A Study of the Pari-Mutuel Model from the Point of View of Imprecise Probabilities

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Abstract

The Pari-Mutuel model is a distortion model that has its origin in horse racing. Since then, it has been applied in many different fields, such as finance or risk analysis. In this paper we investigate the properties of the Pari-Mutuel model within the framework of Imprecise Probabilities. Since a Pari-Mutuel model induces (2-monotone) coherent lower and upper probabilities, we investigate its connections with other relevant models within this theory, such as probability intervals and belief functions. We also determine the number of extreme points of the credal set induced by the Pari-Mutuel model and study how to combine the information given by multiple Pari-Mutuel models.

Keywords: pari-mutuel bets; credal sets; probability intervals; belief functions; information fusion.

1. Introduction

The Pari-Mutuel model (PMM, for short) is a betting scheme originated in horse racing, that has been used in other fields like economics, risk analysis or life insurance. It considers a probability P_0 which models the fair price for a bet fixed by an agent, usually called House. In order to ensure a positive gain, House transforms this fair gain into a slightly greater value given by $(1+\delta)P_0$, where $\delta > 0$ is interpreted as the taxation from House. We refer to (Gerber, 1979; Peters et al., 2007; Terrell, 1994; Thaler and Ziemba, 1988) for some detailed studies on the PMM.

Using this interpretation, the PMM can be embedded into the Theory of Imprecise Probabilities: it determines lower and upper bounds for the probability of any event. These lower and upper probabilities satisfy the usual consistency requirement of coherence (Walley, 1991), and therefore they can be equivalently represented by means of the set of probability measures they bound. This set is a convex set of probabilities usually called credal set. Furthermore, the PMM satisfies the additional property of 2-monotonicity that offers computational advantages (Destercke, 2013).

To the best of our knowledge, there are few studies of the PMM from the point of view of imprecise probabilities. For example, (Pelessoni et al., 2010) studied the PMM as a risk measure and how to extend it from events to gambles, and (Utkin and Wiencierz, 2013) investigated how to use the PMM in classification problems.

In this paper, we further investigate the PMM from the point of view of Imprecise Probabilities. The rest of the paper is organized as follows: Section 2 recalls the definition and basic properties of the PMM. In Section 3 we investigate the connections between the PMM and other models from

Imprecise Probability Theory. In particular, we first prove that a PMM can be represented by means of a probability interval, and secondly we characterize the conditions a PMM must satisfy in order for its lower probability to be not only 2-monotone but also a belief function. Then, Section 4 studies some properties of the extreme points of the credal set induced by a PMM. On the one hand we investigate the form and the maximal number of extreme points of the credal set; on the other hand we give an upper bound of the number of extreme points. A number of procedures for merging different sources of information in the context of PMMs are investigated in Section 5. Due to space limitations, proofs as well as some less relevant explanations have been omitted.

2. Basic Notions About the Pari-Mutuel Model

Let $\mathcal{X} = \{x_1, \ldots, x_n\}$ denote a finite universe and let P_0 be a probability measure defined on $\mathcal{P}(\mathcal{X})$. We shall assume throughout that $P_0(\{x_i\}) > 0$ for $i = 1, \ldots, n$; the results generalize easily to the case where some elements have probability zero.

Given $\delta > 0$, the *pari-mutuel model* (PMM, for short) induced by P_0, δ, that we shall denote (P_0, δ), is given by the following lower and upper probabilities:

$$\underline{P}(A) = \max\{(1+\delta)P_0(A) - \delta, 0\} \text{ and } \overline{P}(A) = \min\{(1+\delta)P_0(A), 1\} \; \forall A \subseteq \mathcal{X}. \quad (1)$$

The functions $\underline{P}, \overline{P}$ are conjugate, meaning that $\overline{P}(A) = 1 - \underline{P}(A^c) \; \forall A \subseteq \mathcal{X}$. Also, since $P_0(\{x_i\}) > 0 \; \forall i = 1, \ldots, n$, it holds that $\overline{P}(A) \geq P_0(A) > 0$ for every $A \subseteq \mathcal{X}$.

The interpretation of the parameter δ can be found in (Walley, 1991, Sec. 2.9.3). There, it is proven that $\overline{P}(A) - \underline{P}(A) \leq \delta$ for any A, and the equality is attained if and only if $\frac{1}{1+\delta} \leq P_0(A) \leq \frac{\delta}{1+\delta}$. In particular, this condition holds when $0 < \underline{P}(A) < \overline{P}(A) < 1$. Therefore, δ may be understood in terms of the imprecision allowed in the definition of $P_0(A)$.

Note also that, since the lower probability of a PMM can be obtained as a convex transformation of a probability measure, it follows (Denneberg, 1994) that $\underline{P}$ is 2-monotone, meaning that

$$\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B)$$

for any $A, B \subseteq \mathcal{X}$. As a consequence (Walley, 1991), $\underline{P}, \overline{P}$ are *coherent*, that is, they are respectively the lower and upper envelopes of the *credal set* associated with the PMM, given by

$$\mathcal{M}(P_0, \delta) = \{P \text{ probability} \mid \underline{P}(A) \leq P(A) \leq \overline{P}(A) \;\; \forall A \subseteq \mathcal{X}\}. \quad (2)$$

3. PMM and Other Imprecise Probability Models

In this section, we study the connection between the PMM and other relevant imprecise probability models. In particular, we show that PMMs in a finite setting are particular instances of probability intervals, and study the conditions a PMM must satisfy in order to induce a belief function.

3.1 Connection Between PMM and Probability Intervals

Probability intervals on $\mathcal{X}$ (de Campos et al., 1994; Tessem, 1992) are just lower probabilities defined on the singletons and their complementaries. Specifically, a probability interval is given by:

$$\mathcal{I} = \{[l_i, u_i] : i = 1, \ldots, n\},$$

where it is assumed that $l_i \leq u_i$ and where the interpretation of $[l_i, u_i]$ is that the unknown or imprecisely specified probability of x_i belongs to the interval $[l_i, u_i]$. A probability interval determines a credal set by:

$$\mathcal{M}(\mathcal{I}) = \{P \text{ probability} \mid l_i \leq P(\{x_i\}) \leq u_i, \quad i = 1, \dots, n\}, \tag{3}$$

and the lower and upper envelopes of $\mathcal{M}(\mathcal{I})$ determine coherent lower and upper probabilities by:

$$l(A) = \inf_{P \in \mathcal{M}(\mathcal{I})} P(A) \text{ and } u(A) = \sup_{P \in \mathcal{M}(\mathcal{I})} P(A)\ \forall A \subseteq \mathcal{X}. \tag{4}$$

A probability interval $\mathcal{I}$ is called *reachable* (coherent in the terminology of Walley (1991)) whenever the functionals l, u determined by Eq. (4) satisfy $l(\{x_i\}) = l_i$ and $u(\{x_i\}) = u_i$ for all $i = 1, \dots, n$. This is equivalent to the following inequalities:

$$\sum_{j \neq i} l_j + u_i \leq 1 \text{ and } \sum_{j \neq i} u_j + l_i \geq 1\ \ \forall i = 1, \dots, n. \tag{5}$$

For a detailed study on probability intervals, we refer to (de Campos et al., 1994). See also (Guo and Tanaka, 2010; Skulj, 2009; Tanaka et al., 2004) for other relevant works on this topic.

By considering the restrictions to singletons of the lower and upper probabilities associated with a PMM, we can associate a reachable probability interval with any PMM. Interestingly, this probability interval keeps all the information about the PMM, in the sense that both determine the same credal set. In other words, PMMs are particular cases of reachable probability intervals, as our next result shows:

Theorem 1 *Let P_0 be a probability measure on $\mathcal{P}(\mathcal{X})$, $\delta > 0$ and (P_0, δ) the PMM they induce. Define the probability interval $\mathcal{I} = \{[l_i, u_i] : i = 1, \dots, n\}$ by $l_i = \underline{P}(\{x_i\})$ and $u_i = \overline{P}(\{x_i\})$, where $\underline{P}, \overline{P}$ are given by Eq. (1). Then, if $\mathcal{M}(\mathcal{I})$ denotes the credal set associated with $\mathcal{I}$ by means of Eq. (3), it holds that:*

1. *The probability interval $\mathcal{I} = \{[l_i, u_i] : i = 1, \dots, n\}$ is reachable.*
2. *$\mathcal{M}(\mathcal{I}) = \mathcal{M}(P_0, \delta)$, or equivalently, $\underline{P}(A) = l(A)$ and $\overline{P}(A) = u(A)$ for any $A \subseteq \mathcal{X}$.*

Thus, the PMM is a particular case of probability interval. On the other hand, the latter model is more general, in the sense that not every reachable probability interval can be expressed in terms of a PMM.

Example 1 *Consider the four-element space $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ and the probability interval $\mathcal{I} = \{[l_i, u_i] : i = 1, \dots, 4\}$ given by:*

	x_1	x_2	x_3	x_4
l_i	0.2	0.1	0.3	0.2
u_i	0.4	0.2	0.5	0.4

which can be shown to be reachable using Eq. (5). To see that $\mathcal{I}$ is not representable by a PMM (P_0, δ), note that from the comments in Section 2, any set A such that $0 < \underline{P}(A) < \overline{P}(A) < 1$ should satisfy $\overline{P}(A) - \underline{P}(A) = \delta$. However, in this example it holds that:

$$0 < l(\{x_1\}) = l_1 = 0.2 < 0.4 = u_1 = u(\{x_1\}) < 1 \textit{ and}$$
$$0 < l(\{x_2\}) = l_2 = 0.1 < 0.2 = u_2 = u(\{x_2\}) < 1,$$

whence $u(\{x_1\}) - l(\{x_1\}) = 0.2$ and $u(\{x_2\}) - l(\{x_2\}) = 0.1$. Thus, the difference is not constant, and therefore l, u cannot be represented by means of a PMM. ♦

3.2 Connection Between PMM and Belief Functions

As we mentioned in Section 2, the lower probability of a PMM is 2-monotone. In this section we study under which conditions it is moreover *completely* monotone. Complete monotonicity means that for any $p \in \mathbb{N}$ and any sets $A_1, \ldots, A_p \subseteq \mathcal{X}$, it holds that

$$\underline{P}(\cup_{i=1}^{p} A_i) \geq \sum_{J \subseteq \{1,\ldots,n\}} (-1)^{|J|-1} \underline{P}(\cap_{i \in J} A_i).$$

A completely monotone lower probability is usually called a *belief function*. Belief functions (Shafer, 1976) are determined by their Möbius inverse $m : \mathcal{P}(\mathcal{X}) \to [0,1]$, which is a mass function on the subsets of $\mathcal{X}$, by means of the formula $\underline{P}(A) = \sum_{B \subseteq A} m(B)$. The sets $A \subseteq \mathcal{X}$ such that $m(A) > 0$ are called the *focal elements* of $\underline{P}$. Conversely, the Möbius inverse m of a lower probability $\underline{P}$ is determined by the formula

$$m(B) = \sum_{A \subseteq B} (-1)^{|B \setminus A|} \underline{P}(A), \tag{6}$$

and $\underline{P}$ is a belief function if and only if the function m given by Eq. (6) satisfies $m(A) \geq 0$ for every $A \subseteq \mathcal{X}$.

We start with a simple result from which we deduce that in general the PMM is not 3-monotone, and therefore it is not completely monotone either.

Proposition 2 *Let $\underline{P}$ be the lower probability associated with a PMM (P_0, δ), with $|\mathcal{X}| \geq 3$. If there are different x_i, x_j, x_k such that $\underline{P}(\{x_i\}), \underline{P}(\{x_j\}), \underline{P}(\{x_k\}) > 0$, then $\underline{P}$ is not 3-monotone.*

To see that the hypotheses of this proposition may be satisfied, let P_0 be the uniform distribution on $\{x_1, x_2, x_3\}$ and take $\delta = \frac{1}{3}$: it follows from Eq. (1) that $\underline{P}(\{x_1\}) = \underline{P}(\{x_2\}) = \underline{P}(\{x_3\}) = \frac{1}{9}$.

Next, we establish necessary and sufficient conditions for the PMM to induce a belief function. For this aim we define the *non-vacuity index* of a PMM as $k = \min\{|A| : \underline{P}(A) > 0\}$.

Theorem 3 *Let $\underline{P}$ be the lower probability induced by a PMM (P_0, δ) by Eq. (1), and denote by k its non-vacuity index. $\underline{P}$ is a belief function if and only if one of the following conditions is satisfied:*

(B1) $k = n$.

(B2) $k = n - 1$ *and* $\sum_{i=1}^{n} \underline{P}(\mathcal{X} \setminus \{x_i\}) \leq 1$.

(B3) $k < n - 1$, *there exists a unique B with $|B| = k$ and $\underline{P}(B) > 0$, and $\underline{P}(A) > 0$ if and only if $B \subseteq A$.*

(B4) $k < n - 1$, *there exists a unique B with $|B| = k - 1$ and $\delta = \frac{P_0(B)}{1 - P_0(B)}$, and $\underline{P}(A) > 0$ if and only if $B \subset A$.*

Proof (Sketch) For sufficiency, it suffices to take into account that cases (B1)–(B4) determine a belief function with focal elements:

(a) $\mathcal{X}$, in the case of (B1);

(b) $\{\mathcal{X}, \mathcal{X} \setminus \{x\} : \forall x \in \mathcal{X}\}$, in the case of (B2);

(c) $\{B, B \cup \{x\} : \forall x \notin B\}$, in the case of (B3); and

(d) $\{B \cup \{x\} : \forall x \notin B\}$, in the case of (B4).

For necessity, if $\underline{P}$ is a belief function and we consider its associated non-vacuity index, we prove that the cases $k = n, k = n-1$ and $k < n-1$ determine the focal elements depicted in (a), (b) or (c)-(d) above, respectively, from which it follows that we are in cases (B1)–(B4) above. ■

Boodgumarn et al. (2013, Thm. 1) established that a sufficient condition for a probability interval to induce a belief function is that

$$\left|\left\{i : u_i + \sum_{j \neq i} l_j < 1\right\}\right| \leq 2. \tag{7}$$

Theorem 3 tells us that this condition is not necessary. Although it holds trivially under condition (B1) (i.e., for PMMs inducing a *vacuous* belief function), it is possible to find PMMs satisfying any of the conditions (B2)–(B4) and not the one in Eq. (7).

4. Extreme points induced by a PMM

Since the coherence of the PMM implies that it is uniquely determined by its (closed and convex) associated credal set, it becomes interesting to determine the extreme points of the set $\mathcal{M}(P_0, \delta)$ given by Eq. (2); this is particularly relevant if we want to use the PMM in some applied contexts, such as credal networks (Antonucci and Cuzzolin, 2010; Cozman, 2005).

Recall that the extreme points of $\mathcal{M}(P_0, \delta)$ are the probability measures $P \in \mathcal{M}(P_0, \delta)$ such that if $P = \alpha P_1 + (1-\alpha) P_2$ for some $\alpha \in (0,1)$, $P_1, P_2 \in \mathcal{M}(P_0, \delta)$, then $P_1 = P_2$.

Since the lower probability of a PMM is 2-monotone, the extreme points of $\mathcal{M}(P_0, \delta)$ are associated with permutations of $\mathcal{X}$ (Chateauneuf and Jaffray, 1989), in the following manner: if σ is a permutation of $\{1, \ldots, n\}$, we consider the probability measure P_σ given by

$$\begin{aligned} &P_\sigma(\{x_{\sigma(1)}\}) = \overline{P}(\{x_{\sigma(1)}\}), \\ &P_\sigma(\{x_{\sigma(k)}\}) = \overline{P}(\{x_{\sigma(1)}, \ldots, x_{\sigma(k)}\}) - \overline{P}(\{x_{\sigma(1)}, \ldots, x_{\sigma(k-1)}\}) \; \forall k = 2, \ldots, n. \end{aligned} \tag{8}$$

Then, the extreme points of $\mathcal{M}(P_0, \delta)$ are $\{P_\sigma : \sigma \in S^n\}$, where S^n denotes the set of permutations of $\{1, \ldots, n\}$. As a consequence, the number of extreme points of $\mathcal{M}(P_0, \delta)$ is bounded above by $n!$, the number of permutations of a n-element space. In this section, we study if this upper bound can be lowered in the particular case of the PMM.

4.1 Maximal Number of Extreme Points

We start our study by establishing two preliminary but helpful properties of the PMM. The first result shows that under some conditions, $\overline{P}$ is not only sub-additive as a coherent upper probability, but also additive.

Lemma 4 *Let $\overline{P}$ be the upper probability induced by a PMM (P_0, δ) by Eq. (1). If $\overline{P}(A) < 1$, then*

$$\overline{P}(A) = \sum_{x \in A} \overline{P}(\{x\}). \tag{9}$$

We deduce that if $\overline{P}(A \cup B) < 1$ and $A \cap B = \emptyset$, then $\overline{P}(A \cup B) = \overline{P}(A) + \overline{P}(B)$. Using Eq. (9), we can prove the second preliminary result, which gives the form of the extreme points in terms of $\underline{P}$ and $\overline{P}$.

Lemma 5 *Consider a PMM (P_0, δ), and let $\underline{P}, \overline{P}$ be given by Eq. (1). The extreme point P_σ associated with the permutation σ by Eq. (8) is given by:*

$$\begin{aligned} P(\{x_i\}) &= \overline{P}(x_i) \quad \forall i = \sigma(1), \dots, \sigma(j-1), \\ P(\{x_{\sigma(j)}\}) &= \underline{P}(\{x_{\sigma(j)}, \dots, x_{\sigma(n)}\}), \\ P(\{x_{\sigma(j+1)}\}) &= \dots = P(\{x_{\sigma(n)}\}) = 0, \end{aligned}$$

where $j \in \{1, \dots, n\}$ satisfies $\overline{P}(\{x_{\sigma(1)}, \dots, x_{\sigma(j-1)}\}) < \overline{P}(\{x_{\sigma(1)}, \dots, x_{\sigma(j)}\}) = 1$.

The above result is illustrated in the following example.

Example 2 *Let $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$, P_0 the uniform probability distribution and $\delta = 0.5$. If we consider the permutation $\sigma = (1, 2, 3, 4)$, we obtain the extreme point P_σ given by:*

$$\begin{aligned} &P_\sigma(\{x_1\}) = \overline{P}(\{x_1\}) = 1.5 \cdot 0.25 = 0.375. \\ &P_\sigma(\{x_2\}) = \overline{P}(\{x_2\}) = 1.5 \cdot 0.25 = 0.375. \\ &P_\sigma(\{x_3\}) = \underline{P}(\{x_3, x_4\}) = 1.5 \cdot 0.5 - 0.5 = 0.25. \\ &P_\sigma(\{x_4\}) = 0. \end{aligned}$$

In fact, it can be proven that the extreme points of $\mathcal{M}(P_0, \delta)$ are given by

$$\begin{aligned} &P(\{x_i\}) = \overline{P}(\{x_i\}) = 0.375, \\ &P(\{x_j\}) = \overline{P}(\{x_j\}) = 0.375, \\ &P(\{x_k\}) = \underline{P}(\{x_k, x_l\}) = 0.25, \\ &P(\{x_l\}) = 0, \end{aligned}$$

for any possible combination of i, j, k, l in $\{1, 2, 3, 4\}$. ♦

Next we use the results above to compute the maximal number of extreme points induced by a PMM. Note that from Theorem 1 we already know that any PMM is in particular a probability interval. This means that the number of extreme points induced by a PMM is upper bounded by the maximal number of extreme points induced by a probability interval. Next theorem shows that this upper bound can be attained.

Theorem 6 *Given a PMM (P_0, δ) on $\mathcal{X}$, the maximal number of extreme points of $\mathcal{M}(P_0, \delta)$ is:*

1. *$\frac{n}{2}\binom{n}{\frac{n}{2}}$ if n is even;*
2. *$\frac{n+1}{2}\binom{n}{\frac{n+1}{2}}$ if n is odd.*

Furthermore, these maxima are attainable, by considering P_0 a uniform distribution and $\delta \in \left(\frac{n-2}{n+2}, 1\right)$, if n is even, or $\delta \in \left(\frac{n-1}{n+1}, \frac{n+1}{n-1}\right)$ if n is odd.

The number of extreme points induced by a PMM (P_0, δ) where P_0 is the uniform probability measure has already been studied in (Utkin, 2014, Sect. 5.2.) and (Utkin and Wiencierz, 2013, Sect. 4.2). In this respect, note that, even if the definition of the PMM considered by Utkin and Wiencierz (2013) is slightly different from the one given in Section 2 (they consider instead $\underline{P}(A) = (1+\delta)P_0(A) - \delta$ and $\overline{P}(A) = (1+\delta)P_0(A)\ \forall A \subseteq \mathcal{X}$), both definitions determine the same credal set: the lower and upper probabilities in Eq. (1) correspond to the natural extensions of the ones considered by Utkin and Wiencierz (2013).

Remark also that the maximal number of extreme points for odd n can equivalently be expressed by $\binom{n+1}{\frac{n+1}{2}}\frac{n+1}{4}$. Therefore, the formula of the maximal number of extreme points of the credal set of a PMM coincides with that of probability intervals (Tessem, 1992).

4.2 Computing the Number of Extreme Points for an Arbitrary PMM

In this section, we establish a simple formula that provides an upper bound on the number of extreme points associated with a PMM. Let (P_0, δ) be a PMM, and define

$$\mathcal{L} = \{A \subseteq \mathcal{X} \mid \overline{P}(A) = 1\}. \tag{10}$$

This is a filter of subsets of $\mathcal{X}$, and as a consequence also a poset with respect to set inclusion. We can use it to bound the number of extreme points of a PMM.

Proposition 7 *Consider a PMM (P_0, δ), and let $\mathcal{L}$ be given by Eq. (10). Then, the number of extreme points of $\mathcal{M}(P_0, \delta)$ is bounded above by:*

$$\sum_{A \in \mathcal{L}} \left| \bigcap_{B \subseteq A, B \in \mathcal{L}} B \right|. \tag{11}$$

Furthermore, the number of extreme points coincides with this upper bound if and only if $P_0(A) > \frac{1}{1+\delta}$ for every $A \in \mathcal{L}$.

The following example illustrates the result.

Example 3 *Consider a four-element space $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ with probabilities 0.1, 0.1, 0.3 and 0.5, respectively, and let $\delta = 0.3$. The poset $(\mathcal{L}, \subseteq)$ is given by*

$$\mathcal{L} = \{\mathcal{X}, \{x_2, x_3, x_4\}, \{x_1, x_3, x_4\}, \{x_3, x_4\}\}$$

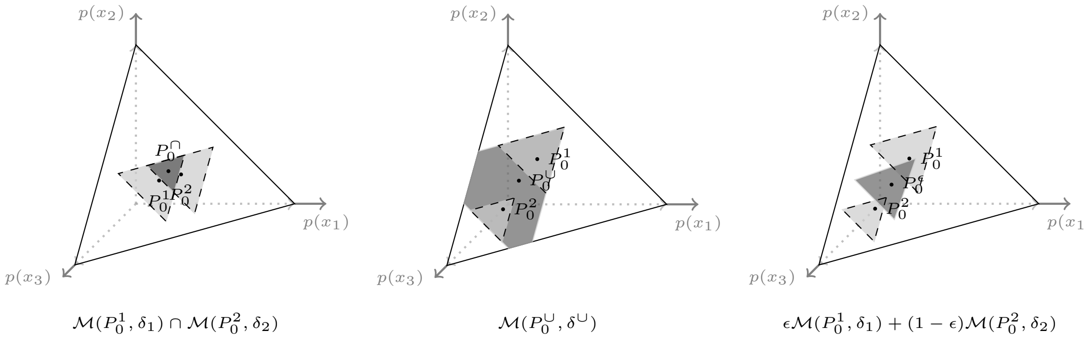
Eq. (11) provides an upper bound for the number of extreme points of $\mathcal{M}(P_0, \delta)$. Specifically, it is easy to see that for any $A \in \mathcal{L}$, it holds that:

$$\left| \bigcap_{B \subseteq A, B \in \mathcal{L}} B \right| = \left| \{x_3, x_4\} \right| = 2;$$

therefore, the number of extreme points of $\mathcal{M}(P_0, \delta)$ is bounded by:

$$\sum_{A \in \mathcal{L}} \left| \bigcap_{B \subseteq A, B \in \mathcal{L}} B \right| = 2 + 2 + 2 + 2 = 8.$$

Moreover, this bound is tight, taking into account that $P_0(A) > \frac{1}{1+\delta}\ \forall A \in \mathcal{L}$, and applying Proposition 7. ♦



(a) Conjunction. (b) Disjunction (approximated). (c) Average.

Figure 1: Illustration of combination rules. Initial credal sets are in light grey and delimited by dashed lines, combination results in dark grey.

However, when the additional condition given in Proposition 7 is not satisfied, the formula of Eq. (11) provides only an upper bound of the number of extreme points.

Example 4 *Take $\mathcal{X} = \{x_1, x_2, x_3\}$, the uniform distribution P_0 on $\mathcal{P}(\mathcal{X})$ and $\delta = 0.5$. It holds that $\mathcal{L} = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \mathcal{X}\}$. By Eq. (11), the number of extreme points is bounded above by:*

$$\sum_{A\in\mathcal{L}}\left|\bigcap_{B\subseteq A, B\in\mathcal{L}} B\right| = 2+2+2+0 = 6.$$

However, $\mathcal{M}(P_0, \delta)$ has only three extreme points: (0.5,0.5,0), (0.5,0,0.5) and (0,0.5,0.5). Thus, the bound given by Eq. (11) is not tight. Note moreover that in this case $P_0(\{x_1, x_2\}) = \frac{2}{3} = \frac{1}{1+\delta}$. ♦

5. Information Fusion of PMMs

When two credal sets $\mathcal{M}(P_0^1, \delta_1)$ and $\mathcal{M}(P_0^2, \delta_2)$ are provided to describe our uncertainty over $\mathcal{X}$, one often needs to combine them into a single model. Three classical ways to achieve such a combination are to consider the conjunction (intersection), the disjunction (union) or the average (convex mixture) of the models. The results of these combinations is illustrated in Figure 1, where the specific used models are described in Examples 5, 6 and 7 for the conjunction, disjunction and average, respectively.

Before studying these three cases, we show a useful result which can be derived from Lemma 4.

Proposition 8 *Let $\mathcal{M}(P_0, \delta)$ denote the credal set associated with a PMM (P_0, δ) by means of Eq. (2). Then, a probability measure P belongs to $\mathcal{M}(P_0, \delta)$ if and only if:*

$$P(\{x\}) \leq (1+\delta)P_0(\{x\}) \quad \forall x \in \mathcal{X}.$$

Thus, the credal set $\mathcal{M}(P_0, \delta)$ is not only determined by the restrictions of the lower and upper probabilities to singletons (as we know from the connection with probability intervals established in Theorem 1) but moreover that only the upper bounds on the singletons are necessary. This fact is instrumental in the derivation of the results of this section.

5.1 Conjunction

Let $\mathcal{M}(P_0^{\cap}, \delta^{\cap}) := \mathcal{M}(P_0^1, \delta_1) \cap \mathcal{M}(P_0^2, \delta_2)$ denote the credal set obtained by conjunctively combining $\mathcal{M}(P_0^1, \delta_1)$ and $\mathcal{M}(P_0^2, \delta_2)$. We then have the following result.

Proposition 9 *The set $\mathcal{M}(P_0^{\cap}, \delta^{\cap})$ is non-empty if and only if*

$$\sum_{x \in \mathcal{X}} \min\left\{(1+\delta_1)P_0^1(\{x\}), (1+\delta_2)P_0^2(\{x\}), 1\right\} \geq 1. \tag{12}$$

In that case, it is induced by the PMM $(P_0^{\cap}, \delta^{\cap})$ such that

$$\delta^{\cap} = \left(\sum_{x \in \mathcal{X}} \min\left\{(1+\delta_1)P_0^1(\{x\}), (1+\delta_2)P_0^2(\{x\})\right\}\right) - 1 \tag{13}$$

$$P_0^{\cap}(\{x\}) = \frac{\min\left\{(1+\delta_1)P_0^1(\{x\}), (1+\delta_2)P_0^2(\{x\})\right\}}{1+\delta^{\cap}}. \tag{14}$$

In the particular case where $P_0^1 = P_0^2$, Eq. (12) is always satisfied because:

$$\begin{aligned}&\sum_{x \in \mathcal{X}} \min\left\{(1+\delta_1)P_0^1(\{x\}), (1+\delta_2)P_0^2(\{x\}), 1\right\} \\ &\qquad = \sum_{x \in \mathcal{X}} \min\left\{(1+\min\{\delta_1, \delta_2\})P_0(\{x\}), 1\right\} \geq \sum_{x \in \mathcal{X}} P_0(\{x\}) = 1,\end{aligned}$$

and the values of $\delta^{\cap}$ and $P_0^{\cap}$ given in Eqs. (13) and (14) become $\delta^{\cap} = \min\{\delta_1, \delta_2\}$ and $P_0^{\cap} = P_0$.

Example 5 *Consider the space $\mathcal{X} = \{x_1, x_2, x_3\}$ and the two models given by $\delta_1 = \delta_2 = 0.3$ and:*

$$P_0^1 = (0.3, 0.3, 0.4), \quad P_0^2 = (0.4, 0.3, 0.3),$$

that are such that $\mathcal{M}(P_0^1, \delta_1) \cap \mathcal{M}(P_0^2, \delta_2) \neq \emptyset$. Their conjunction is given by $P_0^{\cap} = (1/3, 1/3, 1/3)$ and $\delta^{\cap} = 0.17$. The result is illustrated on Figure 1a, where the initial two PMMs are in light gray, and the resulting conjunction is in dark gray. ♦

5.2 Disjunction

When the intersection of two credal sets is empty (they are conflicting), an alternative is to consider their union, that is to consider $\mathcal{M}(P_0^1, \delta_1) \cup \mathcal{M}(P_0^2, \delta_2)$ or its convex hull, since $\mathcal{M}(P_0^1, \delta_1) \cup \mathcal{M}(P_0^2, \delta_2)$ will not be convex in general.

The convex hull $\mathrm{conv}(\mathcal{M}(\mathrm{P}_0^1, \delta_1) \cup \mathcal{M}(\mathrm{P}_0^2, \delta_2))$ will also not be induced by a PMM in general. However, we can easily provide a best outer-approximating PMM $(P_0^{\cup}, \delta^{\cup})$ using the fact that any outer-approximation of $\mathcal{M}(P_0^1, \delta_1) \cup \mathcal{M}(P_0^2, \delta_2)$ must satisfy the constraint

$$\max\left\{(1+\delta_1)P_0^1(\{x\}), (1+\delta_2)P_0^2(\{x\})\right\} \geq P(\{x\}) \ \ \forall x \in \mathcal{X}.$$

Indeed, using the same arguments as in Proposition 9, we can define

$$\delta^{\cup} = \left(\sum_{x \in \mathcal{X}} \max\left\{(1+\delta_1)P_0^1(\{x\}), (1+\delta_2)P_0^2(\{x\})\right\}\right) - 1$$

and

$$P_0^{\cup}(\{x\}) = \frac{\max\left\{(1+\delta_1)P_0^1(\{x\}), (1+\delta_2)P_0^2(\{x\})\right\}}{1+\delta^{\cup}}$$

so that $\mathcal{M}(P_0^{\cup}, \delta^{\cup}) \supseteq \mathcal{M}(P_0^1, \delta_1) \cup \mathcal{M}(P_0^2, \delta_2)$. To see that this inclusion holds, simply note that for any event A, we have

$$\sum_{x\in A} \max\left\{\overline{P}^1(x), \overline{P}^2(x)\right\} \geq \max\left\{\sum_{x\in A}\overline{P}^1(x), \sum_{x\in A}\overline{P}^2(x)\right\}$$

where $\overline{P}^1, \overline{P}^2$ are the upper probabilities induced by (P_0^1, δ_1) and (P_0^2, δ_2), respectively.

Example 6 *Consider the space $\mathcal{X} = \{x_1, x_2, x_3\}$ and the two models given by $\delta_1 = 0.2$, $\delta_2 = 0.3$ and:*

$$P_0^1 = (0.3, 0.4, 0.3), \quad P_0^2 = (0.2, 0.2, 0.6),$$

for which $\mathcal{M}(P_0^1, \delta_1) \cap \mathcal{M}(P_0^2, \delta_2) = \emptyset$. Their outer-approximation is $P_0^{\cup} = (0.222, 0.297, 0.481)$ and $\delta^{\cup} = 0.62$. The result is illustrated on Figure 1b, where the initial two PMMs are in light gray, and the resulting outer-approximation of the disjunction is in dark gray. ♦

5.3 Mixture

The mixture of two PMMs, that is, the computation of

$$\mathcal{M}(P_0^{\epsilon}, \delta_{\epsilon}) := \epsilon\mathcal{M}(P_0^1, \delta_1) + (1-\epsilon)\mathcal{M}(P_0^2, \delta_2)$$

for a given $\epsilon \in (0,1)$ is straightforward when applying results established by Moral and del Sagrado (1998) for probability intervals. In particular, the model $\mathcal{M}(P_0^{\epsilon}, \delta_{\epsilon})$ is described by the constraints

$$\epsilon(1+\delta_1)P_0^1(\{x\}) + (1-\epsilon)(1+\delta_2)P_0^2(\{x\})) \geq P(\{x\}) \;\; \forall x \in \mathcal{X}$$

on a probability measure P. From this, we easily deduce that

$$\begin{aligned} 1+\delta_{\epsilon} &= \sum_{x\in\mathcal{X}} \epsilon(1+\delta_1)P_0^1(\{x\}) + (1-\epsilon)(1+\delta_2)P_0^2(\{x\}) \\ &= \epsilon(1+\delta_1)\sum_{x\in\mathcal{X}} P_0^1(\{x\}) + (1-\epsilon)(1+\delta_2)\sum_{x\in\mathcal{X}} P_0^2(\{x\}) = \epsilon(1+\delta_1) + (1-\epsilon)(1+\delta_2) \end{aligned}$$

and $P_0^{\epsilon}(\{x\}) = \frac{\epsilon(1+\delta_1)P_0^1(\{x\})+(1-\epsilon)(1+\delta_2)P_0^2(\{x\})}{1+\delta_{\epsilon}}$.

Example 7 *Consider the initial models of Example 6 with $\epsilon = 0.5$. We obtain the model $p_0^{\epsilon} = (0.248, 0.296, 0.456)$ and $\delta^{\epsilon} = 0.25$. The result is illustrated on Figure 1c, where the initial two PMMs are in light gray, and the resulting average is in dark gray.* ♦

Other, more elaborate combinations can be derived from these basic ones; see for example (Moral and del Sagrado, 1998; Walley, 1982).

6. Conclusion

This paper presents some advances on the study of the PMM as a model within Imprecise Probability Theory. Our results show that the PMM is a particular type of probability interval (Thm. 1). This means that any property satisfied by a probability interval is also satisfied by a PMM. In this paper, we have studied the extreme points of the credal set induced by a PMM, and proven that the maximal number of extreme points coincides with that of probability intervals (Thm. 6). In addition, we have established a formula that gives an upper bound for the number of extreme points and that is somewhat easier to apply.

With respect to the connection with other imprecise probability models, we have also given necessary and sufficient conditions for a PMM to induce a belief function, improving upon some results from the literature. Our results show that those belief functions that are attained as a PMM are quite specific, since the PMM imposes strong constraints on the focal elements. Although not reported here, from this it is easy to characterize in which cases the lower probability of a PMM is a minitive function. However, this only happens in even more particular scenarios.

Finally, we have also investigated the properties of the PMM when merging different sources of information, each providing a PMM. In particular, we have seen that the conjunction or the mixture of PMMs give rise to other PMM, while the disjunction of PMMs can be outer-approximated by a PMM. This gives simple tools to perform such combinations.

There are other practical aspects of uncertainty models that we did not study in the present paper, but that would deserve some attention, such as what happens when combine into a joint model PMM models issued from marginal variables. In particular, it would be worth checking whether such operations can be performed efficiently and preserve the form of the initial model, i.e., is the result still a PMM?

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