To do so, we start from the risk inequality (11). From the definition of $\varepsilon_t$ (see (16)), we get
\[
\text{Risk}(\hat{\theta}_1) \leq \frac{\alpha \varepsilon_t^2}{8d_0}, \quad t \geq 1.
\] (18)

Thus, by definition of $\hat{\theta}_1 := \theta_1 \text{arg min}_{s \leq t, \varepsilon_t} \varepsilon_t$, we have
\[
\text{Risk}(\hat{\theta}_1) \leq \frac{\alpha \min_{s \leq t} \varepsilon_s^2}{8d_0}.
\] (19)

We conclude the proof of (17) with the following lemma proved in Appendix B.4

**Lemma 7.** Let $i \geq 0$. Let $t_i - 1 \leq t \leq t_{i+1}$, then
\[
\min_{s \leq t} \varepsilon_s \leq U \left( \frac{\sqrt{2} \gamma a'_s}{\sqrt{t}} + \frac{2 + 4 \gamma b'_s}{t} \right),
\]
where $\gamma := 2^i d_0 B/(\alpha U)$.

Let $i \geq 0$ such that $t_i - 1 \leq t \leq t_{i+1}$. Lemma 7 together with (19) and $(x+y)^2 \leq 2x^2 + 2y^2$ for $x, y \geq 0$, yields
\[
\text{Risk}(\hat{\theta}_1) \leq \frac{\alpha U^2 \gamma^2}{8d_0} \left( \frac{\sqrt{2} a'_s}{\sqrt{t}} + \frac{2 \gamma - 1 + 4b'_s}{t} \right)^2
\] (20)
\[
\leq \frac{\alpha U^2 \gamma^2}{d_0} \left( \frac{a'_s^2}{2t} + \frac{2 \gamma - 1 + 4b'_s}{t^2} \right).
\] (21)

Now, remark that if $i \geq 2 \log t$, then $\varepsilon_{t_i - 1} \leq U^2 - i \leq U/t$ and from (19), $\text{Risk}(\hat{\theta}_1) \leq \alpha U^2/(8d_0 t^2)$. Together, with (21), we get
\[
\text{Risk}(\hat{\theta}_1) \leq \frac{\alpha U^2 \gamma^2}{d_0} \left( \frac{a'_s^2}{2t} + \frac{2 \gamma - 1 + 4b'_s}{t^2} \right),
\]
with $a' = a_{[2 \log_2 t]}$ and $b' = b_{[2 \log_2 t]}$. Substituting $\gamma = 2^i d_0 B/(\alpha U)$ concludes the proof of Inequality (17).

**Step 3. Slow rate for the excess risk of $\hat{\theta}_1$.** Now, we prove that
\[
\text{Risk}(\hat{\theta}_1) \leq UB \left( \frac{a'_s}{\sqrt{t/2}} + \frac{4b'_s}{t^2} \right) + \alpha U^2 \frac{d_0}{8d_0 t^2}, \quad t \geq 1.
\] (22)

For small values of $t$, the slow rate will be satisfied from the initial bound of the subroutine during the first session. At some time $\tau > 0$, the fast rate becomes better than the slow rate. This splitting time is defined as the solution of the equality
\[
\frac{\text{Err}_{t_i - 1} - 1}{t_i - 1} = B \left( \frac{\sqrt{2} a'_s}{\sqrt{\tau}} + \frac{2 \gamma - 1 + 4b'_s}{\tau} \right).
\] (23)

Let $t \geq 1$. To control $\text{Risk}(\hat{\theta}_1)$, we distinguish three cases:

- if $t \leq t_i - 1$, then, since by definition of $\varepsilon_s$

\[
\arg \min_{s \leq t} \text{Err}_s = \arg \min_{s \leq t} \varepsilon_s,
\]

we get from Inequality (11) that
\[
\text{Risk}(\hat{\theta}_1) = \text{Risk}(\hat{\theta}_1 \text{arg min}_{s \leq t} \varepsilon_s)
\]
\[
\leq U^2 0/2 \frac{\text{Err}_s}{s} \leq U \frac{\text{Err}_t}{t}.
\]

By definition of $\text{Err}_t$ (see (8)) and upper-bounding the gradients by $B$, we get
\[
\text{Risk}(\hat{\theta}_1) \leq UB \left( \frac{a'_s}{\sqrt{t}} + \frac{b'_s}{t^2} \right).
\]

- if $t_i \leq t \leq \tau$, then following the same reasoning as above, we have

\[
\text{Risk}(\hat{\theta}_1) \leq UB \frac{\text{Err}_{t_i - 1}}{t_i - 1}.
\]
which yields by definition of $\tau$ (see Equality (23)) and by using $t \leq \tau$:
\[
\text{Risk}(\hat{\theta}_t) \leq UB \left( \frac{\sqrt{2a'}}{\sqrt{\tau}} + \frac{2\gamma^{-1} + 4b'}{\tau} \right) \\
\leq UB \left( \frac{\sqrt{2a'}}{\sqrt{\tau}} + \frac{2\gamma^{-1} + 4b'}{t} \right).
\]

- if $\tau \leq t$, since by definition of $t_1$ (see SAEW), $\varepsilon_{t_1-1} \leq U/2$, then by definition of $\varepsilon_{t_1-1}$ (see (16)),
\[
2\sqrt{2d_0\alpha^{-1}U} \frac{\text{Err}_{t_1-1}}{t_1-1} \leq \frac{U}{2},
\]

and thus taking the square and rearranging the terms
\[
\frac{d_0}{\alpha} \leq \frac{U}{2\varepsilon_{t_1-1}}.
\]

Using the definition of $\gamma = 2^4d_0B/(\alpha U)$ and substituting $\text{Err}_{t_1-1}$ with Equality (23), this yields
\[
a U^2\gamma^2 \leq \frac{2^5d_0B^2}{\alpha} \leq UB \left( \frac{\sqrt{2a'}}{\sqrt{\tau}} + \frac{2\gamma^{-1} + 4b'}{t} \right)^{-1}.
\]

Finally from Inequality (20), and using $\tau \leq t$
\[
\text{Risk}(\hat{\theta}_t) \leq UB \left( \frac{\sqrt{2a'}}{\sqrt{\tau}} + \frac{2\gamma^{-1} + 4b'}{t} \right).
\]

Combining the three cases together and substituting $\gamma = 2^4d_0B/(\alpha U)$, concludes the proof of Inequality (22).

**Step 4. Conclusion of the proof** Combining Inequalities (17) and (22), we get the risk inequality stated in the Theorem 1 for $\theta_t$. It only remains to choose $\delta_j = \delta/(j + 1)^2$ so that $\sum_{j=1}^{\infty} \delta_j \leq \delta$ and to control $a' = a'_{(2 \log_2 t)}$ and $b' = b'_{(2 \log_2 t)}$. From (9), we can use $\delta_{(2 \log_2 t) + 1} \leq \delta/(1 + 2 \log_2 t)$ and $T_1 \leq t$. Simple calculation yields that $a' - a$ is lower than
\[
\sqrt{2\left( \log\left( 1 + 1/2 \log(t/2) \right) - \log \delta + 2 \log(1 + 2 \log_2 t) \right)} \\
\leq \sqrt{6 \log(1 + 3 \log t) - 2 \log \delta}.
\]

Similarly, for $b' - b$. It is upper-bounded by
\[
\frac{1}{2} + \log\left( 1 + (1/2) \log(t/2) \right) - \log \delta + 2 \log(1 + 2 \log_2 t) \\
\leq 1/2 + 3 \log(1 + 3 \log t) - \log \delta.
\]

This concludes the proof.

**B.3 Proof of Lemma 6**

Since by assumption $B \geq \max_{t} \|\theta_t\|_{\infty} \leq 2U$ a.s. it suffices to show that $\|\hat{\theta}_t - \theta\| \leq 2U$. By definition of the session $S_t$,
\[
\hat{\theta}_{t-1} \in B_1([\hat{\theta}_{t-1}]_{d_0}, U2^{-t/2}).
\]

Thus:
- if $i = 0$, since $\hat{\theta}_0 \mid d_0 = 0$, $\|\hat{\theta}_{t-1}\|_1 \leq U$.
- if $i = 1$, since $\|\hat{\theta}_{t-1}\|_1 \leq U$ a truncated average of vectors in $B_1(0, U)$, we have
\[
\|\hat{\theta}_{t-1}\|_1 \leq \|\hat{\theta}_{t-1} - \hat{\theta}_{t-1}\|_1 + \|\hat{\theta}_{t-1}\|_1 \\
\leq U/\sqrt{2} + U \leq 2U;
\]
- otherwise, $i \geq 2$ and $\|\hat{\theta}_{t-1}\|_1$ is bounded by
\[
\|\hat{\theta}_{t-1} - \hat{\theta}_{t-1}\|_1 + \|\hat{\theta}_{t-1}\|_1 - \|\theta_t\|_1 + \|\theta_t\|_1 (\text{see } (\ref{eq:gamma})) \\
\leq U2^{-i/2} + U2^{-i/2} + U \leq 2U.
\]

Putting the tree cases together, $\|\hat{\theta}_{t-1}\|_1 \leq 2U$, which concludes the proof.

**B.4 Proof of Lemma 7**

It is enough to control $\varepsilon_{t_{j+1}} \geq \min_{s \leq t} \varepsilon_s$. To do so, we prove that for every $j \geq 0$, $T_j := t_{j+1} - t_j$ cannot be too large, so that at time $t$, $i$ will be at least of order $\log_2 t$.

Let $j \geq 0$. We can assume $t_{j+1} > t_j$, otherwise $T_j = 0$. Thus, from the bound on the gradients (Lemma 6) and from the definition of $\text{Err}_t$ (see (8)) for all $t \in [t_j + 1, t_{j+1})$,
\[
\text{Err}_{t_{j+1}} \leq B(a'_{\sqrt{T_j - t_j}} + b'_{T_j}),
\]

and from the definition of $\varepsilon_{t-1}$ (see (16))
\[
\varepsilon_{t_{j+1}} \leq 2\sqrt{2d_0\alpha^{-1}UB \frac{a'_{\sqrt{T_j - t_j}} + b'_{T_j}}{2^{j/2}(t_j - t_j)}}.
\]

By definition, $t_{j+1}$ is the smallest integer after $t_j$ satisfying $\varepsilon_{t_{j+1} - 1} \leq U2^{-(j+1)/2}$. Hence, we have $\varepsilon_{t_{j+1} - 2} \geq U2^{-(j+1)/2}$, which implies
\[
2\sqrt{2d_0\alpha^{-1}UB \frac{a'_{\sqrt{T_j - t_j}} + b'_{T_j}}{2^{j/2}(T_j - 1)}} \\
\Rightarrow 2^{j/2} 2^4d_0\alpha^{-1}UB \frac{(a'_{\sqrt{T_j - t_j}} + b'_{T_j})}{U2^{-j/2}} \\
\Rightarrow 2^{j/2} 2^4d_0\alpha^{-1}UB \frac{(a'_{\sqrt{T_j - t_j}} + b'_{T_j})}{U2^{-j/2}} \geq T_j - 1
\]

Then, by solving a second order equation in $\sqrt{T_j - 1}$ (see for instance Gaillard et al., 2014, Lemma 10), the above inequality entails
\[
T_j \leq 1 + 2^j \gamma^2 a'_{T_j} + 2^j \gamma b'_{T_j}.
\]
Therefore, summing over \( j = 0, \ldots, i \)
\[
t_{i+1} = \mathcal{Y} + \sum_{j=0}^{i} T_j
\]
\[
\leq \sum_{j=0}^{i} (1 + 2j^2 a_j^2 + 2j/2 \gamma b_j)
\]
\[
\leq 2^{1+i} \gamma a_i^2 + (1 + \sqrt{2}) 2^{(i+1)/2} b_i' + i + 1
\]
\[
\leq 2^{1+i} \gamma a_i^2 + 2^{(i+1)/2} (2 \gamma b_i' + 1),
\]
where the last inequality is because \( 2^{(i+1)/2} \geq \sqrt{2}(i + 1) \) for \( i \geq 0 \). Solving the second-order inequality in \( 2^{(i+1)/2} \) we get
\[
2^{-(i+1)/2} \leq \frac{\gamma a_i'}{t_{i+1}} + \sqrt{2} \frac{1 + 2 \gamma b_i'}{t_{i+1}}.
\]
Thus, since \( \varepsilon_{t_{i-1}} \leq U 2^{-i/2} \), we have
\[
\varepsilon_{t_{i-1}} \leq U \gamma \left( \frac{\sqrt{2} a_i'}{\gamma t_{i+1}} + \frac{2 \gamma - 1 + 4 \gamma b_i'}{t_{i+1}} \right).
\]
The proof of Lemma 7 finally follows using \( t \leq t_{i+1} \).

### B.5 Proof of Theorem 2

With probability \( 1 - \delta \), all inequalities provided in the proof of Theorem 1 are satisfied. We also consider the notation of the previous proof. Let \( t \geq 1 \).

#### Step 1. Slow rate
We remark that for any \( i \geq 0 \),
\[
\sum_{s=t_{i+1}}^{t} E[\ell_s]_{(\hat{\theta}_{s-1})} - E[\ell_s]_{(\theta^*)} \leq U 2^{-i/2} \text{Err}_{t_{i+1}-1} \leq UB 2^{-i/2} (a'_t \sqrt{t} + b'_t)
\]
(24)
where, in the last inequality, we use that \( (t_{i+1} - 1) \wedge t \leq t \) and \( t_i \geq 1 \). We will use this inequality for \( i \leq [2 \log t] \). For \( i \geq [2 \log t] \), we use the fact that the gradients are bounded by \( B \), so that by convexity of the risk
\[
\sum_{s=t_{i+1}}^{t} E[\ell_s]_{(\hat{\theta}_{s-1})} - E[\ell_s]_{(\theta^*)} \leq \sum_{s=t_{i+1}}^{t} \|E[\nabla \ell_s]_{(\hat{\theta}_{s-1})}\|_\infty \|\hat{\theta}_{s-1} - \theta^*\|_1
\]
\[
\leq U B 2^{-i/2} t.
\]
(27)
Summing (26) over \( i = 0, \ldots, [2 \log t] \) and (27) over \( i = [2 \log t], \ldots, \infty \), we get
\[
\text{Risk}_{t} : (\hat{\theta}_{(i+1)}) := \sum_{s=1}^{t} E[\ell_s]_{(\hat{\theta}_{s-1})} - E[\ell_s]_{(\theta^*)}
\]
\[
\leq UBt \sum_{i=0}^{[2 \log t]} 2^{-i/2} (a'_i \sqrt{t} + b'_i)
\]
\[
+ U B t \sum_{i=[2 \log t]}^{\infty} 2^{-i/2}.
\]
(28)
The second sum is controlled as
\[
\sum_{i=[2 \log t]}^{\infty} 2^{-i/2} \leq t^{-1} \sum_{i=0}^{\infty} 2^{-i/2}.
\]
Thus, since \( \sum_{i=0}^{\infty} 2^{-i/2} = 2 + \sqrt{2} \leq 4 \), we have
\[
\text{Risk}_{t} : (\hat{\theta}_{(i+1)}) \leq 4 UB (a'_1 \sqrt{t} + b'_1) + 4 UB,
\]
where we recall that \( a'_i = a'_{[2 \log t]} \), and \( b'_i = b'_{[2 \log t]} \). This concludes Step 1.

#### Step 2. Fast rate
Let us now prove the fast rate
\[
\text{Risk}_{t} : (\hat{\theta}_{(i+1)}) \leq \frac{25 d_0 B^2}{\alpha} U^2 \log t
\]
\[
+ 4BU (1 + b') + U^2 \frac{\alpha}{8d_0},
\]
for all \( t \geq 1 \).

First, we remark that similarly to (18), we get for all \( i \geq 0 \)
\[
\sum_{s=t_i}^{t_{i+1}} E[\ell_s]_{(\hat{\theta}_{s-1})} - E[\ell_s]_{(\theta^*)} \leq \frac{U \text{Err}_{t_{i+1}-1} T_i}{2^{-i/2} T_i}
\]
(16)
\[
\leq \frac{\alpha T_i}{8d_0} T_i
\]
\[
\leq \alpha U 2^{-i/2} \frac{T_i}{16d_0},
\]
(29)
where the last inequality is because \( \varepsilon_{t_{i+1}-1} \leq U 2^{-(i+1)/2} \) by definition of \( t_{i+1} \) (see SAEW). We will use this inequality for \( i \leq [2 \log t] \). Summing (29) over \( i = 0, \ldots, [2 \log t] \) and (27) over \( i = [2 \log t], \ldots, \infty \), we get
\[
\text{Risk}_{t}: (\hat{\theta}_{(i+1)}) := \sum_{s=1}^{t} E[\ell_s]_{(\hat{\theta}_{s-1})} - E[\ell_s]_{(\theta^*)}
\]
\[
\leq \sum_{i=1}^{[2 \log t]} U^2 \alpha \frac{2 \log t}{2 d_0} \sum_{i=0}^{2^{-i/2} t_i} 2^{-i/2} T_i
\]
\[
+ U B t \sum_{i=[2 \log t]}^{\infty} 2^{-i/2}.
\]
(30)
We upper bound both sums. The second one is controlled as we did for (28). The first one is upper-bounded thanks to (25)
\[
\sum_{i=0}^{[2 \log t]} 2^{-i/2} T_i \leq \sum_{i=0}^{[2 \log t]} \left( \gamma^2 a_i^2 + 2^{-i/2} \gamma b_i' + 2^{-i} \right).
\]
\[
\sum_{i=0}^{[2 \log t]} 2^{-i/2} T_i \leq \sum_{i=0}^{[2 \log t]} \left( \gamma^2 a_i^2 + 2^{-i/2} \gamma b_i' + 2^{-i} \right).
\]
that

\[ U^2 \alpha \left( 2\gamma^2 a^2 \log_2 t + 4\gamma b' + 2 \right) + 4UB, \]

which, by substituting \( \gamma = 2^4d_0B/(\alpha U) \), is equal to

\[ \frac{2^5d_0B^2}{\alpha} a^2 \log_2 t + 4BU(1 + b') + \frac{\alpha U^2}{8d_0}. \]

This concludes the proof.

### B.6 Proof of Theorem 3

Let first check that we are indeed in the setting of Theorem 1. The risk is strongly convex because for any \( \theta_1, \theta_2 \in \mathbb{R}^d \)

\[ E[\ell_t(\theta_1) - \ell_t(\theta_2)] = E[(Y_t - X_t^\top \theta_1)^2 - (Y_t - X_t^\top \theta_2)^2] \]

\[ = E[2(Y_t - X_t^\top \theta_1)X_t^\top (\theta_1 - \theta_2) - (Y_t^\top \theta_2^2)] \]

\[ = \nabla E[\ell_t(\theta_1)\top (\theta_1 - \theta_2) - (\theta_1 - \theta_2)\top \nabla E[X_tX_t^\top] (\theta_1 - \theta_2). \]

Assumption (SC) is thus satisfied with \( \alpha = \lambda_{\text{min}}(E[X_tX_t^\top]). \) Besides, for all \( \theta \) such that \( \|\theta\|_1 \leq 2U \), we have

\[ \|\nabla \ell_t(\theta)\|_\infty = \|2(Y_t - X_t^\top \theta)X_t\|_\infty \leq 2(Y + 2UX)X = B. \]

Now, we mimic the proof of Theorem 1. In the rest of the proof, we consider that \( (H_t) \) are satisfied for all \( i \geq 0 \). This occurs with probability \( 1 - \delta \) and all inequalities stated in the proof of Theorem 1 are satisfied.

The proof is based on the following Lemma that we substitute to Inequality (24) from the proof of Theorem 1.

**Lemma 8.** For all \( t \in [t_i, t_{i+1} - 1], \) with probability \( 1 - \delta_{i+1} \),

\[ \text{Err}_{t-1} \leq 2\sqrt{2}X\sigma a'_i(\sqrt{t - t_i} + B\delta'_i), \]

where \( \delta'_i := b'_i + a'_i(\sqrt{\log \delta_{i+1}^{-1} + \sqrt{2}b + 2a}) \). We recall that \( \text{Err}_{t-1} \) is defined in (8).

**Proof of Lemma 8.** In the particular case of the square loss, the gradients are given by \( \nabla \ell_t(\theta) = 2X_t(X_t^\top \theta - Y_t) \), so that

\[ \|\nabla \ell_t(\theta_{t-1})\|_\infty \leq 4X^2 e^{t}{\ell}_t(\theta_{t-1}). \]  

Following Gerchinovitz, 2013, Corollary 2.2, we get from Inequality (5) that

\[ \sum_{t=t_{\text{start}}}^{t_{\text{end}}} \ell_t(\theta_{t-1}) - \ell_t(\theta^*) \leq 2aUX \sqrt{\sum_{t=t_{\text{start}}}^{t_{\text{end}}} \ell_t(\theta_{t-1}) + bUB}. \]

Solving the second-order inequality (see Gaillard et al., 2014, Lemma 10), it yields the improvement for small losses

\[ \sqrt{\sum_{t=t_{\text{start}}}^{t_{\text{end}}} \ell_t(\theta_{t-1})} \leq \sqrt{\sum_{t=t_{\text{start}}}^{t_{\text{end}}} \ell_t(\theta^*) + \sqrt{bUB} + 2aUX}. \]

Thus, from (31),

\[ \sqrt{\sum_{s=t_i}^{t_1} \ell_s(\theta_{s-1})} \leq 2X \sqrt{\sum_{s=t_i}^{t_1} \ell_s(\theta^*) + 2X \sqrt{bUB} + 4aUX}. \]

But, with probability \( 1 - \delta_{i+1}, \) we have from Theorem 9

\[ \sum_{s=t_i}^{t_1} \ell_s(\theta^*) \leq (e - 1) \sum_{s=t_i}^{t_1} E[\ell_s(\theta^*)] + (Y + UX)^2 \log \delta_{i+1}^{-1} \leq 2\sigma^2(t - t_i) + (Y + UX)^2 \log \delta_{i+1}^{-1}, \]

where \( \sigma^2 = E[\ell_t(\theta^*)]. \) Plugging into the previous inequality and using \( \sqrt{x + y} \leq \sqrt{x} + \sqrt{y} \) for \( x, y > 0, \) this yields

\[ 2^{-1}X^{-1} \sqrt{\sum_{s=t_i}^{t_1} \|\nabla \ell_s(\theta_{s-1})\|_\infty^2} \leq \sqrt{2\sigma \sqrt{t - t_i} + (Y + UX) \sqrt{\log \delta_{i+1}^{-1} + \sqrt{bUB} + 2aUX \leq \sqrt{2\sigma \sqrt{t - t_i} + 2^{1/2}B X^{-1} \left( \sqrt{\log \delta_{i+1}^{-1} + \sqrt{2b + 2a}} \right), \]} \]

where the second inequality is because \( B/(2X) \geq (Y + UX) \geq UX. \) The proof of Lemma 8 is concluded by using the definition of \( \text{Err}_{t-1} \) (see (8)).

The proof of Theorem 3 is then completed following the one of Theorem 1 by using Lemma 8 instead of Inequality (24). Finally, it only suffices to substitute \( ba'_i \) with \( 2\sqrt{2}X\sigma a'_i \) and \( b'_i \) with \( c'_i \) in the final results. At the end, \( b' \) of Theorem 1 must thus be substituted with

\[ c' := b' + a'(\sqrt{2\log(1 + 2 \log_2 T)} - \log \delta + \sqrt{2b + 4a}) \leq \frac{1}{2} + b + 3\log(1 + 3 \log T) - \log \delta \]

\[ + (a + \sqrt{6\log(1 + 3 \log T)} - 2\log \delta \]

\[ \leq \frac{1}{2} + b + 3\log(1 + 3 \log T) - \log \delta + 4a^2 \]

\[ + 2b + 6\log(1 + 3 \log T) - 2\log \delta \]

\[ \leq \frac{1}{2} + 3b + 4a^2 + 9\log(1 + 3 \log T) - 3\log \delta \cdot \]

\[ \leq 1 + b + a^2 + \log T - \log \delta \]
However, in contrast to the bound $B$ on the gradients, Lemma 8 only holds with probability $1 - \delta_i + \epsilon$ (instead of almost surely). A union bound over all events states that the final result only holds with probability $1 - \delta - \sum_{i=1}^\infty \delta_i + \epsilon = 1 - 2\delta$. To get a result with probability $1 - \delta$, $\delta$ must thus be multiplied by 2 in the results.

This gives us, from the risk bound of Theorem 1, with probability $1 - \delta$, $\text{Risk}(\hat{\theta}_t)$ is upper-bounded by

$$\min \left\{ 4U \left( \frac{X\sigma'}{\sqrt{T}} + \frac{Bc'}{T} \right) + \frac{\alpha U^2}{8d_0T}, \frac{d_0}{\alpha} \left( \frac{210X^2\sigma^2a^2}{T} + \frac{211B^2c'^2}{T^2} \right) + \frac{2\alpha U^2}{d_0T^2} \right\},$$

where $a' = 2a + 2\sqrt{6 \log(1 + 3 \log T) + 2 \log(2/\delta)}$ and $c' = 1 + 3b + 4a^2 + 9 \log(1 + 3 \log T) + 3 \log(2/\delta)$.

The bound of the theorem is then obtained by using that $B = 2X(Y + 2XU)$.

### B.7 Proof of Theorem 4

For the sake of clarity, we only perform this proof up to universal constants. Let $B^* = 2X(Y + 2X\|\theta^*\|_1) \geq \max_{\theta \in \Theta}\|\nabla f_t(\theta)\|_\infty$ almost surely. We also define by $\alpha^*$ the maximal number strong convexity parameter that satisfies (SC).

Let $T \geq 1$. Then, by definition (see Alg. 3), $\hat{f}_{T-1} = \hat{f}_j$ for $j = \lfloor \log_2 T \rfloor - 1$.

We aim at controlling the excess risk of the average estimator $\hat{f}_j = \sum_{t=2^j}^{2^{j+1}-1} \hat{f}_t$. To do so, we control the cumulative risk for $t = 2^j, \ldots, 2^{j+1} - 1$

$$\text{Risk}^{(j)} := \sum_{t=2^j}^{2^{j+1}-1} \mathbb{E}_{t-1}\left[ (Y_t - \hat{f}_t(X_t))^2 \right] - \mathbb{E}\left[ (Y_t - X_t^T\theta^*)^2 \right],$$

where $\mathbb{E}_{t-1}[\cdot] = \mathbb{E}\left[ \cdot | (X_1, Y_1), \ldots, (X_{t-1}, Y_{t-1}) \right]$. We will use that

$$\text{Risk}(\hat{f}_j) \leq \text{Risk}^{(j)} 2^{-j} \lesssim \frac{\text{Risk}^{(j)}}{T}. \quad \text{(34)}$$

We first prove that it exists a predictor $f_{p,\epsilon}$ with $\epsilon \in G_j$ that has a small excess risk. Then, we will apply Theorem 4.5 of Wintenberger, 2014 to show that BOA almost achieves this performance.

**Step 1.** Either it exists a predictor $f_{p,\epsilon}$ with small excess risk or $\text{Risk}^{(j)}$ is small. Since all predictions $\hat{f}_t(X_t)$ lie in $[-Y, Y]$ almost surely,

$$\text{Risk}^{(j)} \leq Y^2 2^j \leq Y^2 T. \quad \text{(35)}$$

Let $d_0$ in $G_j$ (i.e., a power of 2) such that $d_0/2 \leq \|\theta^*\|_0 \leq d_0$. We show that if the conditions of Theorem 3 cannot be satisfied with any parameter of the grid $G_j$, the cumulative risk $\text{Risk}^{(j)}$ is small enough. We start with the choice of the parameter $U$, which should be of order $\|\theta^*\|_1$:

a) If $\|\theta^*\|_1 \leq 2^{-2j}$. It exists a predictor in $G_j$ such that $f_{p,\epsilon} = 0$ (consider $d_0 = 0$). In this case,

$$\text{Risk}(f_{p,\epsilon}) = \mathbb{E}[\| Y_t - 0 \|^2] \leq B^* \|\theta^*\|_1 \lesssim B^* 2^{-2j} \lesssim B^* T^{-2},$$

where we used that $2^{-j} \lesssim T^{-1}$.

b) If $\|\theta^*\|_1 > 2^{j+[2\log Y]}$, then $2^{j} \leq Y^{-2} \|\theta^*\|_1 \lesssim 2^{-j}$ and from Inequality (35),

$$\text{Risk}^{(j)} \lesssim \|\theta^*\|_1 2^{-j} \lesssim \|\theta^*\|_0(B^*)^2 \lesssim \frac{\|\theta^*\|_1}{T^2} \lesssim \frac{\|\theta^*\|_0(B^*)^2}{\alpha T^2}.$$
Putting everything together, either (for cases b), d), and e))
\[
\text{Risk}^{(j)} \lesssim \left( B^* + \frac{\|\theta^*\|_0 (B^*)^2}{\alpha^*} \right) T^{-1} \tag{36}
\]
or, for cases a), c), and f), there exists \( p \in \mathcal{G}_j \) such that with high probability
\[
\text{Risk}(f_{p,j}) \lesssim \min \left\{ \frac{1}{\gamma} \left( \frac{X^2 \sigma^2 a^2}{T} + \frac{(B^* c')^2}{T^2} \right) + \gamma \|\theta^*\|^2 \right\}
\]
or
\[
\|\theta^*\|_1 X \left( \frac{\sigma a'}{\sqrt{T}} + \frac{Y + X \|\theta^*\| c'}{T} \right) + \gamma \|\theta^*\|^2 \right\} + \frac{B^*}{T^2}, \tag{37}
\]

**Step 2. Bound of the meta-algorithm.** Using that the square loss is 4\(Y\)-Lipschitz over the domain \([-Y, Y]\) and 2-convex strongly, we can apply Theorem 4.5 of Wintenberger, 2014 with \(C_b = 4Y, C_f = 2\), and \(M = \#\mathcal{G}_j\). We get that with high enough probability
\[
\text{Risk}^{(j)} \lesssim T \min_{p \in \mathcal{G}_j} \text{Risk}(f_{p,j}) + Y^2 (\log \#\mathcal{G}_j + \log(\log T + \log Y) - \log \delta).
\]

Substituting
\[
\#\mathcal{G}_j \lesssim (j + \log Y)^3 \log d \lesssim (\log T + \log Y)^3 \log d,
\]
this yields
\[
\text{Risk}^{(j)} \lesssim \frac{\min_{p \in \mathcal{G}_j} \text{Risk}(f_{p,j})}{T} + Y^2 (\log \log d + \log(\log T + \log Y) - \log \delta).
\]

Combining with Inequality (37), we obtain that \(\text{Risk}^{(j)}\) is at most of order
\[
\text{Risk}^{(j)} \lesssim Y^2 (\log \log d + \log(\log T + \log Y) - \log \delta) + \min \left\{ \frac{1}{\gamma} \left( \frac{X^2 \sigma^2 a^2}{T} + \frac{(B^* c')^2}{T^2} \right) + \gamma \|\theta^*\|^2 \right\}
\]
or
\[
\|\theta^*\|_1 X \left( \frac{\sigma a'}{\sqrt{T}} + \frac{Y + X \|\theta^*\| c'}{T} \right) + \gamma \|\theta^*\|^2 \right\} + \frac{B^*}{T^2}.
\]

Finally, using Inequality (34), keeping only the main asymptotic term in \(1/T\), and substituting \(a' \lesssim \log((d \log T)/\delta)\) concludes the proof.

**C.1 Poissonian inequality**

First, we prove a Poissonian inequality which only works for nonnegative increments.

**Theorem 9.** Let \(T \geq 1\). Let \((X_t)_{t \geq 1}\) be a sequence of random variables such that \(X_t \in [0, B]\) almost surely, then with probability at least \(1 - \delta\)
\[
\sum_{t=1}^{T} X_t \leq (e - 1) \sum_{t=1}^{T} \mathbb{E}_{t-1}[X_t] + B \log(1/\delta).
\]

**Proof.** Let \(Z_t = X_t/B \in [0, 1]\). From Cesa-Bianchi and Lugosi, 2006, Lemma A.3, for all \(t \geq 1\), and all \(s > 0\)
\[
\mathbb{E}_{t-1} \left[ \exp \left( s Z_t - (e^s - 1) \mathbb{E}_{t-1}[Z_t] \right) \right] \leq 1.
\]

Thus,
\[
\mathbb{E} \left[ \exp \left( s \sum_{t=1}^{T} Z_t - (e^s - 1) \sum_{t=1}^{T} \mathbb{E}_{t-1}[Z_t] \right) \right]
\]
\[
= \mathbb{E} \left[ \mathbb{E}_{T-1} \left[ \exp \left( s Z_T - (e^s - 1) \mathbb{E}_{T-1}[Z_T] \right) \right] \right]
\]
\[
\leq \mathbb{E} \left[ \exp \left( s \sum_{t=1}^{T-1} Z_t - (e^s - 1) \sum_{t=1}^{T-1} \mathbb{E}_{t-1}[Z_t] \right) \right]
\]
\[
\leq \mathbb{E} \left[ \exp \left( s \sum_{t=1}^{T-1} Z_t - (e^s - 1) \sum_{t=1}^{T-1} \mathbb{E}_{t-1}[Z_t] \right) \right].
\]

By induction, we get
\[
\mathbb{E} \left[ \exp \left( s \sum_{t=1}^{T} Z_t - (e^s - 1) \sum_{t=1}^{T} \mathbb{E}_{t-1}[Z_t] \right) \right] \leq 1.
\]

We conclude thanks to Markov’s inequality, with probability at least \(1 - \delta\)
\[
\sum_{t=1}^{T} Z_t \leq \frac{e^s - 1}{s} \sum_{t=1}^{T} \mathbb{E}_{t-1}[Z_t] + \frac{1}{s} \log(1/\delta).
\]

The final result is obtained by substituting \(Z_t = X_t/B\) and by choosing \(s = 1\).

**C.2 From cumulative regret to cumulative risk**

**Theorem 10.** Let \(T > 0\). Assume \(\theta^* \in \mathcal{B}_1(\theta_{center}, \varepsilon)\). The cumulative risk of any convex optimization procedure in \(\mathcal{B}_1(\theta_{center}, \varepsilon)\) satisfies, with probability \(1 - \delta\)
\[
\text{Risk}_{1:T}(\hat{\theta}_{0:(T-1)}) - \text{Reg}_{1:T}(\hat{\theta}_{0:(T-1)}) \leq \varepsilon \sqrt{2 \log \frac{2 + \log(T/2)}{2 \delta} \left( \sum_{t=1}^{T} \|\nabla \ell_t(\hat{\theta}_{t-1})\|_{\infty}^2 \right)}.
\]
where $B \geq \max_{\theta \in B_1(\theta_{\text{center}}, \varepsilon)} \|\nabla \ell_t(\hat{\theta}_{t-1})\|_\infty$ almost surely.

Proof. This is a consequence of Theorem 4.1 of Wintenberger, 2014. Let $\delta \in (0, 1)$ and $(\eta_t)_{t \geq 0}$ be a sequence adapted to the filtration $(F_t = \{\ell_1, \ldots, \ell_{t-1}\})_{t \geq 0}$. Then, with the notation $\ell_{j,t}^2 \leq \varepsilon^2 \|\nabla \ell_t(\hat{\theta}_{t-1})\|_\infty^2$, applying Theorem 4.1 of Wintenberger (2014), we get that with probability $1 - \delta$

$$R_T := \text{Risk}_{1:T}(\hat{\theta}_{0:(T-1)}) - \text{Reg}_{1:T}(\hat{\theta}_{0:(T-1)})$$

\[\leq \varepsilon^2 \sum_{t=1}^T \eta_{t-1} \|\nabla \ell_t(\hat{\theta}_{t-1})\|_\infty^2\]

\[+ \log \left(1 + \mathbb{E}[\log(\eta_t/\eta_T)]\right) - \log \frac{\eta_T}{\eta_T}, \quad (38)\]

where $\text{Risk}_{1:T}(\hat{\theta}_{0:(T-1)}) := \sum_{t=1}^T \mathbb{E}[\ell_t(\hat{\theta}_{t-1}) - \ell_t(\theta^*)]$ and $\text{Reg}_{1:T}(\hat{\theta}_{0:(T-1)}) := \sum_{t=1}^T \ell_t(\hat{\theta}_{t-1}) - \ell_t(\theta^*)$.

We obtain the stated inequality from (38), by properly setting the tuning parameters

$$\eta_t := \frac{1}{\varepsilon} \min \left\{ \frac{1}{B} \cdot \frac{c\Gamma}{V_{t-1}} \right\},$$

where $c$ will be set by the analysis and

$$\Gamma := \sqrt{\log(1 + \log(\sqrt{T}/c)) - \log \delta},$$

and

$$V_{t-1} := \sqrt{\sum_{s=1}^{t-1} \|\nabla \ell_s(\hat{\theta}_{s-1})\|_\infty^2}.\]

Indeed, first we use that that $\eta_t/\eta_T \leq \sqrt{T}/c$ so that $\mathbb{E}[\log(\eta_t/\eta_T)] \leq \log(\sqrt{T}/c)$. Then, similarly to the proof of Cesa-Bianchi et al., 2007, Theorem 5, we can show that the first term in the right-hand side of (38) is upper-bounded as

$$\sum_{t=1}^T \eta_{t-1} \|\nabla \ell_t(\hat{\theta}_{t-1})\|_\infty^2 \leq \frac{B}{2\varepsilon} + \frac{c\Gamma}{2\varepsilon^2} V_T.$$

But, by definition of $\eta_T$, the second term is also controlled as

$$\log \left(1 + \mathbb{E}[\log(\eta_t/\eta_T)]\right) - \log \frac{\eta_T}{\eta_T} \leq \varepsilon \Gamma \max \left\{ B\Gamma, \frac{1}{c} V_T \right\}.$$

Plugging these two last inequalities into (38) leads to

$$R_T \leq \frac{B\varepsilon}{2} + \frac{c\Gamma\varepsilon}{2} V_T + \varepsilon \Gamma \max \left\{ B\Gamma, \frac{V_T}{c} \right\}.$$

We then need to distinguish two cases

- if $c\Gamma B \leq V_T$, then optimizing in $c = \sqrt{2}$

$$R_T \leq \frac{B\varepsilon}{2} + \left(\frac{c}{2} + \frac{1}{c}\right)\Gamma\varepsilon V_T \leq \frac{B\varepsilon}{2} + \sqrt{2}\Gamma\varepsilon V_T$$

- if $c\Gamma B \geq V_T$, then

$$R_T \leq \frac{B\varepsilon}{2} + \frac{1}{\sqrt{2}} \Gamma\varepsilon V_T + \varepsilon B\Gamma^2.$$

Therefore, putting the two cases together

$$R_T \leq \frac{B\varepsilon}{2} + \sqrt{2}\Gamma\varepsilon V_T + \varepsilon B\Gamma^2.$$

We conclude the proof by substituting $\Gamma$ and $V_T$ with their definitions. \hfill \Box