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Linear Core-Based Criterion for Testing Extreme Exact Games

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Abstract

The notion of a (discrete) coherent lower probability corresponds to a game-theoretical concept of an exact (cooperative) game. The collection of (standardized) exact games forms a pointed polyhedral cone and the paper is devoted to the extreme rays of that cone, known as extreme exact games. A criterion is introduced for testing whether an exact game is extreme. The criterion leads to solving simple linear equation systems determined by (the vertices of) the core polytope (of the game), which concept corresponds to the notion of an induced credal set in the context of imprecise probabilities. The criterion extends and modifies a former necessary and sufficient condition for the extremity of a supermodular game, which concept corresponds to the notion of a 2-monotone lower probability. The linear condition we give in this paper is shown to be necessary for an exact game to be extreme. We also know that the condition is sufficient for the extremity of an exact game in an important special case. The criterion has been implemented on a computer and we have made a few observations on basis of our computational experiments.

Keywords: extreme exact game; coherent lower probability; core; credal set; supermodular game; 2-monotone lower probability; min-representation; oxytrophic game.

1. Introduction

The notion of a *coherent lower probability* and that of an induced *credal set* (of discrete probability distributions) are traditional topics of interest in the theory of imprecise probabilities. These notions correspond to game-theoretical concepts of an *exact game* and its *core* (polytope), widely used in the context of cooperative coalition games. The analogy is even broader: a lower probability avoiding sure loss corresponds to a weaker concept of a balanced game while a *2-monotone lower probability* (= capacity) corresponds to a stronger concept of a *supermodular game*, named also a convex game.

The discrete case is considered here: the sample space (= frame of discernment) for distributions is a fixed finite set N having at least two elements. The elements of N correspond to players in the context of cooperative game theory and to random variables in yet another context of probabilistic conditional independence structures. The collection of coherent lower probabilities on N, where $n = |N|$, is a polytope in a 2^n-dimensional real vector space, while the set of non-negative exact games is a pointed polyhedral cone whose extreme rays are generated just by extreme points of that polytope. This paper offers a method to test whether a ray is extreme in the cone of exact games, which implicitly gives a method to test extreme coherent lower probabilities.

Some effort to develop criteria to recognize the extremity of an exact game was exerted earlier by Rosenmüller (2000, §4 of chapter 5) in his book on game theory. He offered one necessary and one sufficient condition for the extremity based on a *min-representation* of the exact game; however, these conditions have a limited scope because they are applicable only in quite special situations. Nevertheless, in this paper we follow the idea of min-representation and propose a more general criterion based on the list of vertices of the respective *core*, which provides a *standard min-representation* of any exact game. Our condition is always necessary for the extremity of an exact game and we conjecture it is also sufficient, which is the case in a certain special case.

Being motivated by questions raised by Maass (2003), Quaeghebeur and de Cooman (2008) became interested in *extreme lower probabilities* and computed these in the case of small $n = |N|$. Antonucci and Cuzzolin (2010) considered an enlarging transformation of a credal set with a finite number of extreme points, when the respective (coherent) lower probability is computed and then a larger credal set is induced by the lower probability. Note that their second step, namely representing a coherent lower probability by the vertices of the induced credal set, corresponds to our standard min-representation of an exact game.

It is always useful to be aware of the correspondence between concepts from different areas. For example, Wallner (2005) confirmed a conjecture by Weichselberger that the credal set induced by a (coherent) lower probability has at most $n!$ vertices. However, the same result was achieved already by Derks and Kuipers (2002) in the context of cooperative game theory. They also made an interesting observation that whenever a core of an exact game has $n!$ vertices then it has the maximal number of $2^n - 2$ facets and gave an example of a game in the relative interior of the exact cone whose respective core does not have the maximal number of $n!$ vertices.

The criterion we offer here is a modification of the criterion from (Studený and Kroupa, 2016), where a necessary and sufficient condition was provided for a supermodular game being extreme in the cone of (standardized) supermodular games. That result was motivated by the research on conditional independence structures (Studený, 2005), in which context extreme supermodular games encode submaximal structural conditional independence models. The supermodular criterion leads to solving a simple linear equation system determined by certain combinatorial structure (of the core), which concept was pinpointed earlier by Kuipers et al. (2010). The difference here is that testing the extremity in the supermodular cone leads to one linear equation system, while testing the extremity in the exact cone may require solving several such equation systems.

What is an added value of this contribution is that we have also implemented both criteria and provide a web platform for testing the extremity of a supermodular/exact game in the respective cone for reasonably limited number of players. Of course, this can also be used to test the extremity of coherent lower probabilities. However, we have intentionally chosen to deal with games because this approach allows one to utilize the profits of integer arithmetics implementation.

In our paper we assume that the reader is familiar with basic concepts in polyhedral geometry, namely a polytope (= bounded polyhedron) and its faces/facets/vertices. The structure of the paper is as follows. In the next section (§ 2) we recall basic concepts and facts. In § 3 the concept of a min-representation of an exact game and the question of its uniqueness are discussed. After that our criterion is formulated (§ 4). In Conclusions (§ 5) we give a few remarks based on our computational experiments. The Appendix contains some proofs.

2. Notation, basic definitions and facts

Let N be a finite non-empty set of *variables*, $|N| \geq 2$, and $\mathcal{P}(N) := \{S : S \subseteq N\}$ its power set. The symbol $\mathbb{R}^N$ will denote the set of real vectors whose components are indexed by elements of N. Analogously, $\mathbb{R}^{\mathcal{P}(N)}$ is the collection of real functions on $\mathcal{P}(N)$ (= vectors with components indexed by subsets of N). Given $S \subseteq N$, the vector $\chi_S \in \mathbb{R}^N$ will denote the zero-one indicator of S. Given $v, x \in \mathbb{R}^N$, their scalar product will be $\langle v, x \rangle := \sum_{i \in N} v_i \cdot x_i$.

2.1 Game-theoretic concepts

By a (cooperative) *game* we will understand a set function $m \in \mathbb{R}^{\mathcal{P}(N)}$ with $m(\emptyset) = 0$.

Definition 1 (core, exact game, supermodular game)
Let $m : \mathcal{P}(N) \to \mathbb{R}$, $m(\emptyset) = 0$, be a game. Its *core* is a polytope in $\mathbb{R}^N$ defined by

$$C(m) \;:=\; \{\, x \in \mathbb{R}^N \;:\; \sum_{i\in N} x_i = m(N) \;\;\&\;\; \forall\, S \subseteq N \quad \sum_{i\in S} x_i \geq m(S)\,\}\,.$$

The symbol $\mathsf{ext}\, C(m)$ will be used to denote the set of extreme points (= vertices) of $C(m)$. A game m is *balanced* if $C(m) \neq \emptyset$. A balanced game is called *exact* if

$$\forall\, S \subseteq N \;\; \exists\, x \in C(m) \qquad \sum_{i\in S} x_i = m(S)\,.$$

A game m is *supermodular* if it satisfies the supermodularity inequalities

$$\forall\, C, D \subseteq N \qquad m(C) + m(D) \;\leq\; m(C \cup D) + m(C \cap D)\,.$$

A game m is called ℓ*-standardized* (ℓ stands for "lower"; in game theory = *zero-normalized*) if $m(S) = 0$ for any $S \subseteq N$, $|S| \leq 1$. Denote the class of exact ℓ-standardized games by $\mathsf{E}_\ell(N)$.

A well-known fact is that any supermodular game, named traditionally *convex* in game theory, is exact (Csóka et al., 2011, § 4). A non-negative exact game m normalized by $m(N) = 1$ is nothing but a coherent lower probability; see (Walley, 1991, Corollary 3.3.4).

The fact that, for any $S \subseteq N$, $\{x \in C(m) \,:\, \sum_{i\in S} x_i = m(S)\,\}$ is a face of $C(m)$ allows one to observe that any exact game m satisfies a formally stronger condition

$$\forall\, S \subseteq N \;\; \exists\, x \in \mathsf{ext}\, C(m) \qquad \sum_{i\in S} x_i = m(S)\,. \tag{1}$$

Indeed, every face of a polytope is the convex hull of extreme points of the whole polytope contained in the face. A necessary condition for the exactness of a game m is that it is *superadditive*:

$$\forall\, A, B \subseteq N \;\; A \cap B = \emptyset \quad m(A) + m(B) \leq m(A \cup B)\,.$$

Indeed, given disjoint $A, B \subseteq N$ there exists $x \in C(m)$ with $m(A \cup B) = \sum_{i\in A} x_i + \sum_{i\in B} x_i$ and one has both $m(A) \leq \sum_{i\in A} x_i$ and $m(B) \leq \sum_{i\in B} x_i$. In particular, any ℓ-standardized exact game is non-decreasing with respect to inclusion and non-negative.

It can be derived from results in (Csóka et al., 2011, § 3) that the collection of exact games is a rational polyhedral cone. Thus, non-negative exact games on $\mathcal{P}(N)$ form a pointed rational cone and the same holds for $\mathsf{E}_\ell(N)$. Degenerate non-negative exact games are superset indicators for singletons in N, which correspond to crisp degenerate probabilities in the context of imprecise probabilities. Since any non-negative exact game can be written as the sum of an ℓ-standardized exact game and of a conic combination of these degenerate exact games the question of testing the extremity in the cone of non-negative exact games reduces to testing the extremity in $\mathsf{E}_\ell(N)$.

Definition 2 (extreme exact game)
An ℓ-standardized exact game $m : \mathcal{P}(N) \to \mathbb{R}$ is *extreme* if it generates an extreme ray of $\mathsf{E}_\ell(N)$.

It can be derived from the fact that $\mathsf{E}_\ell(N)$ is a rational cone that any *extreme* ℓ-standardized exact game is a multiple of an integer-valued function $m : \mathcal{P}(N) \to \mathbb{Z}$. In particular, when testing the extremity of an exact game one can limit oneself to integer-valued functions.

3. The concept of a min-representation

A useful property of an exact game is that it can be represented as the minimum of a finite collection of additive games. Specifically, every $x \in \mathbb{R}^N$ defines an additive game

$$\mathbf{x} \in \mathbb{R}^{\mathcal{P}(N)} \text{ by the formula } \qquad \mathbf{x}(S) := \sum_{i \in S} x_i \quad \text{for any } S \subseteq N,$$

and every exact game can be obtained as the set-wise minimum of a finite collection of such additive games. This leads to the following concept.

Definition 3 (regular min-representation)
We say that $m \in \mathbb{R}^{\mathcal{P}(N)}$ has a *min-representation* (by additive functions) if a non-empty finite set $\mathcal{R} \subseteq \mathbb{R}^N$ exists such that

$$\forall\, S \subseteq N \qquad m(S) = \min_{x \in \mathcal{R}} \sum_{i \in S} x_i\,. \tag{2}$$

Every $x \in \mathcal{R}$ is then assigned the corresponding *tightness class* of sets

$$\mathcal{T}_x^m := \{\, S \subseteq N \,:\, m(S) = \sum_{i \in S} x_i \,\}\,. \tag{3}$$

We say that a min-representation $\mathcal{R} \subseteq \mathbb{R}^N$ of a game m is *regular* if, for any $x \in \mathcal{R}$,

(i) $\sum_{i \in N} x_i = m(N)$, and

(ii) the linear hull of $\{\, \chi_S \,:\, S \in \mathcal{T}_x^m \,\} \subseteq \mathbb{R}^N$ is whole $\mathbb{R}^N$.

Note that an equivalent formulation of the regularity condition (ii) is that the only vector in $\mathbb{R}^N$ which is orthogonal to all vectors from $\{\chi_S \,:\, S \in \mathcal{T}_x^m\}$ is the zero vector. There exists at least one regular min-representation for every exact game.

Proposition 4 (min-representations of exact games)
A game $m \in \mathbb{R}^{\mathcal{P}(N)}$ is exact iff it admits a min-representation $\mathcal{R}$ satisfying (i) for any $x \in \mathcal{R}$. Every exact game has a regular min-representation given by the list of vertices of its core: $\overline{\mathcal{R}} = \mathsf{ext}\, C(m)$. A min-representation $\mathcal{R} \subseteq \mathbb{R}^N$ of an exact game m is regular iff $\mathcal{R} \subseteq \mathsf{ext}\, C(m)$.

The proof of Proposition 4 is shifted to Appendix, § A.1. In particular, any exact game m has the largest regular min-representation which we consider to be a kind of *standard min-representation* of m. Note that a simple example of a non-exact game exists which has a min-representation.

3.1 On uniqueness of regular min-representations

In general, one can have several regular min-representations of an exact game. On the other hand, sometimes only one regular min-representation exists, which happens iff the next condition holds.

Definition 5 (oxytrophic game)
We say that an exact game $m : \mathcal{P}(N) \to \mathbb{R}$ is *oxytrophic* if $\;\forall\, x \in \mathsf{ext}\, C(m)$

$$\exists\, S \subseteq N \text{ with } \textstyle\sum_{i \in S} x_i = m(S) \text{ such that } \quad \forall\, y \in \mathsf{ext}\, C(m),\ y \neq x \qquad m(S) < \displaystyle\sum_{i \in S} y_i\,. \tag{4}$$

This relevant mathematical concept has already appeared in the literature and we have simply taken over the terminology by Rosenmüller (2000, § 3 of chapter 5). The following gives an example of an oxytrophic game, which is extreme in $\mathsf{E}_\ell(N)$.

Example 1 Put $N = \{a, b, c, d\}$ and consider $\mathcal{R} \subseteq \mathbb{R}^N$ consisting of 4 vectors (x_a, x_b, x_c, x_d), namely $(1, 1, 1, 1)$, $(2, 2, 0, 0)$, $(2, 0, 2, 0)$, $(0, 2, 2, 0)$. Then the formula (2) gives

$$m(abcd) = 4,\ m(abc) = 3,\ m(abd) = m(acd) = m(bcd) = m(ab) = m(ac) = m(bc) = 2,$$

and $m(S) = 0$ for other $S \subseteq N$. One can verify by computation that $\mathcal{R} = \mathsf{ext}\, C(m)$, which allows one to check the condition (4) for any $x \in \mathsf{ext}\, C(m)$: $(1, 1, 1, 1)$ has one respective set $S = abc$, while $(2, 2, 0, 0)$ has even two respective sets $S = c$ and $S = cd$, etc. In particular, m is oxytrophic. Moreover, m is also an example of an (extreme) exact game which is not supermodular: $m(ac) + m(bc) = 4 > 3 = m(abc) + m(c)$.

An interesting observation is that in case $|N| = 3$ the ℓ-standardized oxytrophic games are just the zero game and extreme exact games. However, in case $|N| = 4$ an extreme exact game exists which is not oxytrophic. The next example is even a supermodular game.

Example 2 Put $N = \{a, b, c, d\}$ and introduce $m(abcd) = 2$, $m(abc) = m(abd) = m(acd) = 1$, and $m(S) = 0$ for other $S \subseteq N$. Then the core $C(m)$ has seven vertices (x_a, x_b, x_c, x_d), namely four substantial ones denoted by

$$\mathcal{R}: \qquad (2, 0, 0, 0), \quad (0, 1, 1, 0), \quad (0, 1, 0, 1), \quad (0, 0, 1, 1),$$

and three additional ones, namely

$$(1, 1, 0, 0), \quad (1, 0, 1, 0), \quad (1, 0, 0, 1).$$

The vectors in $\mathcal{R}$ satisfy (4): $S = bcd$ for $(2, 0, 0, 0)$, $S = ad$ for $(0, 1, 1, 0)$, $S = ac$ for $(0, 1, 0, 1)$ and $S = ab$ for $(0, 0, 1, 1)$. However, the remaining 3 vertices of $C(m)$ do not satisfy (4) and m is not oxytrophic. On the other hand, every regular min-representation involves $\mathcal{R}$ and vectors in $\mathcal{R}$ provide a min-representation of m. Thus, $\mathcal{R}$ is the least regular min-representation of m.

On the other hand, an exact game can have several inclusion-minimal regular min-representations (see later Example 5). The following example shows that an oxytrophic game need not be extreme.

Example 3 Put $N = \{a, b, c, d\}$ and $m(abcd) = 2$, $m(S) = 1$ for $S \subseteq N$, $|S| = 3$, while $m(ab) = m(cd) = 1$, and $m(S) = 0$ for other $S \subseteq N$. Then $\mathcal{R} = \mathsf{ext}\, C(m)$ has four vectors (x_a, x_b, x_c, x_d), namely $(0, 1, 0, 1)$, $(0, 1, 1, 0)$, $(1, 0, 0, 1)$ and $(1, 0, 1, 0)$. The vectors in $\mathcal{R}$ satisfy (4): $S = ac$ for $(0, 1, 0, 1)$, $S = ad$ for $(0, 1, 1, 0)$, $S = bc$ for $(1, 0, 0, 1)$ and $S = bd$ for $(1, 0, 1, 0)$. Thus, m is oxytrophic. On the other hand, m is the sum of two other supermodular games m^1 and m^2, where m^1 is the indicator of supersets of ab and m^2 is the indicator of supersets of cd.

4. The criterion: a conjecture and results

Assume now that $m \in \mathsf{E}_\ell(N)$ is an ℓ-standardized exact game. Then the core $C(m)$ consists of non-negative vectors and the same holds for its vertices: $\mathsf{ext}\, C(m) \subseteq [0, \infty)^N$. In this section we formulate our linear core-based criterion.

4.1 Some arrangement

To formalize our conjecture let us choose and fix an auxiliary index set Υ for the vertices of the core of m and imagine (= have) the vertex set $\mathsf{ext}\, C(m)$ arranged in the form of a real *array* $x \in \mathbb{R}^{\Upsilon \times N}$ whose rows are indexed by Υ and columns by N:

$$x \;\; := \;\; [\, x(\tau, i)\,]_{\tau \in \Upsilon, i \in N} \in \mathbb{R}^{\Upsilon \times N} \qquad \text{where } \mathsf{ext}\, C(m) = \{\, [x(\tau, i)]_{i \in N} : \; \tau \in \Upsilon \,\}.$$

Recall that the minimization formula (2) for $\mathcal{R} = \mathsf{ext}\, C(m)$ means that m is obtained by set-wise minimization in the array x over its rows:

$$\forall\, S \subseteq N \qquad m(S) = \min_{\tau \in \Upsilon} \sum_{i \in S} x(\tau, i)\,.$$

In this context, the tightness classes (3) correspond to elements of Υ:

$$\mathcal{T}_\tau \;\; := \;\; \{\, S \subseteq N \; : \; m(S) = \sum_{i \in S} x(\tau, i) \,\} \qquad \text{for any } \tau \in \Upsilon.$$

For computational and implementation reasons, it is advisable to consider a special big zero-one *tightness array* encoding all tightness classes. This indicator array ι has rows indexed by Υ and columns by subsets of N:

$$\iota \;\; := \;\; [\, \iota(\tau, S)\,]_{\tau \in \Upsilon, S \subseteq N} \in \{0, 1\}^{\Upsilon \times \mathcal{P}(N)} \quad \text{where } \; \iota(\tau, S) = \begin{cases} 1 & \text{if } m(S) = \sum_{i \in S} x(\tau, i), \\ 0 & \text{otherwise.} \end{cases}$$

Note that ι serves as computer encoding of the concept of a combinatorial *core structure* mentioned in (Studený and Kroupa, 2016). By Proposition 4, the concept of a *regular min-representation* of m corresponds in this context to a special subset of the set of rows, namely $\Gamma \subseteq \Upsilon$ satisfying

$$\forall\, S \subseteq N \qquad m(S) = \min_{\tau \in \Gamma} \sum_{i \in S} x(\tau, i)\,. \tag{5}$$

To test whether $\Gamma \subseteq \Upsilon$ satisfies (5) one can consider the restricted tightness array $\iota_\Gamma \in \{0, 1\}^{\Gamma \times \mathcal{P}(N)}$ to rows in Γ and check whether each column in ι_Γ contains least one 1. Thus, a computer can be used to find *all inclusion-minimal* regular min-representations of m on basis of ι.

4.2 The linear equation systems

Every regular min-representation $\Gamma \subseteq \Upsilon$ satisfying (5) can be ascribed a system of linear constraints on the respective sub-array specified by rows in Γ:

$$y_\Gamma \;\; = \;\; [\, y(\tau, i)\,]_{\tau \in \Gamma, i \in N} \in \mathbb{R}^{\Gamma \times N}\,.$$

Specifically, the constraints are as follows:

(a) $\forall\, \tau \in \Gamma \;\; \forall\, i \in N$ with $\{i\} \in \mathcal{T}_\tau \qquad y(\tau, i) = 0\,,$

(b) $\forall\, S \subseteq N,\; |S| \geq 2,\;\; \forall\, \tau, \rho \in \Gamma$ with $S \in \mathcal{T}_\tau \cap \mathcal{T}_\rho \qquad \sum_{i \in S} y(\tau, i) = \sum_{i \in S} y(\rho, i)\,.$

It is not difficult to observe that the starting restricted array $x_\Gamma \in \mathbb{R}^{\Gamma \times N}$ satisfies the constraints (a)-(b); this is because these constraints are determined by x_Γ through ι_Γ. Informally, the conjecture is that the extremity means that, for any min-representation Γ, the equation system (a)-(b) has unique solution up to a real multiple.

Conjecture 6 An ℓ-standardized exact game $m \in \mathsf{E}_\ell(N)$ is *extreme* in $\mathsf{E}_\ell(N)$ iff, for every $\Gamma \subseteq \Upsilon$ satisfying (5), every real solution $y_\Gamma \in \mathbb{R}^{\Gamma\times N}$ to (a)-(b) is a multiple of $x_\Gamma \in \mathbb{R}^{\Gamma\times N}$, that is,

$$\exists\, \beta \in \mathbb{R} \quad \forall\, \tau \in \Gamma \ \ \forall\, i \in N \qquad y(\tau,i) = \beta \cdot x(\tau,i)\,.$$

The constraints (a)-(b) for fixed $\Gamma \subseteq \Upsilon$ can be written in the form of a matrix equality

$\mathbf{C}_\Gamma \cdot y_\Gamma = \mathbf{0}$, where $\mathbf{C}_\Gamma$ is an appropriate *constraint matrix* with entries in $\{-1, 0, +1\}$.

The rows of $\mathbf{C}_\Gamma$ encode the constraints and its columns correspond to the elements of $\Gamma \times N$. The matrix is sparse: every constraint of type (a) is encoded by a row with one non-zero component while any constraint of type (b) for $S \subseteq N$, $|S| \geq 2$, is encoded by a row containing $|S|$-times a component $+1$, $|S|$-times a component -1 and 0 otherwise.

The number of rows can be economized because some of the constraints of type (b) follow from the others. For example, whenever $S \subseteq N$, $|S| \geq 2$, belongs to $\mathcal{T}_\tau \cap \mathcal{T}_\rho \cap \mathcal{T}_\sigma$ for different $\tau, \rho, \sigma \in \Gamma$ then only two constraints

$$\sum_{i\in S} y(\tau,i) - \sum_{i\in S} y(\rho,i) = 0 \quad \text{and} \quad \sum_{i\in S} y(\tau,i) - \sum_{i\in S} y(\sigma,i) = 0$$

are enough. Therefore, if $\lambda(S)$, for $S \subseteq N$, denotes the number of 1's in the respective column of the tightness array ι_Γ, then the economized number of rows in $\mathbf{C}_\Gamma$ is

$$\sum_{S\subseteq N:\,|S|=1} \lambda(S) + \sum_{S\subseteq N:\,|S|\geq 2} [\lambda(S) - 1]\,.$$

Testing of the condition from Conjecture 6 for fixed $\Gamma \subseteq \Upsilon$ can be realized by computing the *nullity* of the matrix $\mathbf{C}_\Gamma$, which is the dimension of the space of solutions y_Γ to $\mathbf{C}_\Gamma \cdot y_\Gamma = \mathbf{0}$. Any solution to (a)-(b) is a multiple of x_Γ iff the nullity is 1; otherwise the nullity exceeds 1.

The following observation is useful to avoid testing all regular min-representations.

Proposition 7 Given an ℓ-standardized exact game $m \in \mathsf{E}_\ell(N)$ assume the situation from § 4.1 and take $\Gamma \subseteq \Upsilon$ satisfying (5). If every real solution $y_\Gamma \in \mathbb{R}^{\Gamma\times N}$ to (a)-(b) is a multiple of $x_\Gamma \in \mathbb{R}^{\Gamma\times N}$ then the same holds in case of any Σ such that $\Gamma \subseteq \Sigma \subseteq \Upsilon$.

The proof of Proposition 7 is shifted to Appendix, § A.2. The consequence of this observation is that to test the condition from Conjecture 6 it is enough to consider only the inclusion-minimal regular min-representations; this simplification may spare the computational time. What we have actually shown in the proof of Proposition 7 is that

whenever $\Gamma \subseteq \Upsilon$ satisfies (5) and $\Gamma \subseteq \Sigma \subseteq \Upsilon$ then $\qquad \text{null}\,(\mathbf{C}_\Gamma) \geq \text{null}\,(\mathbf{C}_\Sigma) \geq 1\,,$

meaning that the nullity (of constraint matrices) achieves its maximal value at one of the inclusion-minimal regular min-representations; see later Example 4 for illustration.

4.3 Our theoretical results

Proposition 8 Given a non-zero game $m \in \mathsf{E}_\ell(N)$, the condition from Conjecture 6, that is,

$$\forall\, \Gamma \subseteq \Upsilon \text{ satisfying (5)}, \ \text{ every real solution } y_\Gamma \in \mathbb{R}^{\Gamma \times N} \text{ to (a)-(b) is a multiple of } x_\Gamma \in \mathbb{R}^{\Gamma \times N},$$

is necessary for m being extreme in $\mathsf{E}_\ell(N)$.

The proof of Proposition 8 is shifted to Appendix, § A.3. Another comment is that, in case m is a supermodular function, a necessary and sufficient condition for its extremity in the supermodular cone is that the condition from Conjecture 6 holds for the largest set $\Gamma = \Upsilon$ (Studený and Kroupa, 2016). The relation is illustrated by the next example.

Example 4 Put $N = \{a, b, c, d\}$, $m(abcd) = 4$, $m(S) = 2$ for $S \subseteq N$ with $|S| = 3$, $m(S) = 1$ for any $S \subseteq N$ with $|S| = 2$ except $m(cd) = 0$ and $m(S) = 0$ for remaining $S \subseteq N$. One can easily verify that m is a supermodular game. The core $\overline{\mathcal{R}} = \mathsf{ext}\, C(m)$ consists of 13 vertices. To confirm that m is extreme in the supermodular cone one can use our method: the nullity of the respective constraint matrix $\mathbf{C}_\Upsilon$ is 1. However, this is not the only $\Gamma \subseteq \Upsilon$ with $\mathrm{null}\,(\mathbf{C}_\Gamma) = 1$; there exists $\Sigma \subset \Upsilon$ with $|\Sigma| = 9$ such that $\mathrm{null}\,(\mathbf{C}_\Gamma) = 1$ iff $\Sigma \subseteq \Gamma \subseteq \Upsilon$.

On the other hand, m is not an extreme exact game for it can be written as the sum of the game

$$m^1(abcd) = 2, \quad m^1(S) = 1 \ \text{ for } S \subseteq N, |S| = 3, \text{ and } \quad m^1(ac) = m^1(bc) = m^1(bd) = 1$$

(vanishing otherwise) and the game

$$m^2(abcd) = 2, \quad m^2(S) = 1 \ \text{ for } S \subseteq N, |S| = 3, \text{ and } \quad m^2(ab) = m^2(ad) = 1$$

(vanishing otherwise). Both m^1 and m^2 appear to be extreme in $\mathsf{E}_\ell(N)$. The core $C(m^1)$ has three vertices (x_a, x_b, x_c, x_d), namely $(1,1,0,0)$, $(0,1,1,0)$ and $(0,0,1,1)$, while $C(m^2)$ has four of them: $(1,1,0,0)$, $(1,0,1,0)$, $(1,0,0,1)$ and $(0,1,0,1)$. The set $\mathcal{R}^* := \mathsf{ext}\,[\,C(m^1) \oplus C(m^2)\,]$, where $X \oplus Y := \{\, x + y : \ x \in X \ \& \ y \in Y \}$ denoted the Minkowski sum of sets $X, Y \subseteq \mathbb{R}^N$, defines a regular min-representation of m. The respective index set $\Gamma^* \subseteq \Upsilon$ has 10 elements and one can construct two different solutions $y_{\Gamma^*} \in \mathbb{R}^{\Gamma^* \times N}$ to (a)-(b) on the basis of the standard min-representations of m^1 and m^2. Nevertheless, one even has $\mathrm{null}\,(\mathbf{C}_{\Gamma^*}) = 4$ in this case.

However, Γ^* does not provide the least regular min-representation of m. We found using a computer 27 inclusion-minimal regular min-representations of 10 permutation types; 8 of them have only four vectors (3 types) and 19 of them have five vectors (7 types). The nullities for the above mentioned inclusion-minimal min-representations are 6 and 7, which is the maximal nullity.

We also achieved the following partial converse result, whose proof we skip due to lack of space.

Proposition 9 Given non-zero $m \in \mathsf{E}_\ell(N)$ such that the least $\mathcal{R} \subseteq \mathsf{ext}\, C(m)$ satisfying (2) exists, the condition from Conjecture 6 is sufficient for m being extreme in $\mathsf{E}_\ell(N)$.

The idea of the proof of Proposition 9 is that different solutions to (a)-(b) are constructed on the basis of standard min-representations of potential summands of m. Note that the condition from Proposition 9 involves the special case of oxytrophic $m \in \mathsf{E}_\ell(N)$. On the other hand, an extreme exact game exists not having the least regular min-representation as the following example shows.

Example 5 Put $N = \{a,b,c,d\}$ and define $m(abcd) = 3$, $m(abc) = m(abd) = m(ab) = 2$, $m(acd) = m(bcd) = m(ac) = m(bc) = 1$ with $m(S) = 0$ for remaining for $S \subseteq N$. Then the set $\overline{\mathcal{R}} = \mathsf{ext}\, C(m)$ has 5 vectors (x_a, x_b, x_c, x_d). Three of them satisfy the oxytrophy condition (4):

$$\mathcal{R}: \quad (2,0,1,0), \quad (0,2,1,0), \quad (1,1,0,1),$$

and two of them not: $(2,1,0,0)$ and $(1,2,0,0)$. Adding of any of two other vectors to $\mathcal{R}$ turns it into an inclusion-minimal regular min-representation.

5. Conclusions

We have prepared a web platform for testing the extremity of an ℓ-standardized integer-valued exact game, available at

http://gogo.utia.cas.cz:3838/exact-and-supermodular/ .

It also allows one to test the extremity of supermodular games in the supermodular cone.

We have tested our criterion on 41 permutation types of 398 extreme ℓ-standardized exact games over 4 variables; these were also earlier listed by Quaeghebeur and de Cooman (2008). What we have found out is that 20 of these types are oxytrophic; one of them is mentioned in Example 1. The remaining types are not, but for 19 of these the least min-representation exists; one of them is mentioned in Example 2. We also found 2 types of extreme exact games for which two inclusion-minimal regular min-representations exist; one of these 2 types is given in Example 5.

In all 41 cases the necessary condition from Proposition 8 is valid: the nullities of the respective constraint matrices are 1. Proposition 9 is applicable in great majority of 39 cases, when the least regular min-representation exists. Thus, in these 39 cases our linear criterion allows one to confirm the extremity. However, in the remaining 2 cases one cannot apply Proposition 9 to confirm the extremity and an open question is whether our condition from Conjecture 6 is sufficient for the extremity of an exact game in general.

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Appendix A. Proofs

A.1 Proof of Proposition 4

If m has a min-representation $\mathcal{R}$ satisfying (i) for any $x \in \mathcal{R}$ then (2) implies $\emptyset \neq \mathcal{R} \subseteq C(m)$ and the condition of exactness for m is evident. Conversely, given an exact game m we put $\overline{\mathcal{R}} = \mathsf{ext}\, C(m)$ and use (1) to observe that (2) holds with $\overline{\mathcal{R}}$ in place of $\mathcal{R}$. The regularity condition (i) for $\overline{\mathcal{R}}$ is evident. To verify (ii) consider a fixed $x \in \overline{\mathcal{R}} = \mathsf{ext}\, C(m)$ and realize that the vectors in $\mathcal{V} := \{\chi_S \in \mathbb{R}^N : S \in \mathcal{T}_x^m\} \cup \{-\chi_N\}$ belong to the (inner) normal cone of (the least face of $C(m)$ containing) the vector x defined by

$$N_x := \{\, v \in \mathbb{R}^N : \quad \forall y \in C(m)\;\; \langle v, y\rangle \geq \langle v, x\rangle \,\} \equiv \{\, v \in \mathbb{R}^N : \; \langle v, x\rangle = \min_{y \in C(m)} \langle v, y\rangle \,\};$$

indeed, for $v = \chi_S$, $S \in \mathcal{T}_x^m$, one has $\langle v, y\rangle = \sum_{i\in S} y_i \geq m(S) = \sum_{i\in S} x_i = \langle v, x\rangle$ for any $y \in C(m)$. The cone N_x is the conic hull of $\mathcal{V}$, which observation can be derived from Farkas's lemma: if $t \in \mathbb{R}^N$ is not in the conic hull of $\mathcal{V}$ then $w \in \mathbb{R}^N$ exists such that $\langle v, w\rangle \geq 0$ for any $v \in \mathcal{V}$ while $\langle t, w\rangle < 0$. The former condition allows one to show that $y^\varepsilon := x + \varepsilon \cdot w$ belongs to $C(m)$ for some small $0 < \varepsilon$. The latter one implies that $\langle t, y^\varepsilon\rangle - \langle t, x\rangle = \langle t, y^\varepsilon - x\rangle = \varepsilon \cdot \langle t, w\rangle < 0$, implying that $t \notin N_x$.

The next observation is that, for any $x \in C(m)$, x is a vertex of $C(m)$ iff its normal cone N_x is full-dimensional. This result holds for any polytope $P \subseteq \mathbb{R}^N$ in place of $C(m)$. To see why this is the case the reader is advised to consult (Ziegler, 1995, § 7.1) for basic facts about the collection of normal cones for a polytope P, named the *normal fan* of the polytope. It is easy to realize that the lattice of normal cones is anti-isomorphic to the face-lattice of P. Specifically, the latter means, for $x, y \in P$, that one has $N_y \subseteq N_x$ iff $F[y] \supseteq F[x]$, where $F[x]$ denotes the least face of P containing $x \in P$. To this end realize that, for any $v \in N_x$ and $z \in P$, $\langle v, x\rangle = \langle v, z\rangle$ iff $v \in N_z$, which allows one to observe $F[x] := \bigcap_{v\in N_x} \{z \in P : \langle v, x\rangle = \langle v, z\rangle\} = \{z \in P : N_x \subseteq N_z\}$. Hence,

$$x \text{ is a vertex of } P \;\Leftrightarrow\; F[x] = \{x\} \;\Leftrightarrow\; N_x \text{ is a maximal cone} \;\Leftrightarrow\; N_x \text{ has the dimension } |N|.$$

By the former observation, the linear hull of N_x is the linear hull of $\{\, \chi_S \,:\, S \in \mathcal{T}_x^m \,\}$, which implies the condition (ii) for $x \in \overline{\mathcal{R}}$.

Thus, it follows from above arguments that any min-representation $\mathcal{R} \subseteq \mathsf{ext}\, C(m)$ is regular. Conversely, given a regular min-representation $\mathcal{R}$ of m, its elements belong to the core of m and the second regularity condition (ii) for $x \in \mathcal{R}$ implies that the respective normal cone N_x is full-dimensional, which happens only in case x is a vertex of $C(m)$.

A.2 Proof of Proposition 7

In case $\Gamma \subseteq \Sigma \subseteq \Upsilon$, it is evident that whenever $y_\Sigma \in \mathbb{R}^{\Sigma\times N}$ satisfies (a)-(b) with Σ then its restriction $y_\Gamma \in \mathbb{R}^{\Gamma\times N}$ to $\Gamma \times N$ satisfies (a)-(b) with Γ. The restriction mapping $y_\Sigma \mapsto y_\Gamma$ is linear and we show that it is one-to-one (under the assumptions from § 4.1). Thus, we assume that $y^1_\Sigma, y^2_\Sigma \in \mathbb{R}^{\Sigma\times N}$ are two solutions to (a)-(b) with Σ such that their restrictions to $\mathbb{R}^{\Gamma\times N}$ coincide, that is $y^1_\Gamma = y^2_\Gamma$, and we are going to show $y^1_\Sigma = y^2_\Sigma$.

Consider a fixed $\tau \in \Sigma \setminus \Gamma$ and our goal is to verify that $y^1(\tau, i) = y^2(\tau, i)$ for any $i \in N$. To this end, we show that, for any $S \in \mathcal{T}_\tau$ one has $\sum_{i\in S} y^1(\tau, i) = \sum_{i\in S} y^2(\tau, i)$ and then apply the fact that the vectors $\{\chi_S : S \in \mathcal{T}_\tau\}$ linearly generate $\mathbb{R}^N$ (see Definition 3 and Proposition 4). In case $S \in \mathcal{T}_\tau$, $S = \{i\}$, use (a) for Σ to observe $y^1(\tau, i) = 0 = y^2(\tau, i)$. In case $S \in \mathcal{T}_\tau$, $|S| \geq 2$, use the assumption that $\Gamma \subseteq \Upsilon$ satisfies (5) and find $\rho \in \Gamma$ such that $S \in \mathcal{T}_\rho$. The constraints (b) with Σ then imply

$$\sum_{i\in S} y^1(\tau, i) \overset{\text{(b)}}{=} \sum_{i\in S} y^1(\rho, i) = \sum_{i\in S} y^2(\rho, i) \overset{\text{(b)}}{=} \sum_{i\in S} y^2(\tau, i)\,,$$

which gives what is desired. Thus, if every solution to (a)-(b) with Γ is a multiple of x_Γ then every solution to (a)-(b) with Σ must be a multiple of x_Σ.

A.3 Proof of Proposition 8

To verify the necessity of the condition it is enough to show that its negation implies that m is a convex combination of $m^1, m^2 \in \mathsf{E}_\ell(N)$ none of which is a multiple of m.

For this purpose assume, under the situation described in § 4.1, that there exists $\Gamma \subseteq \Upsilon$ satisfying (5) such that the equation system (a)-(b) has a solution $y \in \mathbb{R}^{\Gamma\times N}$ which is not a multiple of $x_\Gamma \in \mathbb{R}^{\Gamma\times N}$. Note that the facts $m \neq 0$ and (5) imply that $x_\Gamma \neq \mathbf{0}$.

The first observation is that, for any $y \in \mathbb{R}^{\Gamma\times N}$ satisfying (a)-(b), an ℓ-standardized game $t \in \mathbb{R}^{\mathcal{P}(N)}$ exists such that

$$\forall\, \gamma \in \Gamma \ \ \forall\, S \in \mathcal{T}_\gamma \qquad t(S) = \sum_{i\in S} y(\gamma, i)\,. \tag{6}$$

To this end realize that (a)-(b) for y together imply the next *consistency condition*

$$\forall\, S \subseteq N \ \ \forall\, \tau, \rho \in \Gamma \ \text{ with } S \in \mathcal{T}_\tau \cap \mathcal{T}_\rho \qquad \sum_{i\in S} y(\tau, i) = \sum_{i\in S} y(\rho, i)\,.$$

Since (5) implies $\mathcal{P}(N) = \bigcup_{\gamma\in\Gamma} \mathcal{T}_\gamma$, one can correctly define t using (6). This game t is uniquely determined through (6); moreover, the function $y \in \mathbb{R}^{\Gamma\times N} \mapsto t \in \mathbb{R}^{\mathcal{P}(N)}$ is linear by definition. Finally, the fact that m is ℓ-standardized together with the condition (a) for y imply that t must be ℓ-standardized, too.

Consider the line L in $\mathbb{R}^{\Gamma\times N}$ passing through y and x_Γ, namely the vectors

$$z_\varepsilon := (1-\varepsilon)\cdot x_\Gamma + \varepsilon \cdot y \qquad \text{for any } \varepsilon \in \mathbb{R}\,.$$

Observe that L does not contain the zero vector in $\mathbb{R}^{\Gamma\times N}$ as otherwise y is a multiple of x_Γ. As vectors in L satisfy (a)-(b), ℓ-standardized games q_ε, $\varepsilon \in \mathbb{R}$, exist such that

$$\forall\, \varepsilon \in \mathbb{R} \ \ \forall\, \gamma \in \Gamma \text{ with } S \in \mathcal{T}_\gamma \qquad \sum_{i\in S} z_\varepsilon(\gamma, i) = q_\varepsilon(S)\,.$$

The next step is to show that, for sufficiently small $|\varepsilon|$, one has

$$\forall\, \gamma \in \Gamma \text{ with } S \notin \mathcal{T}_\gamma \qquad \sum_{i\in S} z_\varepsilon(\gamma, i) > q_\varepsilon(S)\,, \tag{7}$$

which implies, for those small $|\varepsilon|$, that

$$q_\varepsilon(S) = \min_{\gamma\in\Gamma} \sum_{i\in S} z_\varepsilon(\gamma, i) \qquad \text{for any } S \subseteq N.$$

This implies that $z_\varepsilon(\gamma, *) \in \mathbb{R}^N$, $\gamma \in \Gamma$, belong to the core $C(q_\varepsilon)$; in particular, $q_\varepsilon \in \mathsf{E}_\ell(N)$.

To ensure (7) for small $|\varepsilon|$, consider a fixed $\gamma \in \Gamma$ and $S \subseteq N$, $S \notin \mathcal{T}_\gamma$, and choose $\pi \in \Gamma$ such that $S \in \mathcal{T}_\pi$, by (5). The definitions of $\mathcal{T}_\gamma$ and $\mathcal{T}_\pi$ then imply

$$0 \;<\; \sum_{i\in S} x_\Gamma(\gamma, i) \;-\; m(S) = \sum_{i\in S} x_\Gamma(\gamma, i) \;-\; \sum_{i\in S} x_\Gamma(\pi, i)\,.$$

This allows one to write

$$\begin{aligned} \sum_{i\in S} z_\varepsilon(\gamma, i) \;-\; q_\varepsilon(S) \;&=\; \sum_{i\in S} z_\varepsilon(\gamma, i) \;-\; \sum_{i\in S} z_\varepsilon(\pi, i) \\ &=\; (1-\varepsilon)\cdot \underbrace{\left(\sum_{i\in S} x_\Gamma(\gamma, i) - \sum_{i\in S} x_\Gamma(\pi, i) \right)}_{>0} \;+\; \varepsilon \cdot \left(\sum_{i\in S} y(\gamma, i) - \sum_{i\in S} y(\pi, i) \right)\,, \end{aligned}$$

and observe that the limit of this expression with ε tending to zero is positive. Therefore, after considering all pairs (γ, S), $\gamma \in \Gamma$, $S \notin \mathcal{T}_\gamma$, (7) is ensured for sufficiently small $|\varepsilon|$.

Thus, there exists $0 < \varepsilon$ such that both $r := (1-\varepsilon)\cdot m + \varepsilon \cdot t$ and $s := (1+\varepsilon)\cdot m - \varepsilon \cdot t$ belong to $\mathsf{E}_\ell(N)$. Clearly, $m = \frac{1}{2}\cdot r + \frac{1}{2}\cdot s$. Neither r nor s is a multiple of m as otherwise the linearity of the one-to-one correspondence $y \in \mathbb{R}^{\Gamma\times N} \leftrightarrow t \in \mathbb{R}^{\mathcal{P}(N)}$ implies that the line L contains the zero vector in $\mathbb{R}^{\Gamma\times N}$, which is not the case.

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