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Game Solutions, Probability Transformations and the Core

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Abstract

We investigate the role of some game solutions, such the Shapley and the Banzhaf values, as probability transformations of lower probabilities. The first one coincides with the pignistic transformation proposed in the Transferable Belief Model; the second one is not efficient in general, leading us to propose a normalized version. We consider a number of particular cases of lower probabilities: minitive measures, coherent lower probabilities, as well as the lower probabilities induced by comparative or distorsion models. For them, we provide some alternative expressions of the transformations and study when they belong to the core of the lower probability.

Keywords: game solutions; probability transformations; lower probabilities; belief functions; core; Shapley value; Banzhaf value; pignistic transformation.

1. Introduction

One important problem within imprecise probability theory is that of eliciting a (precise) probability measure from an imprecise model. This is usually referred to as a *probability transformation*, and has been approached in many different ways: we can consider for instance the probability measure that minimizes (some) distance to the lower probability (Baroni and Vicig, 2005) or that with the maximum entropy (Jaffray, 1995). The problem has been considered with particular attention by the belief function community, and a number of different transformations have been proposed (Smets, 2005; Voorbraak, 1989). Among these, one of the most widely used is the *pignistic* transformation, considered by Smets and proposed earlier by Dubois and Prade (1982) and Williams (1982). It turns out that this transformation coincides with what Shapley proposed in 1953 as a solution for a game. Under this formalism, the possibility space represents a set of players, and the non-additive measure of an event A is interpreted as the gain associated with a coalition from the players in A. The link allows us to obtain the pignistic transformation as the center of gravity (the average of the extreme points) of the set of probabilities associated with the non-additive measure, when the latter is 2-monotone.

Inspired by this result, in this paper we investigate game solutions as probability transformations. On the one hand, we deepen in the properties of the Shapley value, studying if it is also the center of gravity of the core under less restrictive conditions than 2-monotonicity. Moreover, we study for which imprecise probability models we can guarantee the consistency of the Shapley value with the lower probability it is induced from. In addition, we shall also study the role as a probability transformation of another popular solution proposed within game theory: the Banzhaf value.

After introducing some preliminary concepts in Section 2, in Sections 3–6 we investigate the properties of the Shapley and Banzhaf values for some particular types of lower probabilities: minitive measures, 2-monotone lower probabilities, coherent lower probabilities, or lower probabilities induced by comparative or distorsion models. We conclude the paper in Section 7 with some additional remarks. Due to the space limitations, proofs have been omitted.

2. Preliminary Concepts

2.1 Lower Probabilities

Consider a finite possibility space $\Omega = \{1, \ldots, n\}$. A *lower probability* on $\Omega = \{1, 2, \ldots, n\}$ is a function $\underline{P} : \mathcal{P}(\Omega) \to [0, 1]$ that is monotone ($A \subseteq B \Rightarrow \underline{P}(A) \leq \underline{P}(B)$) and normalized ($\underline{P}(\emptyset) = 0, \underline{P}(\Omega) = 1$). Its conjugate upper probability is given by $\overline{P}(A) = 1 - \underline{P}(A^c)$ for every $A \subseteq \Omega$, and its *core* is the set $\mathcal{M}(\underline{P})$ of additive models that are compatible with $\underline{P}$, in the sense that

$$\mathcal{M}(\underline{P}) = \{P : \mathcal{P}(\Omega) \to [0, 1] \text{ probability measure} : P(A) \geq \underline{P}(A)\ \forall A \subseteq \Omega\}.$$

We shall only consider in this paper lower probabilities $\underline{P}$ whose core is non-empty. These are said to *avoid sure loss*. They are called *coherent* if they are moreover the lower envelope of their core, in the sense that $\underline{P}(A) = \min\{P(A) : P \in \mathcal{M}(\underline{P})\}$ for every $A \subseteq \Omega$. One particular family of coherent lower probabilities are the *2-monotone* ones, which are those satisfying $\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B)$ for any pair of subsets A, B of Ω.

This notion can be strengthened by considering *complete monotonicity*, which means that

$$\underline{P}(\cup_{i=1}^n A_i) \geq \sum_{i=1}^n \underline{P}(A_i) - \sum_{i,j \in \{1,\ldots,n\}} \underline{P}(A_i \cap A_j) + \cdots + (-1)^{n+1} \underline{P}(\cap_{i=1}^n A_i)$$

for every $n \in \mathbb{N}$ and every $A_1, \ldots, A_n \subseteq \Omega$.

Completely monotone lower probabilities are also called *belief functions* in the theory of evidence (Shafer, 1976). One of their advantages is that they are uniquely determined by their *basic probability assignment* $m : \mathcal{P}(\Omega) \to [0, 1]$, by means of the formula

$$\underline{P}(A) = \sum_{B \subseteq A} m(B). \tag{1}$$

More generally, any lower probability is determined by its Möbius inverse, given by

$$m(B) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \underline{P}(A),$$

in the sense that this function m determines $\underline{P}$ by means of Eq. (1); this Möbius inverse is non-negative if and only if $\underline{P}$ is a belief function. In that case, the sets B with $m(B) > 0$ are called the *focal elements* of the belief function $\underline{P}$.

2.2 Game Solutions

Within game theory, the possibility space Ω is interpreted as a set of players, and $\underline{P}(A)$ is then regarded as the gain that is guaranteed by the coalition of the players in A. Under the assumption of transferable utility, the core of the game is the set distributions of the total payoff among the players that cannot be improved by a coalition. These distributions are referred to here as *solutions* of the game (they should not be mistaken with the alternative use of the term *solution* in game theory as a multifunction that assigns to each game a set of valid strategies).

Arguably the most important solution of a game is the so-called *Shapley value* (Shapley, 1953, 1971) that, for a player i, is given by

$$\Phi(\underline{P})(i) = \sum_{T \not\supseteq \{i\}} \frac{t!(n-t-1)!}{n!} (\underline{P}(T \cup \{i\}) - \underline{P}(T)), \tag{2}$$

where $t = |T|$. It is the only solution of the game that satisfies the properties of efficiency (in the sense defined below), symmetry, linearity and that is equal to zero on null players.

When the game $\underline{P}$ is 2-monotone, $\Phi(\underline{P})$ corresponds to the center of gravity of the core, that is, the average of the extreme points of $\mathcal{M}(\underline{P})$ (Shapley, 1971). These are related to the permutations of Ω (Chateauneuf and Jaffray, 1989): any permutation σ defines an extreme point by means of the equation

$$P_\sigma(\{\sigma(1), \dots, \sigma(i)\}) := \underline{P}(\{\sigma(1), \dots, \sigma(i)\}) \text{ for } i = 1, \dots, n. \tag{3}$$

Thus, it holds that $\Phi(\underline{P})(i) = \frac{\sum_{\sigma \in S^\Omega} P_\sigma(\{i\})}{n!}$, where S^Ω denotes the set of permutations of Ω.

Interestingly, Shapley value of a belief function coincides with what Smets called its *pignistic transformation* within the Transferable Belief Model (Smets and Kennes, 1994), as shown in (Smets, 2005). This means that we can also compute the Shapley value as:

$$\Phi(\underline{P})(i) = \sum_{i \in B} \frac{m(B)}{|B|}. \tag{4}$$

The equivalence goes beyond belief functions, and as a consequence it can be used to justify the use of the pignistic transformation beyond this framework. See Aregui and Denoeux (2008); Monney et al. (2011) for some works making use of the pignistic transformation.

Another popular solution of a game is the so-called *Banzhaf value* (Banzhaf (1965); see also Webber (1988)), given by

$$B(\underline{P})(i) = \frac{1}{2^{n-1}} \sum_{T \not\supseteq \{i\}} \underline{P}(T \cup \{i\}) - \underline{P}(T). \tag{5}$$

However, and unlike the Shapley value, the equation above does not produce a probability mass function, because we may not have $\sum_{i \in \Omega} B(\underline{P})(i) = 1$ (in the language of game theory, if the sum of the values of the players does not agree with the total payoff $\underline{P}(\Omega)$ it means that the solution is not *efficient*). For this reason, it has been suggested to consider instead the *normalized Banzhaf value*, which is given by

$$\Psi(\underline{P})(i) = \frac{B(\underline{P})(i)}{\sum_{j \in \Omega} B(\underline{P})(j)}. \tag{6}$$

Although the normalized Banzhaf value does not share all the properties of the Banzhaf value (Dubey and Shapley, 1979), it has been axiomatized from the point of view of game theory by Van der Brink and Van der Laan (1998).

In this paper, we shall investigate the properties of the Shapley value and the normalized Banzhaf value as probability transformations of a lower probability. Specifically, we shall study for which types of lower probabilities they are guaranteed to belong to their core, as well as some simpler expressions for a number of particular cases.

3. Minitive Measures

We begin by considering a particular case of belief functions: minitive measures. They are also referred to as *consonant* belief functions, *necessity* measures or minitive lower probabilities.

Definition 1 *A lower probability* $\underline{P} : \mathcal{P}(\Omega) \to [0,1]$ *is called* minitive *when it satisfies*

$$\underline{P}(A \cap B) = \min\{\underline{P}(A), \underline{P}(B)\} \ \forall A, B \subseteq \Omega.$$

It was proven by Nguyen et al. (1997) that any minitive measure is in particular completely monotone, and therefore also 2-monotone. As a consequence, Shapley value can also be obtained in this case as the center of gravity of the elements of the core. On the other hand, the number of vertices of the core is smaller than n! in this case, as it was shown by Miranda et al. (2003) to be equal to 2^{n-1}, at most. The reason for this is that minitive functions correspond to the particular case of completely monotone measures whose focal elements are *nested* (Shafer, 1976), in the sense that they are completely ordered by the inclusion relation, and this makes the extreme points associated with many different permutations of Ω to coincide.

Let $\underline{P}$ be a minitive measure. In this subsection, we shall assume without loss of generality that its focal elements are the sets $\{1, \ldots, j\}$ for $j = 1, \ldots, n$; the results extend easily to the general case. Using the expression in Eq. (4), Dubois and Prade established the following formula:

Proposition 2 *(Dubois and Prade, 2002) Let* $\underline{P}$ *be a minitive measure, and denote by* m *its basic probability assignment. Then its Shapley value is given by:*

$$\Phi(\underline{P})(i) = \sum_{j=i}^{n} \frac{m(\{1, \ldots, j\})}{j} \quad \forall i = 1, \ldots, n.$$

With respect to the normalized Banzhaf value, we have proven the following:

Proposition 3 *Let* $\underline{P}$ *be a minitive measure, and denote by* m *its basic probability assignment. Then its Banzhaf value is given by:*

$$B(\underline{P})(i) = \frac{1}{2^{n-1}} \sum_{j=i}^{n} 2^{n-j} m(\{1, \ldots, j\}) \quad \forall i = 1, \ldots, n,$$

whence its normalized Banzhaf value is:

$$\Psi(\underline{P})(i) = \frac{\sum_{j=i}^{n} 2^{n-j} m(\{1, \ldots, j\})}{\sum_{j=1}^{n} j \cdot 2^{n-j} m(\{1, \ldots, j\})} \quad \forall i = 1, \ldots, n.$$

Moreover, the probability measure $\Psi(\underline{P})$ *belongs to the core* $\mathcal{M}(\underline{P})$ *of the minitive measure* $\underline{P}$.

4. 2-Monotone Lower Probabilities

Next, we study in more detail the case of 2-monotone lower probabilities. As we mentioned before, for them the Shapley value always belongs to the core of $\mathcal{M}(\underline{P})$. Interestingly, the same property does not hold for the normalized Banzhaf value, not even in the particular case where $\underline{P}$ is a belief function, as the following example shows:

Example 1 *Let $\Omega = \{1, 2, 3, 4\}$, and consider the belief function associated with the basic probability assignment given by $m(\{1\}) = m(\{2, 3, 4\}) = 0.5$, and $m(A) = 0$ for any other A. Then it follows from Eq. (5) that $B(\underline{P}) = (\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$. As a consequence, the probability mass function of the normalized Banzhaf value is given by $\Psi(\underline{P}) = (\frac{4}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7})$. However, this does not belong to the core of $\underline{P}$: we have that $\Psi(\underline{P})(\{2, 3, 4\}) = \frac{3}{7} < \frac{1}{2} = \underline{P}(\{2, 3, 4\})$.*

For comparison, in this case Eq. (2) tells us that $\Phi(\underline{P}) = (\frac{3}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. ♦

This means that the result we have established in Proposition 3 does not extend to arbitrary belief functions. It also illustrates the difference between the Shapley and the normalized Banzhaf values. On the other hand, it can be checked that in case of belief functions (and as a consequence also for their subclass of minitive measures) the sum $\sum_{i \in \Omega} B(\underline{P})(i)$ of the values given by Eq, (5) is always smaller than or equal to 1. This does not extend to arbitrary 2-monotone lower probabilities, as Example 2 will show.

Next, we shall investigate the properties of the Shapley and Banzhaf values for some other particular types of 2-monotone lower probabilities.

4.1 2-Monotone Lower Probabilities in a Three Element Space

Let us consider the particular case where the possibility space has three elements. In that case, it has been proven that a lower probability is 2-monotone if and only if it is coherent. Moreover, in the case of cardinality three 2-monotone lower probabilities are particular instances of *probability intervals* (de Campos et al., 1994), that is, they are uniquely determined by the constraints $[\underline{P}(\{i\}), \overline{P}(\{i\})]$ on singletons. In other words, it suffices to know in this case the lower and upper bounds on the gain of each player.

The following proposition gives an alternative expression for the Shapley and normalized Banzhaf values in this case:

Proposition 4 *Given $\Omega = \{1, 2, 3\}$, it holds that, for every $i \in \Omega$,*

$$\Phi(\underline{P})(i) = \frac{1}{3} + \frac{1}{2}[\underline{P}(\{i\}) + \overline{P}(\{i\})] - \frac{1}{6}\sum_{l=1}^{3}[\underline{P}(\{l\}) + \overline{P}(\{l\})],$$

while the normalized Banzhaf value is

$$\Psi(\underline{P})(\{i\}) = \frac{4m(\{i\}) + m(\Omega) + 2\sum_{j \neq i} m(\{i, j\})}{4 - m(\Omega)}.$$

Moreover, $\Psi(\underline{P})$ belongs to the core $\mathcal{M}(\underline{P})$.

As Example 1 shows, $\Psi(\underline{P})$ need not belong to the core for greater cardinalities of Ω.

4.2 Lower Probabilities Induced by a Distortion Model

Two particular cases of 2-monotone lower probabilities are those induced by a Pari-Mutuel Model (PMM for short) or a ε-contamination model; these two cases are usually referred to as *distortion* models. The PMM originated in horse racing. It considers a probability P_0 on $\mathcal{P}(\Omega)$ and a distortion value $\delta > 0$. Using P_0 and δ, the PMM defines a lower probability $\underline{P}$ by (Montes et al., 2017; Pelessoni et al., 2010; Walley, 1991):

$$\underline{P}(A) = \max\{(1+\delta)P_0(A) - \delta, 0\}. \tag{7}$$

From (Montes et al., 2017), the lower probability $\underline{P}$ induced by a PMM is in particular a probability interval, and as a consequence also 2-monotone. Thus, the Shapley value coincides with the center of gravity of the core.

The same applies to ε-contamination models, where we consider a probability P_0 and a contamination value $\varepsilon \in (0,1)$, that represents the distortion made on P_0. The ε-contamination model defines a lower probability by:

$$\underline{P}(A) = (1-\varepsilon)P_0(A) + \varepsilon \underline{P}_\Omega(A), \tag{8}$$

where $\underline{P}_\Omega$ is the *vacuous* lower probability that assigns the value 1 to Ω and 0 otherwise. This lower probability is known to be, not only 2-monotone, but also completely monotone.

Although one may think that for a distorsion based on the probability P_0, the probability transformations associated with the Shapley and normalized Banzhaf values return P_0, our next example shows that this is not the case:

Example 2 *Consider the probability P_0 on $\{1,2,3\}$ given by $P_0(\{1\}) = 0.1$, $P_0(\{2\}) = 0.2$ and $P_0(\{3\}) = 0.7$. Take $\delta = \varepsilon = 0.3$, and denote by $\underline{P}_\delta$ and $\underline{P}_\varepsilon$ the PMM and ε-contamination they induce, respectively. Using Eqs. (7) and (8), these are given by:*

A	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\underline{P}_\delta(A)$	0	0	0.61	0.09	0.74	0.87	1
$\underline{P}_\varepsilon(A)$	0.07	0.14	0.49	0.21	0.56	0.63	1

We deduce from Eq. (3) that the extreme points of $\mathcal{M}(\underline{P}_\delta)$ and $\mathcal{M}(\underline{P}_\varepsilon)$ are given by:

σ	P_σ *for* $\mathcal{M}(\underline{P}_\delta)$	P_σ *for* $\mathcal{M}(\underline{P}_\varepsilon)$
$(1,2,3)$	$(0, 0.09, 0.91)$	$(0.07, 0.14, 0.79)$
$(1,3,2)$	$(0, 0.26, 0.74)$	$(0.07, 0.44, 0.49)$
$(2,1,3)$	$(0.09, 0, 0.91)$	$(0.07, 0.14, 0.79)$
$(2,3,1)$	$(0.13, 0, 0.87)$	$(0.37, 0.14, 0.49)$
$(3,1,2)$	$(0.13, 0.26, 0.61)$	$(0.07, 0.44, 0.49)$
$(3,2,1)$	$(0.13, 0.26, 0.61)$	$(0.37, 0.14, 0.49)$

Thus, the Shapley values are $\Phi(\underline{P}_\delta) = (0.08, 0.145, 0.775)$ and $\Phi(\underline{P}_\epsilon) = (0.17, 0.24, 0.59)$, respectively, and none of them coincide with P_0.

Similarly, the normalized Banzhaf values are given by $\Psi(\underline{P}_\delta) = (\frac{0.35}{4.09}, \frac{0.61}{4.09}, \frac{3.13}{4.09})$ and $\Psi(\underline{P}_\varepsilon) = (\frac{0.58}{3.7}, \frac{0.86}{3.7}, \frac{2.26}{3.7})$, which do not coincide with P_0 either. ♦

We now consider the PMM in the particular case where δ satisfies $\delta < \frac{P_0(\{i\})}{1-P_0(\{i\})}$ for any $i = 1,\ldots,n$. This can be shown (Walley, 1991) to correspond to the case where $\underline{P}$ is strictly positive for any non-empty set. In that case we can give a simple expression for the Shapley and the normalized Banzhaf values.

Proposition 5 *Let $\underline{P}$ be the lower probability associated with the PMM determined by P_0, δ, and assume that $\delta < \frac{P_0(\{i\})}{1-P_0(\{i\})}$ for any $i = 1, \dots, n$. Then the Shapley value is given by $\Phi(\underline{P})(i) = (1+\delta)P_0(\{i\}) - \frac{\delta}{n}$, while*

$$\Psi(\underline{P})(i) = \frac{(1+\delta)P_0(\{i\}) - \frac{\delta}{2^{n-1}}}{k}, \text{ where } k = (1+\delta) - \frac{n\delta}{2^{n-1}}.$$

Moreover, both $\Phi(\underline{P}), \Psi(\underline{P})$ belong to the core $\mathcal{M}(\underline{P})$.

Next we establish a similar result for the ε-contamination models:

Proposition 6 *Let $\underline{P}$ be the lower probability associated with the ε-contamination determined by P_0, ε. Then the Shapley value is given by $\Phi(\underline{P})(i) = (1-\varepsilon)P_0(\{i\}) + \frac{\varepsilon}{n}$, while the normalized Banzhaf value is*

$$\Psi(\underline{P})(i) = \frac{(1-\varepsilon)P_0(\{i\}) + \frac{\epsilon}{2^{n-1}}}{k}, \text{ where } k = (1-\varepsilon) + \frac{n\varepsilon}{2^{n-1}}.$$

Moreover, both $\Phi(\underline{P}), \Psi(\underline{P})$ belong to the core $\mathcal{M}(\underline{P})$.

A common choice for P_0 in a distortion model is the uniform distribution; see for example Utkin (2014) and Utkin and Wiencierz (2013). Our next result shows that for the ε-contamination model and for the PMM with small enough values of δ, the Shapley and normalized Banzhaf values coincide with P_0 if and only if P_0 is uniform.

Corollary 7 *Let $\underline{P}$ be the lower probability associated with either the PMM determined by P_0, δ, where δ satisfies $\delta < \frac{P_0(\{i\})}{1-P_0(\{i\})}$ for any $i = 1, \dots, n$ or a ε-contamination model. Then,*

$$\Phi(\underline{P}) = P_0 \iff \Psi(\underline{P}) = P_0 \iff P_0(\{i\}) = \frac{1}{n}\ \forall i \in \Omega.$$

In fact, for the PMM we easily derive from the symmetry axioms satisfied by the Shapley and the Banzhaf values that, if P_0 is the uniform probability measure, then it coincides with the Shapley value of the PMM (P_0, δ) irrespective of the value of δ; to see that the converse is not true in general, i.e., that Φ can be the uniform probability measure for other PMM (P_0, δ), it suffices to consider that $\mathcal{M}(\underline{P})$ is the set of all probability measures for $\underline{P} = (P_0, \delta)$ provided δ is large enough (specifically, when $\delta \geq \frac{1}{P_0(A^c)}$ for every $A \neq \Omega$), and that in that case Φ becomes the uniform distribution. Similar comments apply to the normalized Banzhaf value.

5. Coherent Lower Probabilities

We consider next the case of coherent lower probabilities. It was established by Baroni and Vicig (2005, Proposition 5) in terms of the pignistic transformation that the Shapley value of a coherent lower probability need not be an element of the core, or, in other words, that the result for 2-monotone lower probabilities does not extend to arbitrary coherent lower probabilities. The very same example allows us to show that the normalized Banzhaf value need not belong to the core, either. In fact, as Example 1 shows, the normalized Banzhaf value is not guaranteed to be in the core even in the particular case of belief functions, and in the case of possibility spaces with four elements (the example by Baroni and Vicig considers a space of cardinality five).

In spite of this result, we can guarantee that the Shapley and Banzhaf values belong to the core in a number of particular cases. We begin by considering the case of coherent lower probabilities that are the lower envelope of two probability measures. They may arise for instance when we are aggregating the information from two different sources.

Proposition 8 *Consider two probability measures P_1, P_2 on $\mathcal{P}(\Omega)$ and let $\underline{P}$ be the coherent lower probability they determine. Then $\Psi(\underline{P})(i) = \Phi(\underline{P})(i) = B(\underline{P})(i) = \frac{P_1(\{i\})+P_2(\{i\})}{2}$ for every $i \in \Omega$.*

Interestingly, in the case considered in the proposition above the Banzhaf value is always normalized. On the other hand, the result does not extend to coherent lower probabilities that are the envelope of three probability measures, as Example 1 shows: note that the belief function in that example is the lower envelope of the family of probability measures with mass functions $\{(0.5, 0.5, 0, 0), (0.5, 0, 0.5, 0), (0.5, 0, 0, 0.5)\}$.

Another situation in which we can guarantee that the Shapley value of a coherent lower probability belongs to its core is when the possibility space has cardinality equal to four, as our next result shows:

Proposition 9 *Let $\Omega = \{1, 2, 3, 4\}$ and let $\underline{P} : \mathcal{P}(\Omega) \to [0, 1]$ be a coherent lower probability. Then, $\Phi(\underline{P})$ belongs to $\mathcal{M}(\underline{P})$.*

Example 1 shows that a similar result does not hold for the normalized Banzhaf value.

5.1 Comparative Lower Probabilities

Our attention shifts now to another useful model related to non-additive measures: comparative probabilities. These (de Finetti, 1931; Koopman, 1940) correspond to the case where the available information about the probability of the events is of qualitative nature, in the sense that we can only make statements of the type 'the probability of A is at least as much as that of B'.

The mathematical study of comparative models can be involved, and for instance the existence of an additive model that is compatible with them (Kaplan and Fine, 1977; Kraft et al., 1959) is not guaranteed; we refer to Regoli (1996) for a survey of this topic. In (Miranda and Destercke, 2015), the particular case of *elementary* comparative probabilities was considered, where we only give qualitative assessments about the value of individual players.

With this in mind, given $\mathcal{I} \subseteq \Omega \times \Omega$, we call the *(elementary) comparative model* determined by $\mathcal{I}$ the lower envelope $\underline{P}$ of the set

$$\mathcal{M} := \{P \text{ probability measure} : P(\{i\}) \geq P(\{j\}) \ \forall (i, j) \in \mathcal{I}\}.$$

It was proven by Miranda and Destercke (2015) that the core of these models can be given quite a neat structure, and that we also have at most 2^{n-1} different extreme points. However, the lower probability induced by this core need not be 2-monotone in general (Miranda and Destercke, 2015, Section 4.3). Taking this into account, it is not surprising to see that the Shapley value need not coincide with the center of gravity of the core, as our next example shows:

Example 3 *Let us consider the comparative assessments*

$$P(\{1\}) \geq P(\{2\}), \ \ P(\{1\}) \geq P(\{3\}), \ \ P(\{2\}) \geq P(\{4\}), \ \ P(\{3\}) \geq P(\{4\}).$$

If we consider the set of probability measures compatible with these assessments, it follows from (Miranda and Destercke, 2015) that the extreme points of this set are the probability measures

$$(1,0,0,0), \left(\frac{1}{2},\frac{1}{2},0,0\right), \left(\frac{1}{2},0,\frac{1}{2},0\right), \left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right), \left(\frac{1}{3},\frac{1}{3},\frac{1}{3},0\right).$$

From this we deduce that the lower probability $\underline{P}$ associated with these assessments is given by $\underline{P}(A) = 0$ if $1 \notin A$, and

$$\underline{P}(\{1\}) = \frac{1}{4},\ \underline{P}(\{1,2\}) = \underline{P}(\{1,3\}) = \frac{1}{2},\ \underline{P}(\{1,4\}) = \frac{1}{3},$$
$$\underline{P}(\{1,2,3\}) = \frac{3}{4},\ \underline{P}(\{1,2,4\}) = \underline{P}(\{1,3,4\}) = \frac{1}{2},\ \underline{P}(\Omega) = 1.$$

Now, from Eq. (2), the Shapley value is given by

$$\Phi(\underline{P}) = \left(\frac{41}{72},\frac{13}{72},\frac{13}{72},\frac{5}{72}\right),$$

while the center of gravity of the core is given by $(\frac{31}{60},\frac{13}{60},\frac{13}{60},\frac{3}{60})$. ♦

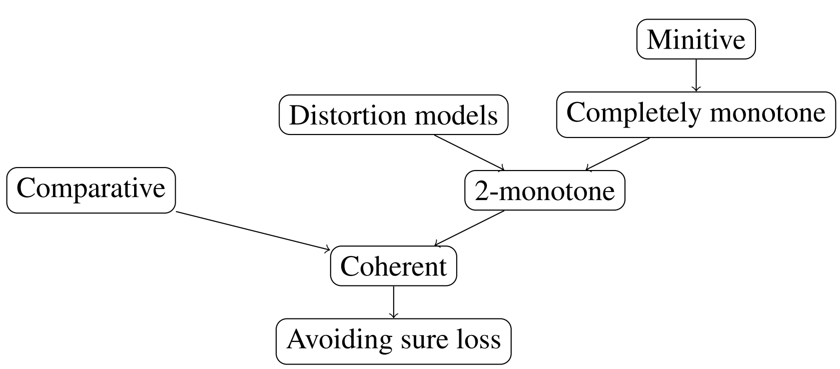
Nevertheless, it is possible to prove that both the Shapley and the normalized Banzhaf values belong to the core in this case:

Proposition 10 *Let $\underline{P}$ be a lower probability determined by elementary comparative probabilities. Then $\Phi(\underline{P})$ and $\Psi(\underline{P})$ belong to the core $\mathcal{M}(\underline{P})$.*

6. Lower Probabilities Avoiding Sure Loss

The most general model of lower probabilities that we shall consider in this paper are those that avoid sure loss. They correspond to *balanced* games within game theory (Shapley, 1967). Recall that a lower probability $\underline{P} : \mathcal{P}(\Omega) \to [0,1]$ is said to *avoid sure loss* (Walley, 1991) when its core $\mathcal{M}(\underline{P})$ is non-empty.

Note that a lower probability that avoids sure loss need not be coherent, because it may not be the lower envelope of its core. The relationship between the different models considered in this paper is summarised by the following figure, where the implication means an inclusion between the families:



Our next example provides an incoherent lower probability that avoids sure loss, and shows that in that case neither of the Shapley and the normalized Banzhaf values need belong to the core:

Example 4 *Consider $\Omega = \{1, 2, 3\}$, and let us consider $\underline{P}$ given by*

$$\underline{P}(\{1\}) = \underline{P}(\{2\}) = \underline{P}(\{3\}) = 0,\ \ \underline{P}(\{1,2\}) = \frac{9}{12},\ \ \underline{P}(\{1,3\}) = \frac{8}{12},\ \ \underline{P}(\{2,3\}) = \frac{7}{12},$$

and of course with $\underline{P}(\Omega) = 1$. The core of $\underline{P}$ is non-empty, as it includes for instance the probability measure P given by $(\frac{5}{12}, \frac{4}{12}, \frac{3}{12})$; in fact it can be checked that $\mathcal{M}(\underline{P})$ consists exactly of this probability measure. As a consequence, we see that the lower probability $\underline{P}$ is not coherent, since for instance $\underline{P}(\{1\}) = 0 < \min\{P(\{1\}) : P \in \mathcal{M}(\underline{P})\}$.

Using Eq. (2), we obtain that the Shapley value of $\underline{P}$ is given by

$$\Phi(\underline{P})(1) = \frac{9}{24},\ \ \Psi(\underline{P})(2) = \frac{8}{24},\ \ \Psi(\underline{P})(3) = \frac{7}{24}.$$

However, $\Phi(\underline{P})$ does not belong to the core of $\underline{P}$: we have that $\Phi(\underline{P})(\{1,2\}) = \frac{17}{24} < \frac{18}{24} = \underline{P}(\{1,2\})$.

By Eq. (6) the normalized Banzhaf value is given by:

$$\Psi(\underline{P})(1) = \frac{11}{30},\ \Psi(\underline{P})(2) = \frac{10}{30},\ \Psi(\underline{P})(3) = \frac{9}{30}.$$

Thus, it does not belong to the core, either. ♦

This example also shows that the result we have established for coherent lower probabilities in Proposition 4 does not extend to those avoiding sure loss, and also that the discussion about possibility spaces of cardinality three in Section 4.1 does not apply when coherence is not satisfied.

7. Conclusions

The results in this paper show that some of the nice properties of the Shapley value can be extended beyond the framework of 2-monotone lower probabilities and belief functions. With respect to the Banzhaf value, although the lack of efficiency leads to the definition of the normalized version, it is also possible to prove its consistency with the lower probability in a number of cases. Although in this paper we have focused on this consistency property, in the future we should deepen into the investigation of the mathematical properties of these models as probability transformations, in the vein of the work carried out by Dezert et al. (2012), so as to be able to compare them properly with the existing models.

More generally, we would like to continue this research by considering the probabilistic solutions of games considered in (Webber, 1988). In addition, we should also study the properties of other probability transformations, such as the maximum entropy one, for some of the imprecise probability models considered in this paper.

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References

A. Aregui and T. Denoeux. Constructing consonant belief functions from sample data using confidence sets of pignistic probabilities. *International Journal of Approximate Reasoning*, 49:575–594, 2008.

J. Banzhaf. Weighted voting does not work: a mathematical analysis. *Rutgers Law Review*, 19: 317–343, 1965.

P. Baroni and P. Vicig. An uncertainty interchange format with imprecise probabilities. *International Journal of Approximate Reasoning*, 40:147–180, 2005.

A. Chateauneuf and J.-Y. Jaffray. Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. *Mathematical Social Sciences*, 17(3):263–283, 1989.

L. M. de Campos, J. F. Huete, and S. Moral. Probability intervals: a tool for uncertain reasoning. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 2:167–196, 1994.

B. de Finetti. Sul significato soggettivo della probabilità. *Fundamenta Mathematicae*, 17:298–329, 1931.

J. Dezert, H. Han, Z.-G. Liu, and J.-M. Tacnet. Hierarchical DSmP transformation for decision-making under uncertainty. In *Proceedings of Fusion'2012*, pages 294–301, 2012.

P. Dubey and L. S. Shapley. Mathematical properties of the Banzhaf power index. *Mathematics of Operations Research*, 4:99–131, 1979.

D. Dubois and H. Prade. On several representations of an uncertain body of evidence. In M. Gupta and E. Sanchez, editors, *Fuzzy Information and Decision Processes*, pages 167–181. North Holland, 1982.

D. Dubois and H. Prade. Quantitative possibility theory and its probabilistic connections. In O. H. P. Grzegorzewski and M. Gil, editors, *Soft Methods in Probability, Statistics and Data Analysis*, pages 3–26. Physica-Verlag, 2002.

J.-Y. Jaffray. On the maximum entropy probability which is consistent with a convex capacity. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 3:27–33, 1995.

M. Kaplan and T. Fine. Joint orders in comparative probability. *Annals of Probability*, 5:161–179, 1977.

B. Koopman. The bases of probability. *Bulletin of the American Mathematical Society*, 46:763–774, 1940.

C. Kraft, J. Pratt, and A. Seidenberg. Intuitive probability on finite sets. *The Annals of Mathematical Statistics*, 30:408–419, 1959.

E. Miranda and S. Destercke. Extreme points of the credal sets generated by comparative probabilities. *Journal of Mathematical Psychology*, 64/65:44–57, 2015.

E. Miranda, I. Couso, and P. Gil. Extreme points of credal sets generated by 2-alternating capacities. *International Journal of Approximate Reasoning*, 33(1):95–115, 2003.

P. Monney, M. Chan, and P. Romberg. A belief function classifier based on information provided by noisy and dependent features. *International Journal of Approximate Reasoning*, 52(3):335–352, 2011.

I. Montes, E. Miranda, and S. Destercke. On the Pari-Mutuel Model seen as imprecise probabilities. 2017. Submitted for publication.

H. T. Nguyen, N. T. Nguyen, and T. Wang. On capacity functionals in interval probabilities. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 5:359–377, 1997.

R. Pelessoni, P. Vicig, and M. Zaffalon. Inference and risk measurement with the pari-mutuel model. *International Journal of Approximate Reasoning*, 51(9):1145–1158, 2010.

G. Regoli. Comparative probability and robustness. *Lecture Notes- Monograph Series*, 29:343–352, 1996.

G. Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, Princeton, NJ, 1976.

L. S. Shapley. A value for n-person games. *Annals of Mathematical Studies*, 28:307–317, 1953.

L. S. Shapley. On balanced sets and cores. *Naval Research Logistic Quarterly*, 14:453–460, 1967.

L. S. Shapley. Cores of convex games. *International Journal of Game Theory*, 1:11–26, 1971.

P. Smets. Decision making in the TBM: the necessity of the pignistic transformation. *International Journal of Approximate Reasoning*, 38:133–147, 2005.

P. Smets and R. Kennes. The transferable belief model. *Artificial Intelligence*, 66(2):191–234, 1994.

L. Utkin. A framework for imprecise robust one-class classification models. *Journal of Machine Learning Research and Cybernetics*, 5(3):379–393, 2014.

L. Utkin and A. Wiencierz. An imprecise boosting-like approach to regression. In *Proceedings of the 8th International Symposium on Imprecise Probability: Theories and Applications*, pages 345–354, 2013.

R. Van der Brink and G. Van der Laan. Axiomatizations of the normalized Banzhaf value and the Shapley value. *Social Choice and Welfare*, 15:567–582, 1998.

F. Voorbraak. A computationally efficient approximation of dempster-shafer theory. *International Journal of Man-Machine Studies*, 30:525–536, 1989.

P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991.

R. Webber. Probabilistic values for games. In A. Roth, editor, *The Shapley value. Essays in honour of L.S. Shapley*, pages 101–119. Cambridge University Press, 1988.

P. Williams. Discussion of 'Belief functions and parametric models', by G. Shafer. *Journal of the Royal Statistical Society, Series B*, 44(3):341–343, 1982.