A Proofs of technical results

A.1 Proof of Proposition 3.1

Lemma A.1 *There exists a finite family of polytopes $(\mathcal{X}^\ell)_{\ell\in\mathcal{L}}$ such that*

(i) $\Delta(\mathcal{I}) = \bigcup_{\ell\in\mathcal{L}} \mathcal{X}^\ell$;

(ii) For each $\ell \in \mathcal{L}$ and $f \in \mathcal{F}$, $\mathbf{r}(\,\cdot\,, f)$ is affine on $\mathcal{X}^\ell$.

Proof. Let $1 \leqslant n \leqslant d$ and $b \in \mathcal{B}$. Let us first prove that $\mathbf{r}^n(\,\cdot\,, b)$ is piecewise affine. The map $\mathbf{f}$ being affine and defined on $\Delta(\mathcal{J})$, the set $\mathbf{f}^{-1}(b)$ is a polytope. Denote $y_{b,1}, \dots, y_{b,q}$ its vertices. Let $x \in \Delta(\mathcal{I})$. By linearity of $\mathbf{g}(x, \cdot)$, $\mathbf{r}^n(x, b)$ can then be written

$$\mathbf{r}^n(x,b) = \max \mathbf{g}^n(x, \mathbf{f}^{-1}(b)) = \max_{1\leqslant p\leqslant q} \mathbf{g}^n(x, y_{b,p}).$$

$\mathbf{r}^n(\,\cdot\,, b)$ now appears as the maximum of a finite family $(\mathbf{g}^n(\,\cdot\,, y_{b,p}))_{1\leqslant p\leqslant q}$ of linear functions. It is therefore piecewise affine and so is $\mathbf{r}(\,\cdot\,, b)$. Therefore, for each $b \in \mathcal{B}$ there exists a decomposition of $\Delta(\mathcal{I})$ into polytopes on each of which $\mathbf{r}(\,\cdot\,, b)$ is affine. $\mathcal{B}$ being finite, we can consider the decomposition $(\mathcal{X}^\ell)_{\ell\in\mathcal{L}}$ which refines all of them. $\mathbf{r}(\,\cdot\,, b)$ is therefore affine on each polytope $\mathcal{X}^\ell$ for all $b \in \mathcal{B}$. Let us now prove that $\mathbf{r}(\,\cdot\,, f)$ is affine on each polytope $\mathcal{X}^\ell$ for all $f \in \mathcal{F}$.

Let $f \in \mathcal{F}$, $\ell \in \mathcal{L}$, $x_1, x_2 \in \mathcal{X}^\ell$ and $\lambda \in [0,1]$. We consider the unique decomposition $f = \sum_{b\in\mathcal{B}} \mu^b \cdot b$ and $k \in \mathcal{K}$ such that $\operatorname{supp}\mu \subset \mathcal{F}^k$. Using the definition of $\mathbf{r}$ and the affinity of $\mathbf{r}(\,\cdot\,, b)$ on $\mathcal{X}^\ell$, we have

$$\begin{aligned}
&\mathbf{r}(\lambda x_1 + (1-\lambda)x_2, f)\\
&\quad= \sum_{b\in\mathcal{B}} \mu^b \cdot \mathbf{r}(\lambda x_1 + (1-\lambda)x_2, b)\\
&\quad= \sum_{b\in\mathcal{B}} \mu^b \left(\lambda \mathbf{r}(x_1, b) + (1-\lambda)\mathbf{r}(x_2, b)\right)\\
&\quad= \lambda \sum_{b\in\mathcal{B}} \mu^b \cdot \mathbf{r}(x_1, b) + (1-\lambda)\sum_{b\in\mathcal{B}} \mu^b \cdot \mathbf{r}(x_2, b)\\
&\quad= \lambda \mathbf{r}(x_1, f) + (1-\lambda)\mathbf{r}(x_2, f),
\end{aligned}$$

where the last equality stands because of the uniqueness of the decomposition of f lets us recognize the definitions of $\mathbf{r}(x_1, b)$ and $\mathbf{r}(x_2, b)$ from Equation (4). □

Proof. of Proposition 3.1 (i) Let $x \in \Delta(\mathcal{I})$ and $y \in \Delta(\mathcal{J})$. Denote $f = \mathbf{f}(y)$. We consider the unique decomposition $f = \sum_{b\in\mathcal{B}} \mu^b \cdot b$ and $k \in \mathcal{K}$ such that $\operatorname{supp}\mu \subset \mathcal{F}^k$. $\mathbf{f}^{-1}$ being affine on $\mathcal{F}^k$, we have

$$\begin{aligned}
\mathbf{g}(x,y) &\in \mathbf{g}(x, \mathbf{f}^{-1}(f))\\
&= \mathbf{g}\left(x, \mathbf{f}^{-1}\left(\sum_{b\in\operatorname{supp}\mu} \mu^b \cdot b\right)\right)\\
&= \mathbf{g}\left(x, \sum_{b\in\mathcal{B}} \mu^b \cdot \mathbf{f}^{-1}(b)\right)\\
&= \sum_{b\in\operatorname{supp}\mu} \mu^b \cdot \mathbf{g}(x, \mathbf{f}^{-1}(b)).
\end{aligned}$$

Then for each $1 \leqslant n \leqslant d$,

$$\begin{aligned}
\mathbf{g}^n(x,y) &\leqslant \max \sum_{b\in\operatorname{supp}\mu} \mu^b \cdot \mathbf{g}(x, \mathbf{f}^{-1}(b))\\
&= \sum_{b\in\mathcal{B}} \mu^b \cdot \max \mathbf{g}^n(x, \mathbf{f}^{-1}(b))\\
&= \sum_{b\in\mathcal{B}} \mu^b \cdot \mathbf{r}^n(x,b) = \mathbf{r}^n(x,f),
\end{aligned}$$

where for the second equality, we recognized the definition of $\mathbf{r}^n(x,b)$ from Equation (3) on page 5, and the the last equality, the definition of $\mathbf{r}^n(x,f)$ from Equation (4).

(ii) Let $f \in \mathcal{F}$. Thanks to the characterization of approachability from Proposition 1, there exists $x \in \Delta(\mathcal{I})$ such that $\mathbf{m}(x,f) \in \mathbb{R}^d_-$. Let $f = \sum_{b\in\mathcal{B}} \mu^b \cdot b$ be the unique decomposition of f given by Lemma 3.1. With the same arguments as above, we have for each $1 \leqslant n \leqslant d$,

$$\begin{aligned}
\mathbf{r}^n(x,f) &= \sum_{b\in\mathcal{B}} \mu^b \cdot \mathbf{r}^n(x,b)\\
&= \sum_{b\in\mathcal{B}} \mu^b \cdot \max \mathbf{g}^n(x, \mathbf{f}^{-1}(b))\\
&= \max \sum_{b\in\mathcal{B}} \mu^b \cdot \mathbf{g}^n(x, \mathbf{f}^{-1}(b))\\
&= \max \mathbf{g}^n\left(x, \mathbf{f}^{-1}\left(\sum_{b\in\mathcal{B}} \mu^b \cdot b\right)\right)\\
&= \max \mathbf{g}^n(x, \mathbf{f}^{-1}(f)) = \max \mathbf{m}^n(x,f) \leqslant 0.
\end{aligned}$$

Therefore, $\mathbf{r}(x,f) \in \mathbb{R}^d_-$.

(iii) Let $x \in \Delta(\mathcal{I})$, $k \in \mathcal{K}$, $f_1, f_2 \in \mathcal{F}^k$ and $\lambda \in [0,1]$. We write $f_1 = \sum_{b\in\mathcal{B}} \mu_1^b \cdot b$ and $f_2 = \sum_{b\in\mathcal{B}} \mu_2^b \cdot b$ with $\operatorname{supp}\mu_1 \subset \mathcal{F}^k$ and $\operatorname{supp}\mu_2 \subset \mathcal{F}^k$. The unique decomposition of $\lambda f_1 + (1-\lambda) f_2$ given by Lemma 3.1 is then

$$\lambda f_1 + (1-\lambda) f_2 = \sum_{b\in\mathcal{B}} (\lambda \mu_1^b + (1-\lambda)\mu_2^b) \cdot b.$$

Therefore, using the definition of $\mathbf{r}$ and the affinity of $\mathbf{r}(x,\,\cdot\,)$ on $\mathcal{F}^k$,

$$\begin{aligned}
&\mathbf{r}(x, \lambda f_1 + (1-\lambda) f_2) \\
&\quad = \mathbf{r}\left(x, \sum_{b\in\mathcal{B}} (\lambda \mu_1^b + (1-\lambda)\mu_2^b)\cdot b\right) \\
&\quad = \sum_{b\in\mathcal{B}} (\lambda \mu_1^b + (1-\lambda)\mu_2^b)\cdot \mathbf{r}(x,b) \\
&\quad = \lambda \sum_{b\in\mathcal{B}} \mu_1^b \cdot \mathbf{r}(x,b) \\
&\qquad + (1-\lambda)\sum_{b\in\mathcal{B}} \mu_2^b \cdot \mathbf{r}(x,b) \\
&\quad = \lambda \mathbf{r}(x,f_1) + (1-\lambda)\cdot \mathbf{r}(x,f_2).
\end{aligned}$$

(iv) is already proved in Lemma A.1. □

A.2 Existence of $\mathbf{r}^{[k]}$

Proposition A.2 *For every $k \in \mathcal{K}$, there exists a map $\mathbf{r}^{[k]} : \Delta(\mathcal{I}) \times \mathbb{R}^{\mathcal{S}\times\mathcal{I}} \to \mathbb{R}^d$ such that*

(i) for all $x \in \Delta(\mathcal{I})$, the map $\mathbf{r}^{[k]}(x,\,\cdot\,) : \mathbb{R}^{\mathcal{S}\times\mathcal{I}} \to \mathbb{R}^d$ is linear;

(ii) for all $x \in \Delta(\mathcal{I})$ and $f \in \mathcal{F}^k$, $\mathbf{r}^{[k]}(x,f) = \mathbf{r}(x,f)$.

Proof. Let $k \in \mathcal{K}$ and $x \in \Delta(\mathcal{I})$. Let us consider $\operatorname{span}(\mathcal{F}^k) \subset \mathbb{R}^{\mathcal{S}\times\mathcal{I}}$, the linear span of $\mathcal{F}^k$. There exists a basis $(f_1,\ldots,f_q)$ of $\operatorname{span}(\mathcal{F}^k)$ such that f_p belongs to $\mathcal{F}^k$ for each $1 \leqslant p \leqslant q$. We now define $\mathbf{r}^{[k]}(x,\,\cdot\,)$ on $\operatorname{span}(\mathcal{F}^k)$ by setting

$$\mathbf{r}^{[k]}(x,f_p) := \mathbf{r}(x,f_p), \text{ for each element } f_p \text{ of the basis,}$$

and extending linearly. $\mathbf{r}^{[k]}(x,\,\cdot\,)$ can then be further extended to the whole space $\mathbb{R}^{\mathcal{S}\times\mathcal{I}}$ by setting its value to zero on some complementary subspace of $\operatorname{span}(\mathcal{F}^k)$.

Let us now prove that $\mathbf{r}^{[k]}(x,\,\cdot\,)$ coincides with $\mathbf{r}(x,\,\cdot\,)$ on $\mathcal{F}^k$. Let $f \in \mathcal{F}^k$. In particular, f belongs to $\operatorname{span}(\mathcal{F}^k)$ and can be uniquely written

$$f = \sum_{p=1}^{q} \lambda_p f_p, \quad \text{where} \quad \lambda_1,\ldots,\lambda_q \in \mathbb{R}.$$

The application $\mathbf{r}^{[k]}(x,\,\cdot\,)$ being linear by definition, we have

$$\mathbf{r}^{[k]}(x,f) = \sum_{p=1}^{q} \lambda_p \mathbf{r}(x,f_p).$$

We now aim at proving that the above sum is equal to $\mathbf{r}(x,f)$. This cannot be done by directly applying the affinity of $\mathbf{r}(x,\,\cdot\,)$ (property (iii) in Lemma 3.1) because some of the above coefficients λ_p may be negative. To overcome this, we first separate the terms according to the signs of the coefficients λ_p. We denote Λ^+ (resp. Λ^-) the sum of all positive (resp. negative) coefficients λ_p and write

$$\begin{aligned}
\mathbf{r}^{[k]}(x,f) &= \sum_{\lambda_p>0} \lambda_p \mathbf{r}(x,f_p) + \sum_{\lambda_p<0} \lambda_p \mathbf{r}(x,f_p) \\
&= \Lambda^+ \sum_{\lambda_p>0} \left(\frac{\lambda_p}{\Lambda^+}\right) \mathbf{r}(x,f_p) \\
&\qquad + \Lambda^- \sum_{\lambda_p<0} \left(\frac{\lambda_p}{\Lambda^-}\right) \mathbf{r}(x,f_p).
\end{aligned}$$

Since each of the above sum is now a convex combination, we can apply the affinity of $\mathbf{r}(x,\,\cdot\,)$:

$$\begin{aligned}
\mathbf{r}^{[k]}(x,f) &= \Lambda^+ \cdot \mathbf{r}\left(x, \sum_{\lambda_p>0}\left(\frac{\lambda_p}{\Lambda^+}\right) f_p\right) \\
&\qquad + \Lambda^- \mathbf{r}\left(x, \sum_{\lambda_p<0}\left(\frac{\lambda_p}{\Lambda^-}\right) f_p\right).
\end{aligned}$$

Let us prove that

$$\begin{aligned}
\mathbf{r}(x,f) - \Lambda^- \mathbf{r}&\left(x, \sum_{\lambda_p<0}\left(\frac{\lambda_p}{\Lambda^-}\right) f_p\right) \\
&= \Lambda^+ \cdot \mathbf{r}\left(x, \sum_{\lambda_p>0}\left(\frac{\lambda_p}{\Lambda^+}\right) f_p\right). \quad (5)
\end{aligned}$$

This will prove that $\mathbf{r}^{[k]}(x,f) = \mathbf{r}(x,f)$.

$$\begin{aligned}
&\mathbf{r}(x,f) - \Lambda^- \mathbf{r}\left(x, \sum_{\lambda_p<0}\left(\frac{\lambda_p}{\Lambda^-}\right) f_p\right) \\
&= (1-\Lambda^-)\left(\frac{1}{1-\Lambda^-}\mathbf{r}(x,f)\right. \\
&\qquad \left. + \frac{-\Lambda^-}{1-\Lambda^-}\mathbf{r}\left(x, \sum_{\lambda_p<0}\left(\frac{\lambda_p}{\Lambda^-}\right) f_p\right)\right) \\
&= (1-\Lambda^-)\cdot \mathbf{r}\left(x, \frac{1}{1-\Lambda^-} f + \sum_{\lambda_p<0}\left(-\frac{\lambda_p}{1-\Lambda^-}\right) f_p\right) \\
&= (1-\Lambda^-)\cdot \mathbf{r}\left(x, \frac{1}{1-\Lambda^-}\left(f - \sum_{\lambda_p<0} \lambda_p f_p\right)\right) \\
&= (1-\Lambda^-)\cdot \mathbf{r}\left(x, \sum_{\lambda_p>0}\left(\frac{\lambda_p}{1-\Lambda^-}\right) f_p\right).
\end{aligned}$$

For relation (5) to be true, it is now enough to prove that $\Lambda^+ + \Lambda^- = 1$. Since $\mathcal{F}^k \subset \mathcal{F} \subset \Delta(\mathcal{S})^{\mathcal{I}}$, for any $f_0 = (f_0^{is})_{\substack{s\in\mathcal{S}\\ i\in\mathcal{I}}} \in \mathcal{F}^k$, we have

$$\sum_{\substack{s\in\mathcal{S}\\ i\in\mathcal{I}}} f_0^{is} = \sum_{i\in\mathcal{I}}\sum_{s\in\mathcal{S}} f_0^{is} = \sum_{i\in\mathcal{I}} 1 = |\mathcal{I}|\,.$$

By applying the above to f and the f_p, we get

$$\begin{aligned}|\mathcal{I}| = \sum_{\substack{s\in\mathcal{S}\\ i\in\mathcal{I}}} f^{is} &= \sum_{\substack{s\in\mathcal{S}\\ i\in\mathcal{I}}}\left(\sum_{\lambda_p>0}\lambda_p f_p^{is} + \sum_{\lambda_p<0}\lambda_p f_p^{is}\right)\\ &= \sum_{\lambda_p>0}\lambda_p \sum_{\substack{s\in\mathcal{S}\\ i\in\mathcal{I}}} f_p^{is} + \sum_{\lambda_p<0}\lambda_p \sum_{\substack{s\in\mathcal{S}\\ i\in\mathcal{I}}} f_p^{is}\\ &= \Lambda^+ |\mathcal{I}| + \Lambda^- |\mathcal{I}|\,,\end{aligned}$$

and we indeed get $\Lambda^+ + \Lambda^- = 1$ by dividing by $|\mathcal{I}|$, which concludes the proof. ☐

A.3 A lemma on $L_{\mathbf{r}}$

Recall that $L_{\mathbf{r}}$ was defined as the maximal Lipschitz constant of mappings $\mathbf{r}(x,\cdot)$. Then it is also the maximal operator norm of the linear maps $\mathbf{r}^{[k]}(a,\,\cdot\,)$:

$$L_{\mathbf{r}} := \max_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} \max_{\substack{f\in\mathbb{R}^{\mathcal{S}\times\mathcal{I}}\\ f\neq 0}} \frac{\left\|\mathbf{r}^{[k]}(a,f)\right\|_2}{\|f\|_2}.$$

Lemma A.3 *$L_{\mathbf{r}}$ is a common Lipschitz constant to $\mathbf{r}(a,\,\cdot\,)$ and $\mathbf{r}^{[k]}(a,\,\cdot\,)$ ($k \in \mathcal{K}$ and $a \in \mathcal{A}$). In other words, for all $k\in\mathcal{K}$ and $a\in\mathcal{A}$, we have*

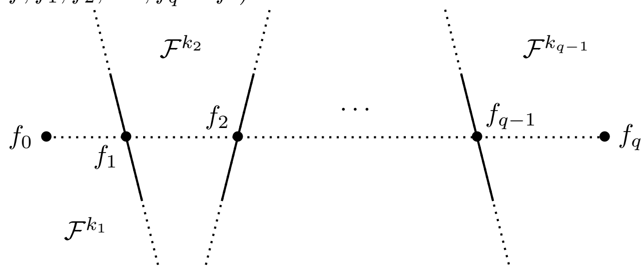
(i) for all $f, f' \in \mathbb{R}^{\mathcal{S}\times\mathcal{I}}$, $\left\|\mathbf{r}^{[k]}(a,f) - \mathbf{r}^{[k]}(a,f')\right\|_2 \leqslant L_{\mathbf{r}}\|f-f'\|_2$;

(ii) for all $f, f' \in \mathcal{F}$, $\|\mathbf{r}(a,f) - \mathbf{r}(a,f')\|_2 \leqslant L_{\mathbf{r}}\|f-f'\|_2$.

Proof. Property (i) follows from the definition of $L_{\mathbf{r}}$ and the linearity of the map $\mathbf{r}^{[k]}(a,\,\cdot\,)$.

(ii) Let $k\in\mathcal{K}$, $a\in\mathcal{A}$ and $f, f' \in \mathcal{F}$. $(\mathcal{F}^k)_{k\in\mathcal{K}}$ being a finite decomposition of $\mathcal{F}$ into convex polytopes, there exists a finite sequence $(k_1, k_2, \dots, k_q)$ in $\mathcal{K}$ such that the k_p's are all different and a sequence $(f_0 = f, f_1, f_2, \dots, f_q = f')$ in the affine segment $[f, f']$ such that $[f_{p-1}, f_p] \subset \mathcal{F}^{k_p}$ for each $1 \leqslant p \leqslant q$, see Figure 1. Therefore, using the fact that $\mathbf{r}^{[k']}(a,\,\cdot\,)$ and

Figure 1: An illustrative figure of the sequence $(f_0 = f, f_1, f_2, \dots, f_q = f')$



$\mathbf{r}(a,\,\cdot\,)$ coincide on $\mathcal{F}^{k'}$ for all $k'\in\mathcal{K}$, we can write

$$\begin{aligned}&\|\mathbf{r}(a,f) - \mathbf{r}(a,f')\|_2\\ &\qquad= \left\|\sum_{p=1}^{q}\left(\mathbf{r}(a,f_{p-1}) - \mathbf{r}(a,f_p)\right)\right\|_2\\ &\qquad= \left\|\sum_{p=1}^{q}\mathbf{r}^{[k_p]}(a,f_{p-1}) - \mathbf{r}^{[k_p]}(a,f_p)\right\|_2\\ &\qquad\leqslant \sum_{p=1}^{q}\left\|\mathbf{r}^{[k_p]}(a,f_{p-1}) - \mathbf{r}^{[k_p]}(a,f_p)\right\|_2\\ &\qquad\leqslant L_{\mathbf{r}}\sum_{p=1}^{q}\|f_{p-1} - f_p\|_2\\ &\qquad= L_{\mathbf{r}}\|f-f'\|_2\,,\end{aligned}$$

where the last equality holds because the points $f_0, \dots, f_q$ are aligned and ordered. ☐

A.4 Proof of Proposition 3.2

Proof. Using the definition of $\mathbf{R}$,

$$\begin{aligned}&\mathbf{R}\left(\left(\mathbb{1}_{\{k_0=k\}}\lambda^a\cdot f\right)_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}}\right)\\ &\qquad= \sum_{k\in\mathcal{K}}\sum_{a\in\mathcal{A}}\mathbf{r}^{[k]}(a, \mathbb{1}_{\{k_0=k\}}\lambda^a\cdot f)\\ &\qquad= \sum_{a\in\mathcal{A}}\lambda^a\cdot\mathbf{r}^{[k_0]}(a,f)\\ &\qquad= \sum_{a\in\mathcal{A}}\lambda^a\cdot\mathbf{r}(a,f) = \mathbf{r}(x,f),\end{aligned}$$

where the second equality holds because by linearity of $\mathbf{r}^{[k]}(a,\,\cdot\,)$ (Proposition A.2), the fourth because $\mathbf{r}^{[k_0]}(x,\,\cdot\,)$ and $\mathbf{r}(x,\,\cdot\,)$ coincide on $\mathcal{F}^{k_0}$ (property (ii) in Proposition A.2, and the last by affinity of $\mathbf{r}(\,\cdot\,,f)$ on $\mathcal{X}^\ell$ (property (iv) in Proposition 3.1). ☐

A.5 Proof of Proposition 3.3

(i) Let $k \in \mathcal{K}$. $\mathbf{R}_k^{-1}(\mathbb{R}_-^d)$ is a closed convex cone as the inverse image via a linear application of the closed convex cone $\mathbb{R}_-^d$ (Proposition C.5). $\mathcal{F}_c^k$ is a closed convex cone by definition, and $(\mathcal{F}_c^k)^{\mathcal{A}}$ is thus a closed convex cone as a Cartesian product of closed convex cones. Therefore, $\tilde{\mathcal{C}}^k = \mathbf{R}_k^{-1}(\mathbb{R}_-^d) \cap (\mathcal{F}_c^k)^{\mathcal{A}}$ is also a closed convex cone as the intersection of two closed convex cones. Then, $\tilde{\mathcal{C}}$ is also a closed convex cone as a Cartesian product of closed convex cones.

(ii) Let $\tilde{g} = (\tilde{g}^{ka})_{\substack{k\in\mathcal{K}\\a\in\mathcal{A}}} \in \tilde{\mathcal{C}}$. By definition of $\tilde{\mathcal{C}}$, for each $k \in \mathcal{K}$, $(\tilde{g}^{ka})_{a\in\mathcal{A}}$ belongs to $\tilde{\mathcal{C}}^k$ and thus to $(\mathcal{F}_c^k)^{\mathcal{A}}$. Therefore, $\tilde{g} \in \prod_{k\in\mathcal{K}}(\mathcal{F}_c^k)^{\mathcal{A}}$. Moreover,

$$\mathbf{R}(\tilde{g}) = \sum_{k\in\mathcal{K}} \mathbf{R}_k\left((\tilde{g}^{ka})_{a\in\mathcal{A}}\right)$$

belongs to $\mathbb{R}_-^d$. Indeed, each term of the above sum belongs to $\mathbb{R}_-^d$ because for all $k \in \mathcal{K}$, $(\tilde{g}^{ka})_{a\in\mathcal{A}} \in \tilde{\mathcal{C}}^k \subset \mathbf{R}_k^{-1}(\mathbb{R}_-^d)$.

(iii) This full information game has convex compact decision sets and a bilinear payoff function. Thanks to the characterization of approachable closed convex cones in full information, the statement of the proposition is then equivalent to Blackwell condition:

$$\forall f \in \mathcal{F},\ \exists \tilde{x} \in \Delta(\mathcal{K}\times\mathcal{A}),\quad \tilde{\mathbf{g}}(\tilde{x}, f) \in \tilde{\mathcal{C}},$$

which we now aim at proving. Let $f \in \mathcal{F}$ and $k_0 \in \mathcal{K}$ such that $f \in \mathcal{F}^{k_0}$. According to property (ii) in Proposition 3.1, there exists $x \in \Delta(\mathcal{I})$ such that such that $\mathbf{r}(x, f) \in \mathbb{R}_-^d$. By Proposition 3.1, there exists $\ell \in \mathcal{L}$ such that $x \in \mathcal{X}^\ell$ and we can write x as a convex combination of the vertices of $\mathcal{X}^\ell$:

$$x = \sum_{a\in\mathcal{A}} \lambda^a \cdot a \quad \text{where} \quad \begin{cases}(\lambda^a)_{a\in\mathcal{A}} \in \Delta(\mathcal{A})\\ \operatorname{supp}(\lambda^a)_{a\in\mathcal{A}} \subset \mathcal{X}^\ell.\end{cases}$$

Now consider the random decision

$$\tilde{x} := \left(\mathbb{1}_{\{k=k_0\}}\lambda^a\right)_{\substack{k\in\mathcal{K}\\a\in\mathcal{A}}} \quad \in \Delta(\mathcal{K}\times\mathcal{A})$$

and let us prove that $\tilde{\mathbf{g}}(\tilde{x}, f) \in \tilde{\mathcal{C}}$. We have by definition of $\tilde{\mathbf{g}}$:

$$\tilde{\mathbf{g}}(\tilde{x}, f) = \left(\mathbb{1}_{\{k=k_0\}}\lambda^a \cdot f\right)_{\substack{k\in\mathcal{K}\\a\in\mathcal{A}}},$$

and since $\tilde{\mathcal{C}} = \prod_{k\in\mathcal{K}}\tilde{\mathcal{C}}^k$, we only have to check that $(\lambda^a f)_{a\in\mathcal{A}}$ belongs to $\tilde{\mathcal{C}}^{k_0} = \mathbf{R}_{k_0}^{-1}(\mathbb{R}_-^d) \cap (\mathcal{F}_c^{k_0})^{\mathcal{A}}$. First, because $f \in \mathcal{F}^{k_0}$, $\lambda^a f$ belongs to the closed convex cone $\mathcal{F}_c^{k_0} = \mathbb{R}_+\mathcal{F}^{k_0}$ and we have indeed $(\lambda^a f)_{a\in\mathcal{A}} \in (\mathcal{F}_c^{k_0})^{\mathcal{A}}$. Then, let us prove that $\mathbf{R}_{k_0}\left((\lambda^a f)_{a\in\mathcal{A}}\right) \in \mathbb{R}_-^d$. Using Proposition 3.2,

$$\begin{aligned}\mathbf{R}_{k_0}((\lambda^a f)_{a\in\mathcal{A}}) &= \mathbf{R}\left(\left(\mathbb{1}_{\{k=k_0\}}\lambda^a \cdot f\right)_{\substack{k\in\mathcal{K}\\a\in\mathcal{A}}}\right)\\ &= \mathbf{r}(x, f) \in \mathbb{R}_-^d.\end{aligned}$$

Therefore, we have proved that $(\lambda^a f)_{a\in\mathcal{A}}$ belongs to $\tilde{\mathcal{C}}^{k_0} = \mathbf{R}_{k_0}^{-1}(\mathbb{R}_-^d) \cap (\mathcal{F}_c^{k_0})^{\mathcal{A}}$, and thus, that $\tilde{\mathbf{g}}(\tilde{x}, f) \in \tilde{\mathcal{C}}$, which concludes the proof. □

A.6 Properties of the estimate $\hat{f}_t$.

Lemma A.4 *For all $t \geqslant 1$,*

(i) $\mathbb{E}\left[\hat{f}_t \,\middle|\, \mathcal{G}_t\right] = \mathbb{E}\left[f_t \,|\, \mathcal{G}_t\right]$;

(ii) $\mathbb{E}\left[\left\|\hat{f}_t\right\|_2^2 \,\middle|\, \mathcal{G}_t\right] \leqslant \dfrac{|\mathcal{I}|^2}{\gamma}$;

(iii) $\left\|\hat{f}_t\right\|_2^2 \leqslant \dfrac{|\mathcal{I}|^2}{\gamma^2}$.

Proof. (i) Let $i \in \mathcal{I}$. Using the conditional expectation with respect to event $\{i_t = i\}$, we have

$$\begin{aligned}\mathbb{E}\left[\hat{f}_t^i \,\middle|\, \mathcal{G}_t\right] &= \mathbb{E}\left[\frac{\mathbb{1}_{\{i_t=i\}}}{\mathbb{P}\left[i_t = i \,|\, \mathcal{G}_t\right]}\delta_{s_t} \,\middle|\, \mathcal{G}_t\right]\\ &= \mathbb{P}\left[i_t = i \,|\, \mathcal{G}_t\right]\mathbb{E}\left[\frac{\delta_{s_t}}{\mathbb{P}\left[i_t = i \,|\, \mathcal{G}_t\right]} \,\middle|\, \mathcal{G}_t, \{i_t = i\}\right]\\ &= \mathbb{E}\left[\delta_{s_t} \,|\, \mathcal{G}_t, \{i_t = i\}\right]\\ &= \mathbb{E}\left[\mathbb{E}\left[\delta_{s_t} \,|\, y_t, \mathcal{G}_t, \{i_t = i\}\right] \,|\, \mathcal{G}_t, \{i_t = i\}\right]\\ &= \mathbb{E}\left[\mathbf{s}(i, y_t) \,|\, \mathcal{G}_t, \{i_t = i\}\right]\\ &= \mathbb{E}\left[\mathbf{s}(i, y_t) \,|\, \mathcal{G}_t\right]\\ &= \mathbb{E}\left[f_t^i \,|\, \mathcal{G}_t\right],\end{aligned}$$

hence the result.

(ii) We write

$$\begin{aligned}\mathbb{E}\left[\left\|\hat{f}_t\right\|_2^2 \,\middle|\, \mathcal{G}_t\right] &= \mathbb{E}\left[\sum_{i\in\mathcal{I}}\left\|\frac{\mathbb{1}_{\{i_t=i\}}}{\mathbb{P}\left[i_t = i \,|\, \mathcal{G}_t\right]}\delta_{s_t}\right\|_2^2 \,\middle|\, \mathcal{G}_t\right]\\ &= \mathbb{P}\left[i_t = i \,|\, \mathcal{G}_t\right]\\ &\quad\times \mathbb{E}\left[\sum_{i\in\mathcal{I}}\left\|\frac{\delta_{s_t}}{\mathbb{P}\left[i_t = i \,|\, \mathcal{G}_t\right]}\right\|_2^2 \,\middle|\, \mathcal{G}_t, \{i_t = i\}\right]\\ &= \sum_{i\in\mathcal{I}}\frac{1}{\mathbb{P}\left[i_t = i \,|\, \mathcal{G}_t\right]}\mathbb{E}\left[\left\|\delta_{s_t}\right\|_2^2 \,\middle|\, \mathcal{G}_t, \{i_t = i\}\right]\\ &= \sum_{i\in\mathcal{I}}\frac{1}{\mathbb{P}\left[i_t = i \,|\, \mathcal{G}_t\right]}\\ &\leqslant \frac{|\mathcal{I}|^2}{\gamma},\end{aligned}$$

where the last inequality stands because $\mathbb{P}\left[i_t = i \,|\, \mathcal{G}_t\right] \geqslant \gamma/|\mathcal{I}|$ by definition of the algorithm.

(iii) We have

$$\left\|\hat{f}_t\right\|_2^2 = \sum_{i\in\mathcal{I}} \left\| \frac{\mathbb{1}_{\{i_t=i\}}}{\mathbb{P}\left[i_t = i \,|\, \mathcal{G}_t\right]}\delta_{s_t} \right\|_2^2 = \sum_{i\in\mathcal{I}} \mathbb{1}_{\{i_t=i\}} \frac{\left\|\delta_{s_t}\right\|_2^2}{\mathbb{P}\left[i_t = i \,|\, \mathcal{G}_t\right]^2} \leqslant \frac{\left|\mathcal{I}\right|^2}{\gamma^2} \sum_{i\in\mathcal{I}} \mathbb{1}_{\{i_t=i\}} = \frac{\left|\mathcal{I}\right|^2}{\gamma^2}.$$

□

B Proof of Theorem 4.1

B.1 Average auxiliary payoff $\bar{\tilde{g}}_T$ is close to auxiliary target set $\tilde{\mathcal{C}}$

Lemma B.1

$$\mathbb{E}\left[\mathbf{d}_2\left(\bar{\tilde{g}}_T,\ \tilde{\mathcal{C}}\right)\right] \leqslant \frac{1}{2\eta T} + \frac{\eta\left|\mathcal{I}\right|^2}{2\gamma}.$$

Proof. For $t \geqslant 1$, we can write

$$\tilde{z}_t = \mathbf{P}_{\tilde{\mathcal{Z}}}\left(\eta\sum_{s=1}^{t-1}\tilde{g}_s\right) = \arg\min_{\tilde{z}\in\tilde{\mathcal{Z}}}\left\|\tilde{z} - \eta\sum_{s=1}^{t-1}\tilde{g}_s\right\|_2^2 = \arg\max_{\tilde{z}\in\tilde{\mathcal{Z}}}\left\{\left\langle \eta\sum_{s=1}^{t-1}\tilde{g}_s \middle| \tilde{z}\right\rangle - \frac{1}{2}\left\|\tilde{z}\right\|_2^2\right\}.$$

Then, Theorem D.1 together with the fact that $\left\|\tilde{\mathcal{Z}}\right\|_2 = \left\|\tilde{\mathcal{C}}^\circ \cap \mathcal{B}_2\right\|_2 \leqslant 1$ gives

$$\max_{\tilde{z}\in\tilde{\mathcal{Z}}}\sum_{t=1}^T \left\langle \tilde{g}_t | \tilde{z}\right\rangle - \sum_{t=1}^T \left\langle \tilde{g}_t | \tilde{z}_t\right\rangle \leqslant \frac{1}{2\eta} + \frac{\eta}{2}\sum_{t=1}^T \left\|\tilde{g}_t\right\|_2^2.$$

By taking the expectation and dividing by T, we get

$$\mathbb{E}\left[\max_{\tilde{z}\in\tilde{\mathcal{Z}}}\left\langle \bar{\tilde{g}}_T | \tilde{z}\right\rangle\right] \leqslant \frac{1}{2\eta T} + \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^T\left\langle \tilde{g}_t | \tilde{z}_t\right\rangle\right] + \frac{\eta}{2T}\mathbb{E}\left[\sum_{t=1}^T\left\|\tilde{g}_t\right\|_2^2\right].$$

We first analyze the first sum of the right-hand side. Let us prove that each scalar product $\left\langle \tilde{g}_t | \tilde{z}_t\right\rangle$ is nonpositive in expectation. For all $1 \leqslant t \leqslant T$, we replace $\tilde{g}_t$ by its definition:

$$\mathbb{E}\left[\left\langle \tilde{g}_t | \tilde{z}_t\right\rangle\right] = \mathbb{E}\left[\left\langle \tilde{\mathbf{g}}((k_t,a_t),\hat{f}_t) \middle| \tilde{z}_t\right\rangle\right].$$

We then consider the conditional expectation with respect to $\mathcal{G}_t$. The application $\tilde{\mathbf{g}}((k_t,a_t),\cdot)$ being linear, and the variables k_t, a_t and $\tilde{z}_t$ being measurable with respect to $\mathcal{G}_t$, we can make $\mathbb{E}\left[\hat{f}_t \middle| \mathcal{G}_t\right]$ appear as follows:

$$\begin{aligned}
\mathbb{E}\left[\left\langle \tilde{g}_t | \tilde{z}_t\right\rangle\right] &= \mathbb{E}\left[\mathbb{E}\left[\left\langle \tilde{\mathbf{g}}((k_t,a_t),\hat{f}_t) \middle| \tilde{z}_t\right\rangle \middle| \mathcal{G}_t\right]\right] \\
&= \mathbb{E}\left[\left\langle \tilde{\mathbf{g}}\left((k_t,a_t),\mathbb{E}\left[\hat{f}_t \middle| \mathcal{G}_t\right]\right) \middle| \tilde{z}_t\right\rangle\right] \\
&= \mathbb{E}\left[\left\langle \tilde{\mathbf{g}}((k_t,a_t),\mathbb{E}\left[f_t \,|\, \mathcal{G}_t\right]) | \tilde{z}_t\right\rangle\right] \\
&= \mathbb{E}\left[\left\langle \tilde{\mathbf{g}}((k_t,a_t),f_t) | \tilde{z}_t\right\rangle\right],
\end{aligned}$$

where we used Lemma A.4 to replace the conditional expectation of $\hat{f}_t$ by the conditional expectation of f_t. Now consider the sigma-algebra $\mathcal{H}_t$ generated by

$$(k_1,a_1,i_1,s_1,\dots,k_{t-1},a_{t-1},i_{t-1},s_{t-1}).$$

By definition of the algorithm, the law of random variable (k_t,a_t) knowing $\mathcal{H}_t$ is $\tilde{x}_t$. We now resume the above computation by introducing the conditional expectation with respect to $\mathcal{H}_t$ and f_t:

$$\begin{aligned}
\mathbb{E}\left[\left\langle \tilde{g}_t | \tilde{z}_t\right\rangle\right] &= \mathbb{E}\left[\left\langle \tilde{\mathbf{g}}((k_t,a_t),f_t) | \tilde{z}_t\right\rangle\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left\langle \tilde{\mathbf{g}}((k_t,a_t),f_t) | \tilde{z}_t\right\rangle \,|\, \mathcal{H}_t,f_t\right]\right] \\
&= \mathbb{E}\left[\left\langle \tilde{\mathbf{g}}\left(\mathbb{E}\left[(k_t,a_t) \,|\, \mathcal{H}_t,f_t\right],f_t\right) | \tilde{z}_t\right\rangle\right] \\
&= \mathbb{E}\left[\left\langle \tilde{\mathbf{g}}\left(\mathbb{E}\left[(k_t,a_t) \,|\, \mathcal{H}_t\right],f_t\right) | \tilde{z}_t\right\rangle\right] \\
&= \mathbb{E}\left[\left\langle \tilde{\mathbf{g}}(\tilde{x}_t,f_t) | \tilde{z}_t\right\rangle\right].
\end{aligned}$$

By definition of the algorithm, $\tilde{x}_t = \tilde{\mathbf{x}}(\tilde{z}_t)$. In other words (see Proposition 3.3), for all $f \in \mathcal{F}$, the scalar product $\left\langle \tilde{\mathbf{g}}(\tilde{x}_t,f) | \tilde{z}_t\right\rangle$ is nonpositive. This is in particular true for $f = f_t$. Therefore, $\mathbb{E}\left[\left\langle \tilde{g}_t | \tilde{z}_t\right\rangle\right] \leqslant 0$.

We now turn to the second sum that involves the squared norms $\left\|\tilde{g}_t\right\|_2^2$. For $1 \leqslant t \leqslant T$, using the definition of $\tilde{\mathbf{g}}$,

$$\begin{aligned}
\left\|\tilde{g}_t\right\|_2^2 &= \left\|\tilde{\mathbf{g}}((k_t,a_t),\hat{f}_t)\right\|_2^2 \\
&= \left\|\left(\mathbb{1}_{\{k=k_t\}}\mathbb{1}_{\{a=a_t\}}\hat{f}_t\right)_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}}\right\|_2^2 \\
&= \sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}}\left\|\mathbb{1}_{\{k=k_t\}}\mathbb{1}_{\{a=a_t\}}\hat{f}_t\right\|_2^2 = \left\|\hat{f}_t\right\|_2^2.
\end{aligned}$$

Using (ii) from Lemma B.1, we have

$$\mathbb{E}\left[\left\|\tilde{g}_t\right\|_2^2\right] = \mathbb{E}\left[\left\|\hat{f}_t\right\|_2^2\right] = \mathbb{E}\left[\mathbb{E}\left[\left\|\hat{f}_t\right\|_2^2 \middle| \mathcal{G}_t\right]\right] \leqslant \frac{\left|\mathcal{I}\right|^2}{\gamma}.$$

Putting everything together, we obtain in expectation the following bound on the distance from $\bar{\tilde{g}}_T$ to $\tilde{\mathcal{C}}$:

$$\mathbb{E}\left[\mathbf{d}_2\left(\bar{\tilde{g}}_T,\ \tilde{\mathcal{C}}\right)\right] = \mathbb{E}\left[\max_{\tilde{z}\in\tilde{\mathcal{Z}}}\left\langle \bar{\tilde{g}}_T | \tilde{z}\right\rangle\right] \leqslant \frac{1}{2\eta T} + \frac{\eta\left|\mathcal{I}\right|^2}{2\gamma},$$

where the above equality comes from the expression of the Euclidean distance to $\tilde{\mathcal{C}}$ given by Proposition C.6. □

B.2 From $\bar{\tilde{g}}_T$ in the auxiliary space to $\mathbf{R}(\bar{\tilde{g}}_T)$ in the initial space

Lemma B.2

$$\mathbf{d}_2\left(\mathbf{R}(\bar{\tilde{g}}_T),\ \mathbb{R}_-^d\right) \leqslant (L_\mathbf{r}\sqrt{|\mathcal{K}|\,|\mathcal{A}|}) \cdot \mathbf{d}_2\left(\bar{\tilde{g}}_T,\ \tilde{\mathcal{C}}\right).$$

Proof. It follows from property (ii) in Proposition 3.3 that $\tilde{\mathcal{C}} \subset \mathbf{R}^{-1}(\mathbb{R}_-^d)$. Therefore, we can write

$$\begin{aligned}
\mathbf{d}_2(\mathbf{R}(\bar{\tilde{g}}_T),\ \mathbb{R}_-^d) &= \min_{g' \in \mathbb{R}_-^d} \|\mathbf{R}(\bar{\tilde{g}}_T) - g'\|_2 \\
&\leqslant \min_{\tilde{g} \in \mathbf{R}^{-1}(\mathbb{R}_-^d)} \|\mathbf{R}(\bar{\tilde{g}}_T) - \mathbf{R}(\tilde{g})\|_2 \\
&\leqslant \min_{\tilde{g} \in \tilde{\mathcal{C}}} \|\mathbf{R}(\bar{\tilde{g}}_T) - \mathbf{R}(\tilde{g})\|_2 \\
&\leqslant \|\mathbf{R}\| \cdot \min_{\tilde{g} \in \tilde{\mathcal{C}}} \|\bar{\tilde{g}}_T - \tilde{g}\|_2 \\
&= \|\mathbf{R}\| \cdot \mathbf{d}_2\left(\bar{\tilde{g}}_T,\ \tilde{\mathcal{C}}\right),
\end{aligned}$$

where $\|\mathbf{R}\|$ is the operator norm of $\mathbf{R}$. To conclude the proof, let us prove that the latter is bounded from above by $L_\mathbf{r}\sqrt{|\mathcal{K}|\,|\mathcal{A}|}$. Let $\tilde{g} \in (\mathbb{R}^{\mathcal{S}\times\mathcal{I}})^{\mathcal{K}\times\mathcal{A}}$. By definition of $\mathbf{R}$, and using the Lipschitz constant $L_\mathbf{r}$ from Lemma A.3 which is common to the linear applications $\mathbf{r}^{[k]}(a,\cdot)$, we have

$$\begin{aligned}
\|\mathbf{R}(\tilde{g})\|_2 &= \left\| \sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} \mathbf{r}^{[k]}(a, \tilde{g}^{ka}) \right\|_2 \leqslant \sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} \left\|\mathbf{r}^{[k]}(a, \tilde{g}^{ka})\right\|_2 \\
&\leqslant \sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} L_\mathbf{r} \left\|\tilde{g}^{ka}\right\|_2 \leqslant L_\mathbf{r} \sqrt{|\mathcal{K}|\,|\mathcal{A}| \sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} \|\tilde{g}^{ka}\|_2^2} \\
&= L_\mathbf{r}\sqrt{|\mathcal{K}|\,|\mathcal{A}|} \cdot \|\tilde{g}\|_2,
\end{aligned}$$

which concludes the proof. □

B.3 Decomposition of $\mathbf{R}(\bar{\tilde{g}}_T)$

We have the following expression of the image by $\mathbf{R}$ of the average auxiliary payoff $\bar{\tilde{g}}_T$.

Lemma B.3

$$\mathbf{R}(\bar{\tilde{g}}_T) = \mathbf{R}\left(\frac{1}{T}\sum_{t=1}^T \tilde{g}_t\right) = \sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} \lambda_T(k,a)\cdot\mathbf{r}^{[k]}(a, \bar{\hat{f}}_T(k,a)).$$

Proof. Using the definitions of $\mathbf{R}$, $\tilde{g}_t$, $\tilde{\mathbf{g}}$, and the linearity of $\mathbf{R}$ and $\mathbf{r}^{[k]}(a,\cdot)$, we can write

$$\begin{aligned}
\mathbf{R}\left(\frac{1}{T}\sum_{t=1}^T \tilde{g}_t\right) &= \frac{1}{T}\sum_{t=1}^T \mathbf{R}(\tilde{g}_t) = \frac{1}{T}\sum_{t=1}^T \sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} \mathbf{r}^{[k]}(a, \tilde{g}_t^{ka}) \\
&= \frac{1}{T}\sum_{t=1}^T \sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} \mathbf{r}^{[k]}\left(a, \mathbb{1}_{\{k=k_t\}}\mathbb{1}_{\{a=a_t\}}\hat{f}_t\right) \\
&= \sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} \lambda_T(k,a)\cdot\mathbf{r}^{[k]}(a, \bar{\hat{f}}_T(k,a)).
\end{aligned}$$

□

B.4 Average estimator $\bar{\hat{f}}_T(k,a)$ is close to average flag $\bar{f}_T(k,a)$

Lemma B.4

$$\mathbb{E}\left[\sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} \lambda_T(k,a) \left\|\bar{\hat{f}}_T(k,a) - \bar{f}_T(k,a)\right\|_2\right] \leqslant |\mathcal{I}|\,|\mathcal{K}|\,|\mathcal{A}|\left(\frac{8}{\sqrt{T\gamma}} + \frac{8}{3T\gamma}\right).$$

Proof. Let $k\in\mathcal{K}$ and $a\in\mathcal{A}$. Consider the random process $(X_t(k,a))_{t\geqslant 1}$ defined by

$$X_t(k,a) := \mathbb{1}_{\{k_t=k,\ a_t=a\}}\left(\hat{f}_t - f_t\right),$$

and to which we are aiming at applying Corollary E.4. $(X_t(k,a))_{t\geqslant 1}$ is a martingale difference sequence with respect to filtration $(\mathcal{G}_t)_{t\geqslant 1}$. Indeed, since $\mathbb{1}_{\{k_t=k,\ a_t=a\}}$ is measurable with respect to $\mathcal{G}_t$,

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\{k_t=k,\ a_t=a\}}\left(\hat{f}_t - f_t\right)\,\middle|\,\mathcal{G}_t\right] \\
&\quad = \mathbb{1}_{\{k_t=k,\ a_t=a\}}\mathbb{E}\left[\hat{f}_t - f_t\,\middle|\,\mathcal{G}_t\right] = 0.
\end{aligned}$$

where the last equality follows from (i) in Lemma A.4. Moreover, using (iii) from Lemma A.4, we bound each $X_t(k,a)$ as follows.

$$\begin{aligned}
\|X_t(k,a)\|_2 &\leqslant \left\|\hat{f}_t - f_t\right\|_2 \leqslant \left\|\hat{f}_t\right\|_2 + \|f_t\|_2 \\
&\leqslant \frac{|\mathcal{I}|}{\gamma} + \left\|(\mathbf{s}(i,y_t))_{i\in\mathcal{I}}\right\|_2 \\
&= \frac{|\mathcal{I}|}{\gamma} + \sqrt{\sum_{i\in\mathcal{I}} \|\mathbf{s}(i,y_t)\|_2^2} \\
&\leqslant \frac{|\mathcal{I}|}{\gamma} + \sqrt{|\mathcal{I}|} \leqslant \frac{2\,|\mathcal{I}|}{\gamma},
\end{aligned}$$

where we used the fact that $\gamma \geqslant 1$ for the last inequality. As far as the conditional variances are concerned, we have

$$\begin{aligned}\mathbb{E}\left[\left\|X_t(k,a)\right\|_2^2 \,\middle|\, \mathcal{G}_t\right] &= \mathbb{E}\left[\mathbb{1}_{\{k_t=k,\ a_t=a\}}\left\|\hat{f}_t - f_t\right\|_2^2 \,\middle|\, \mathcal{G}_t\right]\\ &\leqslant \mathbb{E}\left[\left\|\hat{f}_t - f_t\right\|_2^2 \,\middle|\, \mathcal{G}_t\right]\\ &\leqslant \mathbb{E}\left[\left\|\hat{f}_t\right\|_2^2 \,\middle|\, \mathcal{G}_t\right] + \mathbb{E}\left[\left\|f_t\right\|_2^2 \,\middle|\, \mathcal{G}_t\right]\\ &\leqslant \frac{|\mathcal{I}|^2}{\gamma} + |\mathcal{I}| \leqslant \frac{2\,|\mathcal{I}|^2}{\gamma}.\end{aligned}$$

where the first term of the second line has been bounded using property (ii) from Lemma A.4, whereas the second term is bounded by $|\mathcal{I}|$ since

$$\left\|f_t\right\|_2^2 = \left\|(\mathbf{s}(i,y_t))_{i\in\mathcal{I}}\right\|_2^2 = \sum_{i\in\mathcal{I}} \left\|\mathbf{s}(i,y_t)\right\|_2^2 \leqslant |\mathcal{I}|\,.$$

Therefore we have

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\left\|X_t(k,a)\right\|_2^2 \,\middle|\, \mathcal{G}_t\right] \leqslant \frac{2\,|\mathcal{I}|^2}{\gamma}.$$

We can now apply Corollary E.4 with $M = 2\,|\mathcal{I}|/\gamma$ and $V = 2\,|\mathcal{I}|^2/\gamma$ to get:

$$\mathbb{E}\left[\left\|\frac{1}{T}\sum_{t=1}^{T} X_t(k,a)\right\|_2\right] \leqslant \frac{8\,|\mathcal{I}|}{\sqrt{T\gamma}} + \frac{8\,|\mathcal{I}|}{3T\gamma}.$$

Besides, it follows from the definition of $X_t(k,a)$ that

$$\frac{1}{T}\sum_{t=1}^{T} X_t(k,a) = \lambda_T(k,a)\left(\bar{\hat{f}}_T(k,a) - \bar{f}_T(k,a)\right).$$

Finally, by summing over k and a, we obtain:

$$\begin{aligned}&\mathbb{E}\left[\sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} \lambda_T(k,a)\left\|\left(\bar{\hat{f}}_T(k,a) - \bar{f}_T(k,a)\right)\right\|_2\right]\\ &\qquad\qquad \leqslant |\mathcal{I}|\,|\mathcal{K}|\,|\mathcal{A}|\left(\frac{8}{\sqrt{T\gamma}} + \frac{8}{3T\gamma}\right).\end{aligned}$$

□

B.5 Average estimator $\bar{\hat{f}}_T(k,a)$ is close to $\mathcal{F}_c^k$

Lemma B.5

$$\mathbb{E}\left[\sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} \mathbf{d}_2\left(\bar{\tilde{g}}_T^{ka},\ \mathcal{F}_c^k\right)\right] \leqslant \sqrt{|\mathcal{K}|\,|\mathcal{A}|}\left(\frac{1}{2\eta T} + \frac{\eta\,|\mathcal{I}|^2}{2\gamma}\right)$$

Proof. Consider the set $\tilde{\mathcal{Z}}_0$ defined by

$$\tilde{\mathcal{Z}}_0 := \prod_{k\in\mathcal{K}} \left((\mathcal{F}_c^k)^\circ \cap \mathcal{B}_2\right)^{\mathcal{A}},$$

and let us assume for the moment that the following inclusion holds:

$$\tilde{\mathcal{Z}}_0 \subset \sqrt{|\mathcal{K}|\,|\mathcal{A}|}\cdot\tilde{\mathcal{Z}}. \tag{6}$$

For each $k\in\mathcal{K}$ and $a\in\mathcal{A}$, $\mathcal{F}_c^k$ being a closed convex cone, Proposition C.6 gives the following expression of the distance of $\bar{\tilde{g}}_T^{ka}$ to $\mathcal{F}_c^k$:

$$\mathbf{d}_2\left(\bar{\tilde{g}}_T^{ka},\ \mathcal{F}_c^k\right) = \max_{\tilde{z}^{ka}\in(\mathcal{F}_c^k)^\circ\cap\mathcal{B}_2} \left\langle \bar{\tilde{g}}_T^{ka} \middle| \tilde{z}^{ka}\right\rangle.$$

By summing over k and a, we have:

$$\begin{aligned}\sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} \mathbf{d}_2\left(\bar{\tilde{g}}_T^{ka},\ \mathcal{F}_c^k\right) &= \sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} \max_{\tilde{z}^{ka}\in(\mathcal{F}_c^k)^\circ\cap\mathcal{B}_2} \left\langle \bar{\tilde{g}}_T^{ka} \middle| \tilde{z}^{ka}\right\rangle\\ &= \max_{\tilde{z}\in\tilde{\mathcal{Z}}_0} \sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} \left\langle \bar{\tilde{g}}_T \middle| \tilde{z}\right\rangle\\ &\leqslant \sqrt{|\mathcal{K}|\,|\mathcal{A}|}\cdot \max_{\tilde{z}\in\tilde{\mathcal{Z}}} \left\langle \bar{\tilde{g}}_T \middle| \tilde{z}\right\rangle\\ &= \sqrt{|\mathcal{K}|\,|\mathcal{A}|}\cdot \mathbf{d}_2\left(\bar{\tilde{g}}_T,\ \tilde{\mathcal{C}}\right),\end{aligned}$$

where for the inequality we used inclusion (6), and for the last equality Proposition C.6 together with the fact that $\tilde{\mathcal{Z}} = \tilde{\mathcal{C}}^\circ \cap \mathcal{B}_2$ by definition. Taking the expectation and substituting distance $\mathbf{d}_2(\bar{\tilde{g}}_T,\ \tilde{\mathcal{C}})$ by the bound from Lemma B.1 yields the result.

Let us now prove inclusion (6). Let $\tilde{z} = (\tilde{z}^{ka})_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} \in \tilde{\mathcal{Z}}_0$. First, let us prove that $\tilde{z}\in\tilde{\mathcal{C}}^\circ$. Let $\tilde{g}\in\tilde{\mathcal{C}}$. We can write

$$\left\langle \tilde{g} \middle| \tilde{z}\right\rangle = \sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} \left\langle \tilde{z}^{ka} \middle| \tilde{g}^{ka}\right\rangle.$$

But for each $k\in\mathcal{K}$ and $a\in\mathcal{A}$, by definition of $\tilde{\mathcal{Z}}_0$, we have $\tilde{z}^{ka}\in(\mathcal{F}_c^k)^\circ$, and since $\tilde{\mathcal{C}} \subset \prod_{k\in\mathcal{K}}(\mathcal{F}_c^k)^{\mathcal{A}}$ by definition, we also have $\tilde{g}^{ka}\in\mathcal{F}_c^k$. Therefore, $\left\langle \tilde{g}^{ka} \middle| \tilde{z}^{ka}\right\rangle \leqslant 0$ and consequently, $\left\langle \tilde{g} \middle| \tilde{z}\right\rangle \leqslant 0$. This proves $\tilde{\mathcal{Z}}_0 \subset \tilde{\mathcal{C}}^\circ$.

Let $\tilde{z}\in\tilde{\mathcal{Z}}_0$. By definition of $\tilde{\mathcal{Z}}_0$, we have $\left\|\tilde{z}^{ka}\right\|_2 \leqslant 1$ for all $k\in\mathcal{K}$ and $a\in\mathcal{A}$. Thus

$$\left\|\tilde{z}\right\|_2 = \sqrt{\sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}} \left\|\tilde{z}^{ka}\right\|_2^2} \leqslant \sqrt{|\mathcal{K}|\,|\mathcal{A}|},$$

and therefore $\tilde{\mathcal{Z}}_0 \subset \sqrt{|\mathcal{K}|\,|\mathcal{A}|}\cdot\mathcal{B}_2$. Finally, we have

$$\tilde{\mathcal{Z}}_0 \subset \tilde{\mathcal{C}}^\circ \cap \sqrt{|\mathcal{K}|\,|\mathcal{A}|}\cdot\mathcal{B}_2 = \sqrt{|\mathcal{K}|\,|\mathcal{A}|}\cdot\tilde{\mathcal{Z}}.$$

□

B.6 $\mathbf{r}^{[k]}(a, \bar{\hat{f}}_T(k,a))$ is close to $\mathbf{r}(a, \bar{f}_T(k,a))$

Lemma B.6

$$\mathbb{E}\left[\sum_{\substack{k\in\mathcal{K}\\a\in\mathcal{A}}} \lambda_T(k,a)\left\|\mathbf{r}(a,\bar{f}_T(k,a)) - \mathbf{r}^{[k]}(a,\bar{\hat{f}}_T(k,a))\right\|_2\right]$$
$$\leqslant L_{\mathbf{r}}\,|\mathcal{I}|\,|\mathcal{K}|\,|\mathcal{A}|\left(\frac{8}{\sqrt{T\gamma}} + \frac{8}{3T\gamma}\right)$$
$$+ L_{\mathbf{r}}\sqrt{|\mathcal{K}|\,|\mathcal{A}|}\left(\frac{1}{\eta T} + \frac{\eta\,|\mathcal{I}|^2}{\gamma}\right).$$

Proof. Let $(k,a) \in \mathcal{K}\times\mathcal{A}$ and denote $f := \bar{f}_T(k,a)$ and $\hat{f} := \bar{\hat{f}}_T(k,a)$ to alleviate notation. Denote $\mathbf{P}^{[k]}$ the Euclidean projection onto $\mathcal{F}_c^k$. Then of course $\mathbf{P}^{[k]}(\hat{f})$ belongs to $\mathcal{F}_c^k$, and since $\mathbf{r}(a,\,\cdot\,)$ and $\mathbf{r}^{[k]}(a,\,\cdot\,)$ coincide on $\mathcal{F}_c^k$ by Proposition A.2, we can write

$$\mathbf{r}(a,f) - \mathbf{r}^{[k]}(a,\hat{f}) = \mathbf{r}(a,f) - \mathbf{r}(a,\hat{f}) + \mathbf{r}(a,\hat{f})$$
$$- \mathbf{r}(a,\mathbf{P}^{[k]}(\hat{f})) + \mathbf{r}^{[k]}(a,\mathbf{P}^{[k]}(\hat{f})) - \mathbf{r}^{[k]}(a,\hat{f}).$$

Thus, by taking the norm and using the triangle inequality and the Lipschitz constant $L_{\mathbf{r}}$ which is common to $\mathbf{r}(a,\,\cdot\,)$ and $\mathbf{r}^{[k]}(a,\,\cdot\,)$ to get

$$\left\|\mathbf{r}(a,f) - \mathbf{r}^{[k]}(a,\hat{f})\right\|_2$$
$$\leqslant L_{\mathbf{r}}\left(\left\|f-\hat{f}\right\|_2 + 2\cdot\mathbf{d}_2\left(\hat{f},\ \mathcal{F}_c^k\right)\right).$$

We now multiply by $\lambda_T(k,a)$. The last term in the above right-hand side is transformed as

$$2\lambda_T(k,a)\cdot\mathbf{d}_2\left(\hat{f},\ \mathcal{F}_c^k\right) = 2\cdot\mathbf{d}_2\left(\lambda_T(k,a)\hat{f},\ \mathcal{F}_c^k\right)$$
$$= 2\cdot\mathbf{d}_2\left(\bar{\tilde{g}}_T^{ka},\ \mathcal{F}_c^k\right),$$

where used the fact that $\mathcal{F}_c^k$ is a convex cone to push the factor $\lambda_T(k,a)$ into the distance. Therefore,

$$\lambda_T(k,a)\left\|\mathbf{r}(a,f) - \mathbf{r}^{[k]}(a,\hat{f})\right\|_2$$
$$\leqslant L_{\mathbf{r}}\cdot\lambda_T(k,a)\left\|f-\hat{f}\right\|_2 + 2L_{\mathbf{r}}\cdot\mathbf{d}_2\left(\bar{\tilde{g}}_T^{ka},\ \mathcal{F}_c^k\right).$$

Finally, we get the result by taking the expectation, summing over k and a, and plugging Lemmas B.4 and B.5. □

B.7 $\mathbf{g}$ is closer to $\mathbb{R}_-^d$ than $\mathbf{r}$

Lemma B.7

$$\mathbf{d}_2\left(\sum_{\substack{k\in\mathcal{K}\\a\in\mathcal{A}}} \lambda_T(k,a)\cdot\mathbf{g}(a,\bar{y}_T(k,a)),\ \mathbb{R}_-^d\right)$$
$$\leqslant \mathbf{d}_2\left(\sum_{\substack{k\in\mathcal{K}\\a\in\mathcal{A}}} \lambda_T(k,a)\cdot\mathbf{r}(a,\bar{f}_T(k,a)),\ \mathbb{R}_-^d\right).$$

Proof. Let $k\in\mathcal{K}$ and $a\in\mathcal{A}$. First note that $\mathbf{f}(\bar{y}_T(k,a)) = \bar{f}_T(k,a)$. Indeed, using the affinity of $\mathbf{f}$,

$$\mathbf{f}(\bar{y}_T(k,a)) = \mathbf{f}\left(\frac{1}{|N_T(k,a)|}\sum_{t\in N_T(k,a)} y_t\right)$$
$$= \frac{1}{|N_T(k,a)|}\sum_{t\in N_T(k,a)} \mathbf{f}(y_t)$$
$$= \frac{1}{|N_T(k,a)|}\sum_{t\in N_T(k,a)} f_t = \bar{f}_T(k,a).$$

For each component $n \in \{1,\dots,d\}$, we have $\mathbf{g}^n(a,\bar{y}_T(k,a)) \leqslant \mathbf{r}^n(a,\bar{f}_T(k,a))$ by property (i) in Proposition 3.1. Finally, using the explicit expression of the Euclidean distance to $\mathbb{R}_-^d$, we have

$$\mathbf{d}_2\left(\sum_{\substack{k\in\mathcal{K}\\a\in\mathcal{A}}} \lambda_T(k,a)\cdot\mathbf{g}(a,\bar{y}_T(k,a)),\ \mathbb{R}_-^d\right)$$
$$= \sqrt{\sum_{n=1}^d\left(\sum_{\substack{k\in\mathcal{K}\\a\in\mathcal{A}}} \lambda_T(k,a)\cdot\mathbf{g}^n(a,\bar{y}_T(k,a))\right)_+^2}$$
$$\leqslant \sqrt{\sum_{n=1}^d\left(\sum_{\substack{k\in\mathcal{K}\\a\in\mathcal{A}}} \lambda_T(k,a)\cdot\mathbf{r}^n(a,\bar{f}_T(k,a))\right)_+^2}$$
$$= \mathbf{d}_2\left(\sum_{\substack{k\in\mathcal{K}\\a\in\mathcal{A}}} \lambda_T(k,a)\cdot\mathbf{r}(a,\bar{f}_T(k,a)),\ \mathbb{R}_-^d\right).$$

□

B.8 Decomposition of $\mathbf{g}(a_t,y_t)$ with respect to the realized auxiliary decision (k_t,a_t)

Lemma B.8

$$\frac{1}{T}\sum_{t=1}^T \mathbf{g}(a_t,y_t) = \sum_{\substack{k\in\mathcal{K}\\a\in\mathcal{A}}} \lambda_T(k,a)\cdot\mathbf{g}(a,\bar{y}_T(k,a))$$

Proof. Using the definitions of $N_T(k,a)$ and $\lambda_T(k,a)$, and the linearity of $\mathbf{g}(a,\cdot)$, we have

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\mathbf{g}(a_t,y_t) &= \frac{1}{T}\sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}}\sum_{t\in N_T(k,a)}\mathbf{g}(a,y_t)\\ &= \sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}}\frac{|N_T(k,a)|}{T}\cdot\frac{1}{|N_T(k,a)|}\sum_{t\in N_T(k,a)}\mathbf{g}(a,y_t)\\ &= \sum_{\substack{k\in\mathcal{K}\\ a\in\mathcal{A}}}\lambda_T(k,a)\cdot\mathbf{g}(a,\bar{y}_T(k,a)).\end{aligned}$$

□

B.9 From $\mathbf{g}(i_t,j_t)$ to $\mathbf{g}(a_t,y_t)$

Lemma B.9

$$\mathbb{E}\left[\left\|\frac{1}{T}\sum_{t=1}^{T}\mathbf{g}(i_t,j_t)-\frac{1}{T}\sum_{t=1}^{T}\mathbf{g}(a_t,y_t)\right\|_2\right] \leqslant \frac{2\sqrt{\pi}\,\|\mathbf{g}\|_2}{\sqrt{T}}+2\gamma\,\|\mathbf{g}\|_2\,.$$

Proof. Consider the process $(X_t)_{t\geqslant 1}$ defined by

$$X_t = \mathbf{g}(i_t,j_t)-(1-\gamma)\mathbf{g}(a_t,y_t)-\gamma\mathbf{g}(u,y_t),$$

and the filtration $(\mathcal{G}'_t)_{t\geqslant 1}$ where $\mathcal{G}'_t$ is generated by

$$(k_1,a_1,y_1,i_1,s_1,\dots,\; k_{t-1},a_{t-1},y_{t-1},i_{t-1},s_{t-1},k_t,a_t,y_t).$$

$(X_t)_{t\geqslant 1}$ is martingale difference sequence with respect to filtration $(\mathcal{G}'_t)_{t\geqslant 1}$. Indeed, knowing $\mathcal{G}'_t$, the law of i_t is $(1-\gamma)a_t+\gamma u$ by definition of the algorithm, and thus the law of (i_t,j_t) is $((1-\gamma)a_t+\gamma u)\otimes y_t$. We can then write, by bilinearity of $\mathbf{g}$:

$$\mathbb{E}\left[\mathbf{g}(i_t,j_t)\,|\,\mathcal{G}'_t\right] = (1-\gamma)\mathbf{g}(a_t,y_t)+\gamma\mathbf{g}(u,y_t).$$

Moreover, $\|X_t\|_2$ is always bounded by $2\,\|\mathbf{g}\|_2$:

$$\begin{aligned}\|X_t\|_2 &= \|(1-\gamma)\left(\mathbf{g}(i_t,j_t)-\mathbf{g}(a_t,y_t)\right)\\ &\qquad+\gamma(\mathbf{g}(i_t,j_t)-\mathbf{g}(u,y_t))\|_2\\ &\leqslant (1-\gamma)\|\mathbf{g}(i_t,j_t)-\mathbf{g}(a_t,y_t)\|_2\\ &\qquad+\gamma\|\mathbf{g}(i_t,j_t)-\mathbf{g}(u,y_t)\|_2\\ &\leqslant 2\,\|\mathbf{g}\|_2\,.\end{aligned}$$

We can thus apply Corollary E.2 with $M=2\,\|\mathbf{g}\|_2$ to get

$$\mathbb{E}\left[\left\|\frac{1}{T}\sum_{t=1}^{T}X_t\right\|_2\right]\leqslant\frac{2\sqrt{\pi}\,\|\mathbf{g}\|_2}{\sqrt{T}}.$$

Therefore,

$$\begin{aligned}&\left\|\frac{1}{T}\sum_{t=1}^{T}\mathbf{g}(i_t,j_t)-\frac{1}{T}\sum_{t=1}^{T}\mathbf{g}(a_t,y_t)\right\|_2\\ &=\left\|\frac{1}{T}\sum_{t=1}^{T}\left(X_t+\gamma(\mathbf{g}(u,y_t)-\mathbf{g}(a_t,y_t))\right)\right\|_2\\ &\leqslant\left\|\frac{1}{T}\sum_{t=1}^{T}X_t\right\|_2+\left\|\frac{\gamma}{T}\sum_{t=1}^{T}\left(\mathbf{g}(u,y_t)-\mathbf{g}(a_t,y_t)\right)\right\|_2\\ &\leqslant\left\|\frac{1}{T}\sum_{t=1}^{T}X_t\right\|_2+2\gamma\,\|\mathbf{g}\|_2\,,\end{aligned}$$

And taking the expectation:

$$\mathbb{E}\left[\left\|\frac{1}{T}\sum_{t=1}^{T}\mathbf{g}(i_t,j_t)-\frac{1}{T}\sum_{t=1}^{T}\mathbf{g}(a_t,y_t)\right\|_2\right] \leqslant \frac{2\sqrt{\pi}\,\|\mathbf{g}\|_2}{\sqrt{T}}+2\gamma\,\|\mathbf{g}\|_2\,.$$

□

B.10 Final bound

We now combine the above lemmas in the order specified at the beginning of the section to get:

$$\begin{aligned}\mathbb{E}\left[\mathbf{d}_2\left(\bar{g}_T,\ \mathbb{R}^d_-\right)\right]\leqslant&\frac{2\sqrt{\pi}\,\|\mathbf{g}\|_2}{\sqrt{T}}+2\gamma\,\|\mathbf{g}\|_2\\ &+L_{\mathbf{r}}\,|\mathcal{I}|\,|\mathcal{K}|\,|\mathcal{A}|\left(\frac{8}{\sqrt{T\gamma}}+\frac{8}{3T\gamma}\right)\\ &+\frac{3L_{\mathbf{r}}}{2}\sqrt{|\mathcal{K}|\,|\mathcal{A}|}\left(\frac{1}{\eta T}+\frac{\eta\,|\mathcal{I}|^2}{\gamma}\right).\end{aligned}$$

Injecting the values of η and γ yields the result.

C Closed Convex Cones

Throughout the section, $\mathcal{W}$ will be a finite-dimensional vector space and $\mathcal{W}^*$ its dual.

Definition C.1 *A nonempty subset $\mathcal{C}$ of $\mathcal{W}$ is a closed convex cone if it is closed and if for all $w,w'\in\mathcal{C}$ and $\lambda\in\mathbb{R}_+$, we have $w+w'\in\mathcal{C}$ and $\lambda w\in\mathcal{C}$.*

The following proposition gathers a few immediate properties.

Proposition C.2 *(i) A closed convex cone is convex.*

(ii) An intersection of closed convex cones is a closed convex cone.

(iii) A Cartesian product of closed convex cones is a closed convex cone.

(iv) A half-space of the form $\{w \in \mathcal{W} \,|\, \langle z|w\rangle \leqslant 0\}$ *(for some* $z \in \mathcal{W}^*$*) is a closed convex cone.*

Definition C.3 *Let* $\mathcal{A}$ *be a subset of* $\mathcal{W}$*. The polar cone of* $\mathcal{A}$ *is a subset of the dual space* $\mathcal{W}^*$ *defined by*

$$\mathcal{A}^\circ = \{z \in \mathcal{W}^* \,|\, \forall w \in \mathcal{A},\ \langle w|z\rangle \leqslant 0\}\,.$$

The following proposition is an immediate consequence of the Bipolar theorem — see e.g. Theorem 3.3.14 in Borwein and Lewis [2010].

Proposition C.4 *Let* $\mathcal{A}$ *be a subset of* $\mathcal{W}$*.*

(i) $\mathcal{A}^{\circ\circ}$ *is the smallest closed convex cone containing* $\mathcal{A}$*.*

(ii) If $\mathcal{A}$ *is closed and convex, then* $\mathcal{A}^{\circ\circ} = \mathbb{R}_+\mathcal{A}$*.*

(iii) If $\mathcal{A}$ *is a closed convex cone, then* $\mathcal{A}^{\circ\circ} = \mathcal{A}$*.*

Proposition C.5 *Let* $\varphi : \mathcal{W} \to \tilde{\mathcal{W}}$ *be a linear application between two finite-dimensional vector spaces* $\mathcal{W}$ *and* $\tilde{\mathcal{W}}$*,* φ^* *its transpose,* $\mathcal{C}$ *and* $\tilde{\mathcal{C}}$ *closed convex cones in* $\mathcal{W}$ *and* $\tilde{\mathcal{W}}$ *respectively.*

(i) $\varphi(\mathcal{C})$ *is a closed convex cone.*

(ii) Then $\varphi^{-1}(\tilde{\mathcal{C}}) = \varphi^*(\tilde{\mathcal{C}}^\circ)^\circ$*. In particular,* $\varphi^{-1}(\tilde{\mathcal{C}})$ *is a closed convex cone.*

Proof. Property (i) is obvious. We prove property (ii) as follows. For $w \in \mathcal{W}$,

$$\begin{aligned} w \in \varphi^{-1}(\tilde{\mathcal{C}}) \quad &\iff \quad \varphi(w) \in \tilde{\mathcal{C}} \quad \iff \quad \varphi(w) \in \tilde{\mathcal{C}}^{\circ\circ} \\ &\iff \quad \forall \tilde{z} \in \tilde{\mathcal{C}}^\circ, \quad \langle \tilde{z}|\varphi(w)\rangle \leqslant 0 \\ &\iff \quad \forall z \in \tilde{\mathcal{C}}^\circ, \quad \langle \varphi^*(\tilde{z})|w\rangle \leqslant 0 \\ &\iff \quad w \in \varphi^*(\tilde{\mathcal{C}}^\circ)^\circ. \end{aligned}$$

Therefore, $\varphi^{-1}(\tilde{\mathcal{C}})$ is a closed convex cone because it is a polar cone. □

Proposition C.6 *Let* $\mathcal{C}$ *be a closed convex cone in* $\mathbb{R}^n$*. For all point* $w \in \mathbb{R}^n$*, its Euclidean distance to* $\mathcal{C}$ *can be written*

$$\mathbf{d}_2\,(w,\ \mathcal{C}) = \max_{z \in \mathcal{C}^\circ \cap \mathcal{B}_2} \langle w|z\rangle\,.$$

where $\mathcal{B}_2$ *denotes the closed unit Euclidean ball.*

D A regret minimization bound

The following statement is classic in the regret minimization literature—see e.g. Shalev-Shwartz [2011, Theorem 2.4].

Theorem D.1 *Let* $n \geqslant 1$*,* $\mathbb{R}^n$ *endowed with its canonical Euclidean structure,* $\mathcal{Z}$ *a nonempty convex compact subset of* $\mathbb{R}^d$*,* $(u_t)_{t\geqslant 1}$ *a sequence in* $\mathbb{R}^n$*,* $\eta > 0$*, and*

$$z_t = \arg\max_{z \in \mathcal{Z}} \left\{ \left\langle \eta \sum_{s=1}^{t-1} u_s \middle| z \right\rangle - \frac{1}{2}\left\|z\right\|_2^2 \right\}, \quad t \geqslant 1.$$

Then, for all $T \geqslant 1$*,*

$$\max_{z \in \mathcal{Z}} \sum_{t=1}^T \langle u_t|z\rangle - \sum_{t=1}^T \langle u_t|z_t\rangle \leqslant \frac{\left\|\mathcal{Z}\right\|_2^2}{2\eta} + \frac{\eta}{2}\sum_{t=1}^T \left\|u_t\right\|_2^2.$$

E Concentration inequalities

The following result is a generalization to vector-valued martingale differences of Hoeffding–Azuma's inequality and is due to Kallenberg and Sztencel [1991].

Proposition E.1 *Let* $(U_t)_{t\geqslant 1}$ *be a sequence of martingale differences in* $\mathbb{R}^d$*, bounded almost-surely by* $M > 0$*:*

$$\forall t \geqslant 1, \quad \left\|U_t\right\|_2 \leqslant M, \quad a.s.$$

Then, for every $\varepsilon > 0$ *and* $T \geqslant 1$*,*

$$\mathbb{P}\left[\left\|\frac{1}{T}\sum_{t=1}^T U_t\right\|_2 \geqslant \varepsilon\right] \leqslant 2\exp\left(-\frac{T\varepsilon^2}{4M^2}\right).$$

Corollary E.2 *Under the assumptions of Proposition E.1, we have:*

$$\mathbb{E}\left[\left\|\frac{1}{T}\sum_{t=1}^T U_t\right\|_2\right] \leqslant M\sqrt{\frac{\pi}{T}}.$$

Proof. The result follows from Proposition E.1 by integrating the tail of the distribution:

$$\begin{aligned} \mathbb{E}\left[\left\|\bar{U}_T\right\|_2\right] &= \int_0^{+\infty} \mathbb{P}\left[\left\|\bar{U}_T\right\|_2 \geqslant \varepsilon\right]\,\mathrm{d}\varepsilon \\ &\leqslant \int_0^{+\infty} 2e^{-T\varepsilon^2/4M^2}\,\mathrm{d}\varepsilon \\ &= 2\int_0^{+\infty} e^{-\varepsilon^2(T/4M^2)}\,\mathrm{d}\varepsilon = M\sqrt{\frac{\pi}{T}}. \end{aligned}$$

□

The following Bernstein-like inequality is proved in Pinelis [1994]—see also [Tarres and Yao, 2014, Corollary A.2].

Proposition E.3 *Let* $(X_t)_{t\geqslant 1}$ *be a martingale difference sequence in a Hilbert space with respect to a filtration* $(\mathcal{G}_t)_{t\geqslant 0}$*. Suppose that* $\|X_t\| \leqslant M$ *almost-surely, and*

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\|X_t\|^2 \,\middle|\, \mathcal{G}_{t-1}\right] \leqslant V.$$

Then,

$$\mathbb{P}\left[\max_{1\leqslant t\leqslant T}\left\|\sum_{t'=1}^{t} X_{t'}\right\| \geqslant \varepsilon\right] \leqslant 2\exp\left(-\frac{\varepsilon^2}{2TV + 2M\varepsilon/3}\right).$$

Corollary E.4 *Under the assumptions of Proposition E.3,*

$$\mathbb{E}\left[\left\|\frac{1}{T}\sum_{t=1}^{T} X_t\right\|\right] \leqslant 4\sqrt{2}\sqrt{\frac{V}{T}} + \frac{4M}{3T}.$$

Proof. Let $A \geqslant 0$ to be chosen later.

$$\begin{aligned}
\mathbb{E}\left[\|\bar{X}_T\|\right] &= \int_0^{+\infty} \mathbb{P}\left[\|\bar{X}_T\| \geqslant \varepsilon\right] \,\mathrm{d}\varepsilon \\
&\leqslant 2\int_0^{+\infty} \exp\left(-\frac{\varepsilon^2 T^2}{2VT + 2M\varepsilon T/3}\right) \,\mathrm{d}\varepsilon \\
&= 2\int_0^{+\infty} \exp\left(-\frac{\varepsilon^2 T}{2V + 2M\varepsilon/3}\right) \,\mathrm{d}\varepsilon \\
&\leqslant 2\left(A + \int_A^{+\infty} \exp\left(-\frac{\varepsilon^2 T}{2\varepsilon(V/A + M/3)}\right) \,\mathrm{d}\varepsilon\right) \\
&= 2\left(A + \int_A^{+\infty} \exp\left(-\frac{\varepsilon T}{2(V/A + M/3)}\right) \,\mathrm{d}\varepsilon\right) \\
&= 2\left(A + \left[-\frac{2}{T}\left(\frac{V}{A} + \frac{M}{3}\right)\right.\right. \\
&\qquad\qquad \left.\left. \times \exp\left(-\frac{\varepsilon T}{2(V/A + M/3)}\right)\right]_A^{+\infty}\right) \\
&\leqslant 2A + \frac{4}{T}\left(\frac{V}{A} + \frac{M}{3}\right).
\end{aligned}$$

Choosing $A = \sqrt{2V/T}$ gives:

$$\mathbb{E}\left[\|\bar{X}_T\|\right] \leqslant 4\sqrt{2}\sqrt{\frac{V}{T}} + \frac{4M}{3T}.$$

□