Learning Cost-Effective and Interpretable Treatment Regimes (Supplementary)

1 Proof for Theorem 1

Theorem 1: *The objective function defined in Eqn. 7 is NP-hard.*

Proof: Here, we prove that the objective outlined in Eqn. 7 is NP-hard via a reduction to the weighted exact-cover problem.

Recall that our goal is to find a sequence of if-then-else rules (decision list) which maximizes the following objective:

$$\arg\max_{\pi \in C(\mathcal{L}) \times \mathcal{A}} \lambda_1 g_1(\pi) - \lambda_2 g_2(\pi) - \lambda_3 g_3(\pi)$$

where $g_1(\pi)$, $g_2(\pi)$, and $g_3(\pi)$ are as defined in Eqns. 4-6 and correspond to expected outcome, expected assessment cost, and expected treatment cost respectively. $C(\mathcal{L})$ is the set of permutations of all possible subsets (except the null set) of $\mathcal{L} = \mathcal{FP} \times \mathcal{A}$ where FP denotes the set of frequently occurring patterns each of which is a conjunction of one or more predicates of the form (f, o, v) (See section 3.2) and $\mathcal{A}$ is the set of all possible treatments.

The key idea behind this proof lies in demonstrating that the problem of finding an optimal decision list over $C(\mathcal{L}) \times \mathcal{A}$ can be reformulated as the problem of finding a set of independent if-then rules which are 1) non-overlapping 2) *cover* all the subjects in the dataset $\mathcal{D}$, and 3) optimize our objective function. (1) and (2) together imply that the characteristics of each subject in $\mathcal{D}$ should satisfy exactly one of these if-then rules.

To illustrate, let us consider a medical treatment recommendation dataset $\mathcal{D}$ with three characteristics namely, age, gender, and BMI. Let us assume that each subject in this data is assigned to either treatment T1 or T2 and the set $\mathcal{FP}$ comprises of the following two frequently occurring patterns:
(1) Age $\geq$ 40 $\wedge$ Gender $=$ Female;
(2) BMI $=$ High;

The set $\mathcal{L} = \mathcal{FP} \times \mathcal{A}$ for this dataset will consist of the following rules:
(1) (Age $\geq$ 40 $\wedge$ Gender $=$ Female, T1)
(2) (Age $\geq$ 40 $\wedge$ Gender $=$ Female, T2)
(3) (BMI $=$ High, T1)
(4) (BMI $=$ High, T2)

Let us assume that the optimal decision list for this dataset is:

If Age$\geq$ 40 **and** Gender$=$Female **then** T1
Else T2

This list can be rewritten using a set of non-overlapping if-then rules and the negation operator as follows:

If Age$\geq$ 40 **and** Gender$=$Female **then** T1
If $\neg$(Age$\geq$ 40 **and** Gender$=$Female) **then** T2

This simple example shows that a decision list can be easily converted into a set of independent, non-overlapping if-then rules which cover all the subjects in $\mathcal{D}$. In order to reformulate our problem of learning an optimal decision list to that of learning a set of if-then rules, we first need to define the candidate set of if-then rules appropriately.

Let us create a new candidate rule set $\mathcal{L}'$ from $\mathcal{L}$ as follows: For each rule $r = (c, a)$ in $\mathcal{L}$

- add the rule r to $\mathcal{L}'$
- create a new rule $r^c = (\neg c, a)$ and add it to $\mathcal{L}'$
- append all possible conjunctive combinations of the negations of conditions $c' \in \mathcal{FP}$ ($c' \neq c$ and $c' \neq \neg c$) to the condition c in r and add these new rules to $\mathcal{L}'$.
- append all possible conjunctive combinations of negations of conditions $c' \in \mathcal{FP}$ ($c' \neq c$ and $c' \neq \neg c$) to the condition $\neg c$ in r^c and add these new rules to $\mathcal{L}'$.

Following our example above, the new set $\mathcal{L}'$ will comprise of the following rules:
(1) (Age $\geq$ 40 $\wedge$ Gender $=$ Female, T1)
(2) (Age $\geq$ 40 $\wedge$ Gender $=$ Female, T2)
(3) ($\neg$(Age $\geq$ 40 $\wedge$ Gender $=$ Female), T1)
(4) ($\neg$(Age $\geq$ 40 $\wedge$ Gender $=$ Female), T2)
(5) ($\neg$ (BMI $=$ High) $\wedge$ Age $\geq$ 40 $\wedge$ Gender $=$ Female, T1)
(6) ($\neg$ (BMI $=$ High) $\wedge$ Age $\geq$ 40 $\wedge$ Gender $=$ Female, T2)
(7) (BMI $=$ High, T1)
(8) (BMI $=$ High, T2)
(9) ($\neg$(BMI $=$ High), T1)

(10) ($\neg$(BMI = High), T2)
(11) ($\neg$ (Age $\geq$ 40 $\wedge$ Gender = Female) $\wedge$ BMI = High, T1)
(12) ($\neg$ (Age $\geq$ 40 $\wedge$ Gender = Female) $\wedge$ BMI = High, T2)
(13) ($\neg$(Age $\geq$ 40 $\wedge$ Gender = Female) $\wedge$ $\neg$(BMI = High), T1)
(14) ($\neg$(Age $\geq$ 40 $\wedge$ Gender = Female) $\wedge$ $\neg$(BMI = High), T2)

Our problem of learning an optimal decision list can now be solved by choosing a set of if-then rules from the set $\mathcal{L}'$ such that 1) characteristics of each subject in the data satisfy exactly one rule in the solution set 2) the objective function in Eqn. 7 is maximized. This can be formalized as a weighted exact cover problem:

$$\min \sum_{r_j \in \mathcal{L}'} \Psi(r_j)\phi(r_j)$$

$$\text{s.t.} \sum_{\{r_j=(c_j,a_j)\in\mathcal{L}' \mid satisfy(\mathbf{x}_i,c_j)\}} \phi(r_j) = 1 \;\forall\; (\mathbf{x}_i, a_i, y_i) \in \mathcal{D}$$

$$\phi(r_j) \in \{0,1\} \;\forall r_j \in \mathcal{L}'$$

$\phi(r_j)$ is an indicator function which is 1 if the rule $r_j \in \mathcal{L}'$ is chosen to be in the solution set. $\Psi(r_j)$ represents the weight associated with choosing the rule $r_j = (c_j, a_j)$ which is defined as:

$$\Psi(r_j) = \sum_{\{i\in\{1\cdots N\} \mid satisfy(\mathbf{x}_i,c_j)\}} -\frac{\lambda_1}{N} o(i, a_j) + \frac{\lambda_2}{N} \sum_{e\in\mathcal{Q}_j} d(e) + \frac{\lambda_3}{N} d'(a_j)$$

where $\mathcal{Q}_j$ denotes the set of all characteristics present in the condition c_j.

Note that the function Ψ distributes the value of our objective function (Eqn. 7) across the rules in the solution set. Furthermore, weighted exact cover formulation is a minimization problem, so we flip the signs of the terms in our objective (which is a maximization function) when defining the function Ψ. Since the weighted exact cover problem is NP-Hard, our objective function is also NP-Hard.