A Proofs

Proof of Lemma 4.1. Let T_b be the smallest t at which the positive martingale $\Lambda_t \geq b$, with $T_b = \infty$ if the threshold is never crossed. Then T_b is a stopping time. For some number of visitors t, we have

$$\Lambda_{\min\{T_b,t\}} \geq \begin{cases} b & T_b \leq t, \\ 0 & T_b > t. \end{cases}$$

Therefore, we may write

$$1 = \mathbb{E}[\Lambda_{\min\{T_b,t\}}] \geq bP(T_b \leq t),$$

where the equality comes from the *optional stopping theorem.* Taking $t \to \infty$ bounds the probability of ever crossing the threshold at $P(T_b \leq \infty) \leq 1/b$, which means $P(\exists\, t \text{ such that } \Lambda_t \geq b) \leq 1/b$. This completes the proof of this lemma. □

Proof of Corollary 4.2. Based on fundamental arguments in probability theory, we have the following chain of inequalities:

$$\begin{aligned}\mathbb{P}\left(\frac{1}{\max_{n' \leq t} \Lambda_{n'}} \leq \delta\right) &= \mathbb{P}\left(\max_{n' \leq t} \Lambda_{n'} \geq 1/\delta\right) \\ &\leq \mathbb{P}\left(\exists\, t \text{ such that } \Lambda_t \geq 1/\delta\right) \\ &\leq \delta\end{aligned}$$

where the last step follows from Lemma 4.1. This completes the proof of this corollary. □

Proof of Corollary 4.3. We first show that the likelihood ratio is a positive martingale under the simple null hypothesis $H_0 : \theta_0$. If we take the expectation under the null hypothesis H_0 from the likelihood ratio Λ_t, i.e.,

$$\Lambda_t = \frac{\Pr(\mathcal{D}_t|\theta_1^*)}{\Pr(\mathcal{D}_t|\theta_0^*)} \overset{\text{i.i.d.}}{=} \frac{\prod_{t=1}^n P(x_t|\theta_1^*)}{\prod_{t=1}^n P(x_t|\theta_0^*)}$$

conditioned on whatever observed up to time $t-1$, i.e., $\mathcal{F}_{t-1}$, we have

$$\begin{aligned}\mathbb{E}_0[\Lambda_t|\mathcal{F}_{t-1}] &= \mathbb{E}_0\left[\frac{P(x_t|\theta_1^*)}{P(x_t|\theta_0^*)}\right]\prod_{t=1}^{t-1}\frac{P(x_t|\theta_1^*)}{P(x_t|\theta_0^*)} \\ &= \Lambda_{t-1}\underbrace{\int \frac{P(x_t|\theta_1^*)}{P(x_t|\theta_0^*)}P(x_t|\theta_0^*)dx_t}_{=1} \\ &= \Lambda_{t-1}\end{aligned}$$

where $\mathbb{E}_0$ is the expectation under the null hypothesis H_0.

We then show that the Bayes factor is also a test martingale if the null hypothesis is simple. To start with, since the null hypothesis is simple, $\int P(\theta_0|M_0)P(\mathcal{D}_n|\theta_0, M_0)d\theta_0 = P(\mathcal{D}_n|M_0, \theta_0)$. Hence we may write the Bayes factor as

$$\begin{aligned}\Lambda_n &= \frac{\int \Pr(\theta_1|M_1)\Pr(\mathcal{D}_n|\theta_1, M_1)d\theta_1}{\int \Pr(\theta_0|M_0)\Pr(\mathcal{D}_n|\theta_0, M_0)d\theta_0} \\ &= \frac{\int P(\theta_1|M_1)\prod_{t=1}^n P(x_t|\theta_1, M_1)d\theta_1}{\prod_{t=1}^n P(x_t|\theta_0, M_0)}\end{aligned}$$

The expectation under the null hypothesis H_0 from Λ_n conditioned on whatever observed up to time $n-1$, i.e., $\mathcal{F}_{n-1}$, we have

$$\begin{aligned}&\mathbb{E}_0[\Lambda_n|\mathcal{F}_{n-1}] \\ &= \mathbb{E}_0\left[\int \frac{P(\theta_1|M_1)P(x_n|\theta_1, M_1)\prod_{t=1}^{n-1}P(x_t|\theta_1, M_1)}{P(x_n|M_0)\prod_{t=1}^{n-1}P(x_t|M_0)}d\theta_1\right] \\ &= \int P(\theta_1|M_1)\underbrace{\mathbb{E}_0\left[\frac{P(x_n|\theta_1, M_1)}{P(x_n|M_0)}\right]}_{=1}\frac{\prod_{t=1}^{n-1}P(x_t|\theta_1, M_1)}{\prod_{t=1}^{n-1}P(x_t|M_0)}d\theta_1 \\ &= \int P(\theta_1|M_1)\frac{\prod_{t=1}^{n-1}P(x_t|\theta_1, M_1)}{\prod_{t=1}^{n-1}P(x_t|M_0)}d\theta_1 \\ &= \frac{\int P(\theta_1|M_1)P(\mathcal{D}_{n-1}|\theta_1, M_1)d\theta_1}{P(\mathcal{D}_{n-1}|M_0)} = \Lambda_{n-1}\end{aligned}$$

□

Proof of Lemma 5.5. Recall that m_0 is the number of true null hypotheses which we assume have indices $1, \ldots, m_0$; thus, $p_T^1, \ldots, p_T^{m_0}$ are all sequential p-values. This implies that there must exist an increasing function g^k with $g^k(x) \leq x$ such that $\Pr(g^k(p_T^k) \leq \delta) = \delta$; thus, $g^k(p_T^k) \sim \text{Uniform}[0,1]$. If p_T^k is discrete, then we may need to allow g to be random, e.g. $g^k(p_T^k) + \xi$, where ξ is chosen to interpolate between subsequent values of p_T^k.

Define $V = \left|\mathcal{P}\left(g^1(p_T^1), \ldots, g^m(p_t^m)\right) \cap \{1, \ldots, m_0\}\right|$ as the number of true hypotheses rejected by $\mathcal{P}$ on the modified p-values. We will argue that $V_T \leq V$ almost surely.

First, suppose that $\mathcal{P}$ is a step-up procedure and $V = i$; this implies that $g^{(k)}(p^{(k)}) \leq \alpha_i$ for all $k \leq i$ and $\alpha_i < g^{(k)}(p^{(k)})$ for all $k > i$. Since $g(p) \leq p$, we must have

$$\alpha_i < g^{(k)}(p^{(k)}) \leq p^{(k)} \quad \forall k < i$$

and hence $V_t \leq V$ a.s.. This implies that $\mathbb{E}[f(V_T, s)] \leq \mathbb{E}[f(V, s)]$, and using the the guarantee of $\mathcal{P}$ on $f(V, s)$ implies $\mathbb{E}[f(V_T, s)] \leq \mathbb{E}[f(V, s)] \leq q$ for all s, which yields the theorem statement by linearity of expectation.

If $\mathcal{P}$ only have an independent guarantee, note that if $p_T^1, \ldots, p_T^{m_0}$ are independent, then so are

$g^1(p_T^1), \dots, g^{m_0}(p_T^{m_0})$ and we can apply the same reasoning as above to imply the (f, q) guarantee in the independent case. □

Proof of Lemma 5.6. Let $\mathcal{M} = \mathcal{R}(p_{T_1}^1, \dots, p_{T_m}^m)$, $\mathcal{M}' = \mathcal{R}(p_N^1, \dots, p_N^m)$ be the tests rejected by $\mathcal{S}(\mathcal{P})$ and $\mathcal{S}'(\mathcal{P})$ and $R = |\mathcal{M}|$, $R' = |\mathcal{M}'|$ be their cardinally, respectively.

Consider the case when $\mathcal{P}$ is a step-up procedure. We show that $R = R'$. The monotonicity of the sequential p–values implies that $p_{T_k}^k \le p_N^k$ for all k, and thus $R \le R'$; if $p_{T_k}^k$ is rejected, then p_N^k must be as well. We also argue that $R \ge R'$; suppose, instead, that $R < R'$, which implies that $p_{T_{(R)}}^{(R)} \le \alpha_R < p_N^{(R')} \le \alpha_{R'}$, which is a contradiction since $T_k = N$ for all k that were not already stopped. Hence, $R = R'$.

If the two procedures both reject R tests, then $\alpha_R < p_N^{(V+1)}, \dots, p_N^{(m)}$ and $\alpha_R < p_{T_{(R+1)}}^{(R+1)}, \dots, p_{T_{(m)}}^{(m)}$ correspond to the tests that have not been rejected. But we have $T_{(R+1)}, \dots, T_{(m)} = N$, so $\mathcal{M} = \mathcal{M}'$.

Now consider the case when $\mathcal{P}$ is a step-down procedure. If R tests are rejected by $\mathcal{S}(\mathcal{P})$, then $p_{T_{(1)}}^{(1)} \le \alpha_1, \dots, p_{T_{(R)}}^{(R)} \le \alpha_R$. Since $p_N^{(k)} \le p_{T_{(k)}}^{(k)}$, we must have $\mathcal{M} \subseteq \mathcal{M}'$. Now, suppose there exists an index k such that test k is rejected by $\mathcal{S}'(\mathcal{P})$ but not $\mathcal{S}(\mathcal{P})$. This implies that $p_N^k \le \alpha_{R'}$ but $p_{T_k}^k > \alpha_{R'}$. However, by construction of T_k, we have that $T_k < N$ only if test k is rejected, which implies that $p_{T_k}^k > \alpha_{R'}$ cannot happen. Thus, we have $\mathcal{M} = \mathcal{M}'$ for the step-down case as well. □