

# Concepts for Decision Making under Severe Uncertainty with Partial Ordinal and Partial Cardinal Preferences

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## Abstract

We introduce three different approaches for decision making under uncertainty, if (I) there is only partial (both cardinal and ordinal) information on an agent's preferences and (II) the uncertainty about the states of nature is described by a credal set. Particularly, (I) is modeled by a pair of relations, one specifying the partial rank order of the alternatives and the other modeling partial information on the strength of preference. Our first approach relies on criteria that construct complete rankings of the acts based on generalized expectation intervals. Subsequently, we introduce different concepts of global admissibility that construct partial orders by comparing all acts simultaneously. Finally, we define criteria induced by suitable binary relations on the set of acts and, therefore, can be understood as concepts of local admissibility. Whenever suitable, we provide linear programming based algorithms for checking optimality/admissibility of acts.

**Keywords:** partial preferences; ordinality; cardinality; decision making under uncertainty; linear programming; decision criterion; stochastic dominance; utility representation; admissibility.

## 1. Introduction

One of the constantly recurring topics discussed in the *imprecise probabilities* community (and on ISIPTA conferences in particular) is defining meaningful criteria for decision making under complex uncertainty, finding persuading axiomatic justifications for them and providing efficient algorithms capable to deal with them. Examples ranging from early ISIPTA contributions by, e.g., [Jaffray \(1999\)](#) to (most) recent ones by, e.g., [Bradley \(2015\)](#). However, in the vast majority of works in this field, the complexity underlying the decision situation is assumed to solely arise from beliefs on the mechanism generating the states of nature that are expressed by an imprecise probabilistic model. In contrast, the cardinal utility function adequately describing the decision maker's preference structure is often unquestioned and assumed to be precisely given in advance.<sup>1</sup> Our paper generalizes the classical (generalized) setting to situations, in which this assumption is no longer justified. Particularly, we consider the case that the decision maker's preference structure is both partially ordinal and partially cardinal and, therefore, no longer can be characterized by (a set of positive linear transformations of) *one* cardinal utility function.

The paper is structured as follows: In Section 2, we give a brief overview on the background of our work and show how our approach naturally fits into this picture. In Section 3, we introduce the crucial concept of a *preference system* over a set of alternatives that allows for modeling partially ordinal and partially cardinal preference structures. Section 4 introduces three different approaches

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1. Exceptions include [Montes \(2014, Section 4.2.1\)](#), who uses set-valued utility functions.

for decision making with acts taking values in a preference system by proposing decision criteria based on generalized expectation intervals (Section 4.2), on global comparisons of acts (Section 4.3) and on pairwise comparisons of acts (Section 4.4). Whenever suitable, we give linear programming driven algorithms for checking feasibility of acts in finite decision settings. Section 5 concludes.

## 2. Brief Overview on the Fundamentals underlying our Approach

In classical subjective expected utility theory (SEUT), the decision maker is assumed to be able to specify (I) a *cardinal* utility function (unique up to a positive linear transformation) representing his preferences on a set of alternatives and (II) a unique and *precise* subjective probability measure on the space of states of nature specifying his beliefs on the occurrence of the states. Once these ingredients are specified, according to SEUT, the decision maker should choose any act that maximizes expected utility with respect to his utility and his subjective probability measure. However, in practice both assumptions (I) and (II) often are systematically too restrictive. In particular, (I) demands the decision maker to act in accordance with the axioms of von Neumann and Morgenstern, i.e. to be able to specify a complete preference ranking of all simple lotteries that is both independent and continuous (see, e.g., Fishburn, 1970, Ch. 8), whereas (II) requires that the decision maker can completely order the resulting utility-valued acts by preference in accordance with the axioms of de Finetti, i.e. continuous, additive and monotone (see, e.g., Gilboa, 2009, Ch. 9).

Consequently, there exists plenty of literature relaxing these assumptions. If *only* (II) is violated in the sense that there is *partial* probabilistic information on the occurrence of the states of nature together with a cardinal preference structure, the common relaxation is to allow for *imprecise* probabilistic models in order to represent the probabilistic information. In this case, one can define optimality of acts in terms of some imprecise decision criterion such as  $\Gamma$ -*maximin*,  $\Gamma$ -*maximix*, *maximality* or *E-admissibility* that, each in its own way, takes into account the whole set of probabilities for constructing a ranking of the acts (see Huntley et al. (2014) for a survey and, e.g., Kofler and Menges (1976); Levi (1983); Walley (1991); Gilboa and Schmeidler (1989) for original sources). Accordingly, there exists a very well-investigated and established theory as well as efficient and powerful algorithms to deal with this kind of violation of the classical assumptions (see, e.g., Utkin and Augustin, 2005; Kikuti et al., 2011; Hable and Troffaes, 2014).

If (I) is violated in the sense that the decision maker has only complete *ordinal* preferences and (II) is violated in the sense that there is no probabilistic information *at all*, it is nearly unanimously favored to define optimality of acts in terms of Wald's classical *maximin criterion*: Choose whatever act receiving highest possible rank under the worst possible state of nature (see Wald, 1949). However, note that the completeness of the involved ordinal ranking is essential, since, otherwise, the worst consequences of two distinct acts might be incomparable and, therefore, an optimal act with respect to the maximin criterion simply does not exist. Even more severe, also the vacuousness assumption is crucial: Applying the minimax criterion in the presence of (partial) probabilistic information means willingly ignoring information. This seems not reasonable at all (cf. also Example 1 for an illustration). Finally, if *only* (I) is violated in the sense that there is no cardinal information at all and the available ordinal information is possibly incomplete, one commonly applies the concept of first order *stochastic dominance*: Dismiss an act  $X$  taking values in the partially ordered set, if there exists another act  $Y$  such that  $u \circ Y$  dominates  $u \circ X$  in expectation for every real-valued function  $u$  respecting the partial order (see, e.g., Lehmann, 1955; Kamae et al., 1977; Mosler and Scarsini, 1991).

### 3. Preference Systems

In this section we define the crucial concept of a *preference system*. The intuition behind this concept is simple: In many decision problems, the (available information on) the agent's preferences is incomplete. More precisely, it often is the case that some pairs of possible decision outcomes are incomparable, whereas others can be ordered by preference. For some pairs there might even be an idea of the strength of the preference. There are several situations that could lead to such incomplete preferences. For example, if a company wants to analyze the choice behavior of their customers, the information on the customer's preferences will often be given in form of observed choices and/or survey data. In this case, incompleteness is a missing data problem and originates in lacking information. However, also the agent herself might have incomplete preferences. Suppose she knows (e.g. from earlier experience) certain outcomes better than others. Then for pairs involving better known outcomes, she might be able to specify a preference ranking and even some intuition for the strength of the preference, whereas for pairs involving unfamiliar outcomes, she might be able to specify only a ranking or can't make a comparison at all. The following definition captures the intuition just described.

**Definition 1** *Let  $A$  be a non-empty set and let  $R_1 \subset A \times A$  denote a preorder (i.e. reflexive and transitive) on  $A$ . Moreover, let  $R_2 \subset R_1 \times R_1$  denote a preorder on  $R_1$ . Then the triplet  $\mathcal{A} = [A, R_1, R_2]$  is called a **preference system** on  $A$ .*

Except from transitivity, Definition 1 makes no rationality and/or compatibility assumption on the relations  $R_1$  and  $R_2$ . Hence, a preference system in the sense of the above definition needs by no means to be reasonable or rational. In Krantz et al. (1971, Chapter 4), an axiomatic approach for characterizing consistent preference systems is provided for the case that the involved relations are complete. The corresponding axioms then imply the existence of a real valued function representing both relations simultaneously that is unique up to a positive linear transformation. Another axiomatization that uses quaternary relations instead of pairs of relations is established in Pivato (2013), where it is shown that under some quite strong conditions (like, e.g., *solvability*) there exists a *multiutility characterization* of the corresponding quaternary relation. A weaker consistency condition that still applies to settings in which conditions like solvability no longer can be expected is given in the following definition, for which we need some further notation: If  $R$  is a preorder on  $A$ , we denote by  $I_R$  and  $P_R$  its *indifference* and its *strict part*, respectively. More precisely, for  $(a, b) \in A \times A$ , we have  $(a, b) \in I_R :\Leftrightarrow ((a, b) \in R \wedge (b, a) \in R)$  and  $(a, b) \in P_R :\Leftrightarrow ((a, b) \in R \wedge (b, a) \notin R)$ .

**Definition 2** *Let  $\mathcal{A} = [A, R_1, R_2]$  be a preference system. Then  $\mathcal{A}$  is said to be **consistent** if there exists a function  $u : A \rightarrow [0, 1]$  such that for all  $a, b, c, d \in A$  the following two properties hold:*

- i) *If  $(a, b) \in R_1$ , then  $u(a) \geq u(b)$  with equality iff  $(a, b) \in I_{R_1}$ .*
- ii) *If  $((a, b), (c, d)) \in R_2$ , then  $u(a) - u(b) \geq u(c) - u(d)$  with equality iff  $((a, b), (c, d)) \in I_{R_2}$ .*

Every such function  $u$  is then said to (**weakly**<sup>2</sup>) **represent** the preference system  $\mathcal{A}$ . The set of all (weak) representations  $u$  of  $\mathcal{A}$  is denoted by  $\mathcal{U}_{\mathcal{A}}$ . The set of all  $u \in \mathcal{U}_{\mathcal{A}}$  satisfying  $\inf_{a \in A} u(a) = 0$  and  $\sup_{a \in A} u(a) = 1$  is denoted by  $\mathcal{N}_{\mathcal{A}}$ .

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2. Here, the term *weakly* refers to the fact that the representation is meant in the *if* and not the *iff* sense.

The idea behind the set  $\mathcal{N}_{\mathcal{A}}$  in the above definition is the following: For the special case, that the preference system  $\mathcal{A}$  is in accordance with the axioms in Krantz et al. (1971, Chapter 4), the representation is unique up to a positive linear transformation. Hence, the conditions  $\inf_a u(a) = 0$  and  $\sup_a u(a) = 1$  guarantee a unique representation for that special case. For the general case of a consistent preference system  $\mathcal{A}$  with non complete relations  $R_1$  and  $R_2$ , restricting analysis to the set  $\mathcal{N}_{\mathcal{A}}$  ensures that comparison will not be made with respect to equivalent representation which only measure utility on a different scale. Note that for finite  $A$ , the boundedness condition on the utility function implies the existence of alternatives in  $A$  with greatest and lowest utility value, but not necessarily of worst and best alternatives in  $A$  w.r.t. the relation  $R_1$ . The restriction on  $\mathcal{N}_{\mathcal{A}}$ , together with the concept of *granularity* of Definition 3, will prove crucial when comparing acts by means of numerical representation in Section 4.2. Obviously, for a preference system  $\mathcal{A} = [A, R_1, R_2]$  to be consistent, certain compatibility criteria between the relations  $R_1$  and  $R_2$  have to be satisfied. For example it cannot be the case that, for some elements  $a, b, c \in A$ , it simultaneously holds that  $(c, a) \in P_{R_1}$  and  $((a, b), (c, b)) \in R_2$ , since any element  $u \in \mathcal{U}_{\mathcal{A}}$  would have to satisfy  $u(c) > u(a)$  and  $u(a) - u(b) \geq u(c) - u(b)$ . We now provide an algorithm for checking the consistency of a finite preference system. The proof is straightforward and therefore left out.

**Proposition 1** *Let  $\mathcal{A} = [A, R_1, R_2]$  be a preference system, where  $A = \{a_1, \dots, a_n\}$  is a finite and non-empty set. Consider the linear optimization problem*

$$\varepsilon = \langle (0, \dots, 0, 1)', (u_1, \dots, u_n, \varepsilon)' \rangle \longrightarrow \max_{(u_1, \dots, u_n, \varepsilon) \in \mathbb{R}^{n+1}} \quad (1)$$

with constraints  $0 \leq (u_1, \dots, u_n, \varepsilon) \leq 1$  and

- i)  $u_p = u_q$  for all  $(a_p, a_q) \in I_{R_1} \setminus \text{diag}(A)$
- ii)  $u_q + \varepsilon \leq u_p$  for all  $(a_p, a_q) \in P_{R_1}$
- iii)  $u_p - u_q = u_r - u_s$  for all  $((a_p, a_q), (a_r, a_s)) \in I_{R_2} \setminus \text{diag}(R_1)$
- iv)  $u_r - u_s + \varepsilon \leq u_p - u_q$  for all  $((a_p, a_q), (a_r, a_s)) \in P_{R_2}$

Then  $\mathcal{A}$  is consistent if and only if the optimal outcome of (1) is strictly positive.

The linear programming problem (1) possesses  $|R_2| + n + 2$  constraints. Thus, the number of constraints increases with the preciseness of the available information on the agent's preferences. In applications, typically the relation  $R_2$  will be rather sparse, whereas the relation  $R_1$  will be rather dense. This is intuitive: While  $R_1$  is directly observable in the choice behavior of the agent, edges in  $R_2$  need to be gained by hypothetical comparisons in interviews and polls by asking questions like: "Imagine you have objects  $a$  and  $b$ . Would you rather be willing to accept the exchange of  $a$  by  $c$  or the exchange of  $b$  by  $d$ ?" In order to reduce the number of constraints of the problem, note that (weak) representability of a preference system  $\mathcal{A} = [A, R_1, R_2]$  automatically implies transitivity of the relations  $R_1$  and  $R_2$ . Therefore, in the constraints of the above optimization problem it actually suffices to quantify only over the transitive reduction of the relations  $I_{R_1}, P_{R_1}, I_{R_2}$  and  $P_{R_2}$ . Before turning to decision theory with preference system valued acts, we need one further concept:

**Definition 3** *Let  $\mathcal{A} = [A, R_1, R_2]$  be a consistent preference system. Moreover, for  $\delta \in (0, 1)$ , let  $\mathcal{N}_{\mathcal{A}}^{\delta}$  denote the set of all  $u \in \mathcal{N}_{\mathcal{A}}$  satisfying  $u(a) - u(b) \geq \delta$  for all  $(a, b) \in P_{R_1}$  and  $u(c) - u(d) - u(e) + u(f) \geq \delta$  for all  $((c, d), (e, f)) \in P_{R_2}$ . Then,  $\mathcal{N}_{\mathcal{A}}^{\delta}$  is called the **(weak) representation set of granularity** (at least)  $\delta$ .*

The granularity can be given a similar interpretation as the *just noticeable difference* in the context of psychophysics (see [Luce \(1956\)](#) for details): It is the minimal difference in utility that the specific decision maker under consideration is able to notice given that utility is measured on a  $[0, 1]$ -scale. More practically, the restriction to utility functions that reflect the fact that utility differences below some threshold are not distinguishable empirically will play a crucial role when it comes to defining generalized expectations in [Section 4.2](#). For now, it is sufficient to note that the algorithm given in [Proposition 1](#) straightforwardly extends to checking whether the preference system is consistent for a decision maker with granularity  $\delta > 0$ : If  $(u_1^*, \dots, u_n^*, \varepsilon^*)$  is an optimal solution to problem [\(1\)](#), then the system is  $\delta$ -consistent if and only if it holds that  $\delta \leq \varepsilon^*$ .

#### 4. Decision Theory with ps-valued Acts

Differently from axiomatic approaches followed in, e.g., [Seidenfeld et al. \(1995\)](#); [Nau \(2006\)](#); [Galaabaatar and Karni \(2013\)](#), where (multi-)utility and (imprecise) probability representations are obtained by preferences over acts, the aim of the present paper is to obtain preferences on acts given a preference system and some additional probabilistic information. Therefore, we now propose and discuss some first ideas on decision making under uncertainty with acts taking values in some preference system (short: *ps-valued acts*) and partial probabilistic information on the occurrence of the states available. Before turning to these ideas, let us briefly give some intuition why the standard criteria for decision making under uncertainty generally will fail (or at least produce counter-intuitive results) in our context: The classical *maximin criterion*, originally proposed by A. Wald (see [Wald \(1949\)](#)), is the prototypical criterion for decision under complete lack of information. However, applying this criterion in the presence of probabilistic information means willingly ignoring available information and will often lead to counter-intuitive decisions (see [Example 1](#)). On the other hand, the principle of *maximizing expected utility* requires both cardinal utility and precise probabilistic information and, therefore, obviously is not applicable in our situation. Moreover, the common imprecise decision criteria, while explicitly allowing to take into account the incompleteness of the probabilistic information, still require cardinal utility scale. Contrarily, stochastic dominance allows for dealing with non-cardinal utility scales, however, requires precise probabilistic information (for approaches generalizing stochastic dominance to credal sets, see [Montes \(2014, Section 4.1.1\)](#)).

##### 4.1 Basic Setting

We start by defining the central concepts of the theory for the most general case. Let  $S$  denote some non-empty set equipped with some suitable  $\sigma$ -algebra  $\sigma(S)$ . The elements of  $S$  are interpreted as all possible states of nature about whose occurrence the decision maker is uncertain. Moreover, let  $\mathcal{M}$  denote the credal set on the measurable space  $(S, \sigma(S))$ , interpreted as the set of all probabilities that are compatible with the available (partial) probabilistic information and thus describing the uncertainty about the occurrence of the states. For a given consistent preference system  $\mathcal{A}$ , a state space  $S$  and a credal set  $\mathcal{M}$ , a *ps-valued act* is a mapping  $X : S \rightarrow A$  assigning states of nature to values in the preference system. Define the set  $\mathcal{F}_{(\mathcal{A}, \mathcal{M}, S)} \subset A^S := \{f | f : S \rightarrow A\}$  by setting

$$\mathcal{F}_{(\mathcal{A}, \mathcal{M}, S)} := \left\{ X \in A^S : u \circ X \text{ is } \sigma(S)\text{-}\mathcal{B}_{\mathbb{R}}\text{-measurable for all } u \in \mathcal{U}_{\mathcal{A}} \right\} \quad (2)$$

where  $\mathcal{B}_{\mathbb{R}}$  denotes the Borel sigma field on  $\mathbb{R}$ . By construction, the space  $\mathcal{F}_{(\mathcal{A}, \mathcal{M}, S)}$  consists of exactly those acts  $X : S \rightarrow A$  whose expectation exists with respect to all pairs  $(u, \pi) \in \mathcal{U}_{\mathcal{A}} \times \mathcal{M}$

of compatible probability measure and utility representation (since bounded and measurable random variables have finite expectation). Given this notation, we can now define our main object of study:

**Definition 4** *In the situation above, call every subset  $\mathcal{G} \subset \mathcal{F}_{(\mathcal{A}, \mathcal{M}, S)}$  a **decision system** (with information base  $(\mathcal{A}, \mathcal{M})$ ). Moreover, call a decision system  $\mathcal{G}$  finite, if both  $|\mathcal{G}| < \infty$  and  $|S| < \infty$ .*

The elements of a decision system  $\mathcal{G}$  are interpreted as those elements of the space  $\mathcal{F}_{(\mathcal{A}, \mathcal{M}, S)}$  that are available in the specific choice situation under consideration. Given a decision system  $\mathcal{G}$ , we are interested in the following question: How can we utilize the information base  $(\mathcal{A}, \mathcal{M})$  best possibly in order to define meaningful and reasonable choice criteria on the set  $\mathcal{G}$ ? In the following sections, we propose three different classes of approach that address exactly this question.

## 4.2 Criteria based on Generalized Expectation Intervals

In this section, we consider decision criteria that are based on the analysis of generalized expectation intervals. Depending on the attitude towards ambiguity of the decision maker of interest, such intervals give rise to different criteria for decision making. Specifically, for a ps-valued act and a decision maker with granularity  $\delta > 0$ , the corresponding interval will range from the lowest to the highest possible expected value that choosing this act can lead to under some pair  $(u, \pi) \in \mathcal{N}_{\mathcal{A}}^{\delta} \times \mathcal{M}$ . This leads to the definition of the basic quantity of this section.

**Definition 5** *Let  $X \in \mathcal{F}_{(\mathcal{A}, \mathcal{M}, S)}$  and  $\delta \in (0, 1)$ . With  $\mathcal{D}_{\delta} := \mathcal{N}_{\mathcal{A}}^{\delta} \times \mathcal{M}$ , we call the quantity*

$$\mathbb{E}_{\mathcal{D}_{\delta}}(X) := \left[ \underline{\mathbb{E}}_{\mathcal{D}_{\delta}}(X), \overline{\mathbb{E}}_{\mathcal{D}_{\delta}}(X) \right] := \left[ \inf_{(u, \pi) \in \mathcal{D}_{\delta}} \mathbb{E}_{\pi}(u \circ X), \sup_{(u, \pi) \in \mathcal{D}_{\delta}} \mathbb{E}_{\pi}(u \circ X) \right] \quad (3)$$

*the **generalized interval expectation** of  $X$  with respect to  $\mathcal{A}, \mathcal{M}$  and granularity  $\delta$ .*

In the spirit of the theory of imprecise probabilities, the set  $\mathbb{E}_{\mathcal{D}_{\delta}}(X)$  can be given an *epistemic* or an *onthological* interpretation: If the imprecision/ambiguity in the sets arises from lack of information in the sense of e.g. partially observed choice behavior and/or partially known precise probabilities, the set  $\mathbb{E}_{\mathcal{D}_{\delta}}(X)$  is the set of all expectations arising in at least one situation that is compatible with the data. In contrast, if both sets  $\mathcal{N}_{\mathcal{A}}^{\delta}$  and  $\mathcal{M}$  have an onthological interpretation, i.e. are interpreted as holistic entities of their own, the same holds true for the set of expectations  $\mathbb{E}_{\mathcal{D}_{\delta}}(X)$ . Of course, all decision theory that is based on comparisons of the set  $\mathbb{E}_{\mathcal{D}_{\delta}}(X_i)$  of different acts  $X_i$  should reflect the underlying interpretation. The following definition gives three criteria rather relying on an onthological interpretation of the set  $\mathcal{D}_{\delta}$ . Note that all of them are straightforward generalizations of the (complete order inducing) decision criteria commonly used in the theory of imprecise probabilities and reviewed, e.g., in [Huntley et al. \(2014\)](#).

**Definition 6** *Let  $\mathcal{G} \subset \mathcal{F}_{(\mathcal{A}, \mathcal{M}, S)}$  be a decision system and  $\delta, \alpha \in (0, 1)$ . An act  $X \in \mathcal{G}$  is called*

- i)  $\mathcal{D}_{\delta}$ -**maximin** :iff  $\forall Y \in \mathcal{G} : \underline{\mathbb{E}}_{\mathcal{D}_{\delta}}(X) \geq \underline{\mathbb{E}}_{\mathcal{D}_{\delta}}(Y)$
- ii)  $\mathcal{D}_{\delta}$ -**maximax** :iff  $\forall Y \in \mathcal{G} : \overline{\mathbb{E}}_{\mathcal{D}_{\delta}}(X) \geq \overline{\mathbb{E}}_{\mathcal{D}_{\delta}}(Y)$
- iii)  $\mathcal{D}_{\delta}^{\alpha}$ -**maximix** :iff  $\forall Y \in \mathcal{G} : \alpha \underline{\mathbb{E}}_{\mathcal{D}_{\delta}}(X) + (1 - \alpha) \overline{\mathbb{E}}_{\mathcal{D}_{\delta}}(X) \geq \alpha \underline{\mathbb{E}}_{\mathcal{D}_{\delta}}(Y) + (1 - \alpha) \overline{\mathbb{E}}_{\mathcal{D}_{\delta}}(Y)$

*We denote by  $\underline{\mathcal{G}}_{\delta}$ ,  $\overline{\mathcal{G}}_{\delta}$  and  $\mathcal{G}_{\delta}^{\alpha}$  the sets of  $\mathcal{D}_{\delta}$ -maximin,  $\mathcal{D}_{\delta}$ -maximax and  $\mathcal{D}_{\delta}^{\alpha}$ -maximix acts in  $\mathcal{G}$ .*



Independent of its interpretation, we need ways for computing the set  $\mathbb{E}_{\mathcal{D}_\delta}(X)$  in concrete situations. The following proposition gives a linear programming based algorithm for doing so in finite decision systems. However, note that applying the proposition requires the extreme points of the underlying credal set  $\mathcal{M}$  and, therefore, is ideal for situations where the number of extreme points is moderate and where closed formulas for computing the extreme points are available. For credal sets induced by 2-monotone lower/ 2-alternating upper probabilities such formulas exist (cf., [Shapley, 1971](#), Theorem 3, p.19). While generally the number of extreme points could be very high (maximally  $|S|!$  for lower probabilities), convenient cases exist where furthermore efficient enumeration procedures are available (such special cases include *ordinal probabilities* (cf., [Kofler, 1989](#), p. 26), *comparative probabilities* (cf., [Miranda and Destercke, 2015](#)), *necessity measures* (cf., [Schollmeyer, 2015](#)) or *p-boxes* (cf., [Montes and Destercke, 2017](#)).

**Proposition 2** *Let  $\mathcal{A} = [A, R_1, R_2]$  be a consistent preference system, where  $A = \{a_1, \dots, a_n\}$  such that  $(a_1, b), (b, a_n) \in R_1$  for all  $b \in A$  and let  $\varepsilon^*$  denote the optimal outcome of problem (1). Moreover, let  $S = \{s_1, \dots, s_m\}$  be finite,  $\mathcal{M}$  be some polyhedral credal set on  $(S, 2^S)$  with extreme points  $\mathcal{E}(\mathcal{M}) := \{\pi^{(1)}, \dots, \pi^{(T)}\}$  and let  $X \in \mathcal{G}$ . For  $\varepsilon^* \geq \delta > 0$ , consider the collection of linear programs  $LP_1^\delta, \dots, LP_T^\delta$  given by:*

$$\sum_{i=1}^n u_i \cdot \pi^{(t)}(X^{-1}(\{a_i\})) \longrightarrow \min_{(u_1, \dots, u_n) \in \mathbb{R}^n} / \max_{(u_1, \dots, u_n) \in \mathbb{R}^n} \quad (LP_t^\delta)$$

with constraints  $0 \leq (u_1, \dots, u_n) \leq 1$ ,  $u_1 = 1$ ,  $u_n = 0$  and  $i$  to  $iv$ ) as given in Proposition 1 (with  $\varepsilon := \delta$  fixed). Let  $\underline{v}(t, \delta)$  and  $\bar{v}(t, \delta)$  denote the optimal outcomes of problem  $LP_t^\delta$  in minimum and maximum form. Then, we have  $\mathbb{E}_{\mathcal{D}_\delta}(X) = [\min_t \underline{v}(t, \delta), \max_t \bar{v}(t, \delta)]$ .

**Proof.** Let  $X \in \mathcal{G}$  and  $\varepsilon^* \geq \delta > 0$ . Then,  $\mathcal{N}_{\mathcal{A}}^\delta$  is non-empty and we can define the function  $f : \mathcal{D}_\delta \rightarrow \mathbb{R}$ ,  $(u, \pi) \mapsto \mathbb{E}_\pi(u \circ X)$ . For any  $u \in \mathcal{N}_{\mathcal{A}}^\delta$  fixed, the function  $\pi \mapsto f(u, \pi)$  is linear and, therefore, both convex and concave. By applying standard results on families of convex and concave functions, we know that the functions  $\pi \mapsto \inf_u f(u, \pi)$  and  $\pi \mapsto \sup_u f(u, \pi)$  have to be concave and convex, respectively. But concave functions on polyhedral set attain their minimum and convex functions on polyhedral set attain their maximum on the set of extreme points. Hence, in order to find global maximum and minimum of the function  $f$ , it suffices to check on the set  $\mathcal{N}_{\mathcal{A}}^\delta \times \mathcal{E}(\mathcal{M})$ .

Now, let  $(u_1^*, \dots, u_n^*)$  denote an optimal solution to problem  $LP_t^\delta$  in maximum form for fixed  $t \in \{1, \dots, T\}$ . One easily verifies that the constraints imply  $u^* \in \mathcal{N}_{\mathcal{A}}^\delta$ , where  $u^* : A \rightarrow [0, 1]$ ,  $u^*(a_i) := u_i^*$  and  $\bar{v}(t, \delta) = \mathbb{E}_{\pi^{(t)}}(u^* \circ X) = \sup\{\mathbb{E}_{\pi^{(t)}}(u \circ X) : u \in \mathcal{N}_{\mathcal{A}}^\delta\}$ . Analogous reasoning for the problem in minimum form yields  $\underline{v}(t, \delta) = \inf_{u \in \mathcal{N}_{\mathcal{A}}^\delta} \mathbb{E}_{\pi^{(t)}}(u \circ X)$ . Thus, applying our considerations from before yields  $\mathbb{E}_{\mathcal{D}_\delta}(X) = [\min_t \underline{v}(t, \delta), \max_t \bar{v}(t, \delta)]$ .  $\square$

Another way to compute the bounds in (3) in the case of 2-monotone lower probabilities on a finite space  $A$  is to use the Choquet representation of the upper (lower) expectation (cf., e.g., [Denneberg \(1994, Proposition 10.3, p. 126\)](#)): For a fixed utility  $u$  and a 2-alternating upper probability  $\nu$  with associated credal set  $\mathcal{M}_\nu$  the corresponding expected upper utility can be written as  $\overline{\mathbb{E}}_{\{u\} \times \mathcal{M}_\nu}(X) = \sum_{i=1}^n (u_{(i)} - u_{(i-1)}) \cdot \nu(\{s \in S \mid u(X(s)) \geq u_{(i)}\})$ . If  $R_1$  is complete then the expectation is a linear form in the utility  $u$  and the maximization  $\max_{u \in \mathcal{N}_{\mathcal{A}}^\delta} \overline{\mathbb{E}}_{\{u\} \times \mathcal{M}_\nu}(X)$  translates to a simple linear program. If the relation  $R_1$  is not complete then the ordering of the utility values  $u_i$  can change as  $u$  ranges in  $\mathcal{N}_{\mathcal{A}}^\delta$  and one has to compute the expectation separately for every possible ordering of the utility values and then take the maximum. If there are totally comparable values  $u_i$  meaning that

for every  $u_j$  either  $u_i \leq u_j$  or  $u_i > u_j$ , independently from the concrete  $u \in \mathcal{N}_A^\delta$  then one can split the sum in a part containing all utility values below  $u_i$  and a part containing all utility values above  $u_i$  and then analyze every subsum independently which would help in reducing the combinatorial complexity. The criteria from Definition 6 allow for comparing acts *given* the granularity  $\delta$  of the specific decision maker of interest. However, note that knowing the granularity might be a strong assumption if  $R_1$  and  $R_2$  are *partial* orderings, since experimental settings in which this additional parameter could precisely be elicited are not as straightforward as in the complete case. Further possibilities to deal with these issues are treated in the next two sections, where we propose two approaches completely overcoming the choice of a granularity parameter.

### 4.3 Criteria based on Global Comparisons

The decision criteria defined in Section 4.2 all construct complete rankings on the set  $\mathcal{G}$  by comparing numerical representations of parts of the decision system and by somehow ignoring the inherent utility and probability structure. Therefore, when defining optimality of acts in terms of one of the criteria from Definition 6, it makes no difference if the ranking is constructed by pairwise or global comparisons. In the next sections, we turn to two approaches that explicitly take into account a global and local viewpoint for defining optimality of acts, respectively.<sup>3</sup> We start with the global perspective in the sense that we try to find existing utilities (or probabilities, respectively) that can establish a form of global admissibility of a given act  $X$  over all other acts  $Y$  that is valid for every possible underlying probability (or utility, respectively). This is reflected in the fact that in the three admissibility concepts of Definition 7 a  $\forall$  quantifier can follow an  $\exists$  quantifier but not vice versa.

**Definition 7** Let  $\mathcal{G} \subset \mathcal{F}_{(A,M,S)}$  denote a decision system. We call an act  $X \in \mathcal{G}$

- i)  $\mathcal{A}|\mathcal{M}$ -admissible :iff  $\exists u \in \mathcal{U}_A \exists \pi \in \mathcal{M} \forall Y \in \mathcal{G} : \mathbb{E}_\pi(u \circ X) \geq \mathbb{E}_\pi(u \circ Y)$
- ii)  $\mathcal{A}$ -admissible :iff  $\exists u \in \mathcal{U}_A \forall \pi \in \mathcal{M} \forall Y \in \mathcal{G} : \mathbb{E}_\pi(u \circ X) \geq \mathbb{E}_\pi(u \circ Y)$
- iii)  $\mathcal{M}$ -admissible :iff  $\exists \pi \in \mathcal{M} \forall u \in \mathcal{U}_A \forall Y \in \mathcal{G} : \mathbb{E}_\pi(u \circ X) \geq \mathbb{E}_\pi(u \circ Y)$
- iv)  $\mathcal{A}|\mathcal{M}$ -dominant :iff  $\forall u \in \mathcal{U}_A \forall \pi \in \mathcal{M} \forall Y \in \mathcal{G} : \mathbb{E}_\pi(u \circ X) \geq \mathbb{E}_\pi(u \circ Y)$

Denote by  $\mathcal{G}_{\mathcal{A}|\mathcal{M}}$ ,  $\mathcal{G}_A$ ,  $\mathcal{G}_M$  and  $\mathcal{G}_{\mathcal{A}|\mathcal{M}}^d$  the sets of such acts, respectively.

All four act properties just defined rely on the idea that, if there *was* perfect information on both the state probabilities (i.e.  $\mathcal{M} = \{\pi\}$  is a singleton) and the utility values (i.e. the utility representation  $u$  is unique up to a positive linear transformation), then an act  $X$  should be labeled optimal iff  $X$  has greater or equal expected utility than every other act  $Y \in \mathcal{G}$  with respect to  $(u, \pi)$ . However, they differ in the way they handle the ambiguity underlying the involved sets  $\mathcal{M}$  and  $\mathcal{U}_A$ : While  $\mathcal{A}|\mathcal{M}$ -admissibility only demands the existence of at least *one* compatible combination  $(u, \pi)$  with respect to which  $X$  maximizes expected utility,  $\mathcal{A}|\mathcal{M}$ -dominance requires this for *all* compatible combinations.  $\mathcal{M}$ - and  $\mathcal{A}$ -admissibility relax the  $\forall$ -assumption on probability and utility level, respectively. Clearly, it holds that  $\mathcal{G}_A, \mathcal{G}_M, \mathcal{G}_{\mathcal{A}|\mathcal{M}}^d \subset \mathcal{G}_{\mathcal{A}|\mathcal{M}}$  and  $\mathcal{G}_{\mathcal{A}|\mathcal{M}}^d \subset \mathcal{G}_A$  and  $\mathcal{G}_{\mathcal{A}|\mathcal{M}}^d \subset \mathcal{G}_M$ , but in general neither  $\mathcal{G}_A \subset \mathcal{G}_M$  nor  $\mathcal{G}_M \subset \mathcal{G}_A$ . The following example demonstrates that ignoring the available information base and applying the maximin criterion instead leads to counter-intuitive decisions even in very simple situations.

3. Note that in the context of IP decision theory, fundamental differences between global criteria and criteria based on pairwise comparisons have already been discussed (Schervish et al., 2003).



**Example 1** Let  $A = \{a_1, a_2, a_3, a_4\}$ , the (complete) relation  $R_1$  induced by  $a_2 P_{R_1} a_3 P_{R_1} a_4 P_{R_1} a_1$  and  $P_{R_2} = \{(a_2, a_4), (a_3, a_1)\}$  consists of one single edge. Consider the decision system  $\mathcal{G} = \{X_1, X_2\}$ , where the acts  $X_1, X_2 : \{s_1, s_2\} \rightarrow A$  are defined by  $(X_1(s_1), X_1(s_2)) = (a_1, a_2)$  and  $(X_2(s_1), X_2(s_2)) = (a_3, a_4)$ . Moreover, suppose our probabilistic information is given by the credal set  $\mathcal{M} := \{\pi : \pi(\{s_1\}) \leq 0.5\}$ . In this case, act  $X_1$  is  $\mathcal{A}|\mathcal{M}$ -dominant, since it maximizes expected utility w.r.t. every pair  $(u, \pi) \in \mathcal{U}_{\mathcal{A}} \times \mathcal{M}$ . In contrast,  $X_2$  is not even  $\mathcal{A}|\mathcal{M}$ -admissible, although it is the unique optimal act w.r.t. the maximin criterion!

To complete the section, we give a proposition containing a linear programming based approach for checking whether an act  $X$  is  $\mathcal{A}$ -admissible in finite decision settings.

**Proposition 3** Consider again the situation of Proposition 2. Moreover, let  $\mathcal{G} := \{X_1, \dots, X_k\} \subset \mathcal{F}_{(\mathcal{A}, \mathcal{M}, S)}$  denote a finite decision system and let  $X_z \in \mathcal{G}$ . Consider again the linear optimization problem (I) with additional constraints

$$\sum_{i=1}^n u_i \cdot \pi^{(t)}(X_z^{-1}(\{a_i\})) \geq \sum_{i=1}^n u_i \cdot \pi^{(t)}(X_l^{-1}(\{a_i\})) \text{ for all } l = 1, \dots, k \quad (C_t)$$

for every  $t = 1, \dots, T$ . Then  $X_z$  is  $\mathcal{A}$ -admissible if and only if the optimal outcome of this optimization problem is strictly greater than 0.

**Proof.** A similar argument as in the proof of Proposition 1 guarantees the existence of an optimal solution  $(u_1^*, \dots, u_n^*, \varepsilon^*)$  such that  $u : A \rightarrow \mathbb{R}$ ,  $u(a_i) := u_i^*$  for all  $i \in \underline{n}$  (weakly) represents the preference system  $\mathcal{A}$ . Now, let  $\pi \in \mathcal{M}$  be arbitrary. Choose  $\alpha \in \Delta_{T-1}$  such that  $\pi(\cdot) = \sum_{t=1}^T \alpha_t \cdot \pi^{(t)}(\cdot)$ . Then, condition  $(C_t)$  additionally guarantees that for all  $l = 1, \dots, k$  it holds

$$\begin{aligned} \mathbb{E}_{\pi}(u \circ X_z) &= \sum_{i=1}^n u_i^* \cdot \pi(X_z^{-1}(\{a_i\})) &= \sum_{i=1}^n u_i^* \cdot \left( \sum_{t=1}^T \alpha_t \cdot \pi^{(t)}(X_z^{-1}(\{a_i\})) \right) \\ &= \sum_{t=1}^T \alpha_t \left( \sum_{i=1}^n u_i^* \cdot \pi^{(t)}(X_z^{-1}(\{a_i\})) \right) &\geq \sum_{t=1}^T \alpha_t \left( \sum_{i=1}^n u_i^* \cdot \pi^{(t)}(X_l^{-1}(\{a_i\})) \right) \\ &= \sum_{i=1}^n u_i^* \cdot \left( \sum_{t=1}^T \alpha_t \cdot \pi^{(t)}(X_l^{-1}(\{a_i\})) \right) &= \mathbb{E}_{\pi}(u \circ X_l) \end{aligned}$$

Hence,  $X_z$  maximizes expected utility with respect to  $(u, \pi)$ . Since  $\pi \in \mathcal{M}$  was chosen arbitrarily, this implies that  $X_z$  is  $\mathcal{A}$ -admissible.  $\square$

Note that a similar algorithm as given in in Proposition 3 could be used for checking  $\mathcal{M}$ -admissibility of acts. However, this would require the set  $\mathcal{E}(\mathcal{U}_{\mathcal{A}})$  of extreme points of the representation set to be known, which is way less straightforward than assuming  $\mathcal{E}(\mathcal{M})$  to be known.

#### 4.4 Criteria based on Pairwise Comparisons

While the criteria defined in Section 4.3 rather relied on global comparisons of acts in the sense that an act, in order to be labeled admissible, has to dominate all other acts in expectation for (at least one) fixed pair  $(\pi, u)$ , we now turn to criteria induced by pairwise expectation comparisons of acts (i.e. binary relations on the set of acts). Similarly as already seen in the global case, there are

several different ways to define such relations each of which reflecting a different attitude towards the underlying ambiguity. In particular, we define six binary relations  $R_{\exists\exists}$ ,  $R_{\exists\forall}^1$ ,  $R_{\exists\forall}^2$ ,  $R_{\forall\exists}^1$ ,  $R_{\forall\exists}^2$  and  $R_{\forall\forall}$  on  $\mathcal{F}_{(\mathcal{A}, \mathcal{M}, S)}$  by setting for all  $X, Y \in \mathcal{F}_{(\mathcal{A}, \mathcal{M}, S)}$ :

$$(X, Y) \in R_{\exists\exists} \quad :\Leftrightarrow \quad \exists u \in \mathcal{U}_{\mathcal{A}} \exists \pi \in \mathcal{M} : \mathbb{E}_{\pi}(u \circ X) \geq \mathbb{E}_{\pi}(u \circ Y) \quad (4)$$

$$(X, Y) \in R_{\exists\forall}^1 \quad :\Leftrightarrow \quad \exists u \in \mathcal{U}_{\mathcal{A}} \forall \pi \in \mathcal{M} : \mathbb{E}_{\pi}(u \circ X) \geq \mathbb{E}_{\pi}(u \circ Y) \quad (5)$$

$$(X, Y) \in R_{\exists\forall}^2 \quad :\Leftrightarrow \quad \exists \pi \in \mathcal{M} \forall u \in \mathcal{U}_{\mathcal{A}} : \mathbb{E}_{\pi}(u \circ X) \geq \mathbb{E}_{\pi}(u \circ Y) \quad (6)$$

$$(X, Y) \in R_{\forall\exists}^1 \quad :\Leftrightarrow \quad \forall u \in \mathcal{U}_{\mathcal{A}} \exists \pi \in \mathcal{M} : \mathbb{E}_{\pi}(u \circ X) \geq \mathbb{E}_{\pi}(u \circ Y) \quad (7)$$

$$(X, Y) \in R_{\forall\exists}^2 \quad :\Leftrightarrow \quad \forall \pi \in \mathcal{M} \exists u \in \mathcal{U}_{\mathcal{A}} : \mathbb{E}_{\pi}(u \circ X) \geq \mathbb{E}_{\pi}(u \circ Y) \quad (8)$$

$$(X, Y) \in R_{\forall\forall} \quad :\Leftrightarrow \quad \forall \pi \in \mathcal{M} \forall u \in \mathcal{U}_{\mathcal{A}} : \mathbb{E}_{\pi}(u \circ X) \geq \mathbb{E}_{\pi}(u \circ Y) \quad (9)$$

Obviously, it holds that  $R_{\forall\forall}$  is subset of all other relation, whereas  $R_{\exists\exists}$  is a superset of them. For the remaining relations, in general, no sub- or superset relation has to be satisfied. Furthermore, transitivity is only guaranteed for  $R_{\forall\forall}$  in general. Similarly as already discussed in the global case, each of the desirability relations just defined relies on the idea that, given perfect information on utilities and probabilities, maximizing expected utility should be the criterion of choice. Again, the relations differ only in the way they handle the ambiguity on the involved sets  $\mathcal{U}_{\mathcal{A}}$  and  $\mathcal{M}$ . Naturally, each of the relations defined above induces a different criterion of (local) admissibility. These criteria are summarized in the following definition.

**Definition 8** *Let  $R \in \{R_{\exists\exists}, R_{\exists\forall}^1, R_{\exists\forall}^2, R_{\forall\exists}^1, R_{\forall\exists}^2, R_{\forall\forall}\} =: \mathcal{R}_p$ . We call an act  $X \in \mathcal{G}$  **locally admissible** with respect to  $R$ , if it is an element of the set  $\max_R(\mathcal{G}) := \{Y \in \mathcal{G} : \nexists Z \in \mathcal{G} \text{ s.t. } (Z, Y) \in P_R\}$ , that is if it is a maximal element in  $\mathcal{G}$  with respect to the relation  $R \cap (\mathcal{G} \times \mathcal{G})$ .*

So, which of the relations defined above are most important in our context? To address this question, we discuss some special cases: If the credal set  $\mathcal{M}$  is a singleton  $\mathcal{M} = \{\pi\}$  and if  $\mathcal{U}_{\mathcal{A}} = \{a \cdot u_0 + b \mid a > 0, b \in \mathbb{R}\}$  is unique up to a positive linear transformation then all relations  $R \in \mathcal{R}_p$  coincide with the classical expected utility criterion. If  $\mathcal{M}$  is a singleton and  $\mathcal{U}_{\mathcal{A}}$  is the class of all non-decreasing functions then the relations  $R_{\forall\exists}^1$  and  $R_{\forall\forall}$  essentially coincide with the classical concept of first order stochastic dominance (cf., e.g., [Mosler and Scarsini \(1991\)](#); [Lehmann \(1955\)](#); [Kamae et al. \(1977\)](#)) while second order stochastic dominance is obtained if  $\mathcal{U}_{\mathcal{A}}$  is the set of all continuous concave non-decreasing utility functions that are related to the concept of *decreasing returns to scale*. An intermediate case would arise if one has information about decreasing returns to scale only for parts of the preference system. To compute the relations  $R_{\exists\exists}$  and  $R_{\forall\forall}$  in the general case one can use the same technique as in [Proposition 2](#) by noting that  $\mathbb{E}_{\pi}(u \circ X) \geq \mathbb{E}_{\pi}(u \circ Y)$  is equivalent to  $\mathbb{E}_{\pi}(u \circ X - u \circ Y) \geq 0$ . The other relations  $R \in \mathcal{R}_p$  do not appear to be manageable in such a straightforward manner. However, if  $\mathcal{M}$  is the core of a belief function then all  $\pi \in \mathcal{M}$  can be understood as obtained from a mass transfer of probability mass to singleton sets of  $S$ . Since classical first order stochastic dominance can be checked via the solution of a mass transportation problem (cf., [Mosler and Scarsini \(1991, p. 269\)](#)), the computation of  $R_{\exists\forall}^2$  can be done by solving a composite mass transportation problem. The most rigorous relation  $R_{\forall\forall}$  is also discussed in [Montes \(2014, Ch. 4.1\)](#). Note that the locally  $R_{\forall\forall}$ -admissible acts coincide with the  $\mathcal{A}|\mathcal{M}$ -dominant acts. Note also that, in general, the other global concepts of admissibility from [Definition 4.3](#) are not expressible as induced by one of the local criteria from [Definition 4.4](#) (for the special case of a cardinal  $u$  this is discussed in [Schervish et al. \(2003\)](#)).

## 5. Summary and Outlook

We proposed three approaches for decision making under severe uncertainty if acts are ps-valued: The first is based on granularity-dependent expectation intervals, while the other two rely on local and global comparisons of specific expectations of acts. For selected criteria, we gave linear programs. Several challenges should be addressed in future research. Clearly, further algorithms for the remaining criteria need to be explored in order to make the theory computationally feasible and, therefore, applicable in practice. Further, it is certainly worth investigating in more detail how the criteria from the different approaches relate to each other. Finally, designing experimental settings for eliciting the parameter  $\delta$  could help to receive a more canonical interpretation of granularity.

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