

# Exchangeable Choice Functions

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## Abstract

We investigate how to model exchangeability with choice functions. Exchangeability is a structural assessment on a sequence of uncertain variables. We show how such assessments constitute a special kind of indifference assessment, and how this idea leads to a counterpart of de Finetti's Representation Theorem, both in a finite and a countable context.

**Keywords:** exchangeability; choice functions; indifference; sets of desirable gambles; representation.

## 1. Introduction

In this paper, we study how to model exchangeability, a structural assessment for uncertainty models that is important for inference purposes, in the framework of choice functions, an interesting approach to modelling uncertainty. This work builds on earlier results by [De Cooman et al. \(2009\)](#); [De Cooman and Quaeghebeur \(2012\)](#).

Choice functions are related to the fundamental problem in decision theory: how to make a choice from within a set of available options. In their book, [von Neumann and Morgenstern \(1944\)](#) provide an axiomatisation of choice based on a pairwise comparison between options. Later on, many authors ([Arrow, 1951](#); [Uzawa, 1956](#); [Rubin, 1987](#)) generalised this idea and proposed a theory of choice functions based on choice between more than two elements. One of the aspects of [Rubin's \(1987\)](#) theory is that, between any pair of options, the agent either prefers one of them or is indifferent between them, so two options are never incomparable. However, for instance when the available information does not allow for a complete comparison of the options, the agent may be undecided between two options without being indifferent between them; this will for instance typically be the case when there is little or no relevant information available. This is one of the motivations for a theory of imprecise probabilities ([Walley, 1991](#)), where incomparability and indifference are distinguished. [Kadane et al. \(2004\)](#) and [Seidenfeld et al. \(2010\)](#) generalise [Rubin's \(1987\)](#) axioms to allow for incomparability.

Exchangeability is a structural assessment on a sequence of uncertain variables. Loosely speaking, making a judgement of exchangeability means that the order in which the variables are observed is considered irrelevant. This irrelevancy will be modelled through an indifference assessment. The first detailed study of exchangeability was given by [de Finetti \(1937\)](#). We refer to the paper by [De Cooman and Quaeghebeur \(2012, Sec. 1\)](#) for a brief historical overview.

In [Sec. 2](#), we recall the necessary tools for modelling indifference with choice functions. Next, in [Sec. 3](#), we derive de Finetti-like Representation Theorems for a finite sequence that is exchangeable. We take this one step further in [Sec. 4](#), where we consider a countable sequence and de-

rive a representation theorem for such sequences. To compare with earlier work (De Cooman and Quaeghebeur, 2012), we also provide representation theorems for sets of desirable gambles.<sup>1</sup>

## 2. Choice Functions, Desirability and Indifference

Consider a real vector space  $\mathcal{V}$ , provided with the vector addition and scalar multiplication. Elements  $u$  of  $\mathcal{V}$  are intended as abstract representations of *options* amongst which a subject can express his preferences, by specifying, as we will see below, choice functions. Mostly, options will be real-valued maps on the possibility space, interpreted as uncertain rewards, and therefore also called *gambles*. The set of all gambles on the possibility space  $\mathcal{X}$  will be denoted by  $\mathcal{L}(\mathcal{X})$ . However, we will define choice functions on  $\mathcal{V}$  rather than on  $\mathcal{L}(\mathcal{X})$ , because, as we will see later, we will need to define choice functions on *equivalence classes* of gambles, which are no longer gambles themselves, but still constitute a vector space.<sup>2</sup> Given any subset  $O$  of  $\mathcal{V}$ , we will define the *linear hull*  $\text{span}(O) := \{\sum_{k=1}^n \lambda_k u_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}, u_k \in O\} \subseteq \mathcal{V}$  and the *positive hull*  $\text{posi}(O) := \{\sum_{k=1}^n \lambda_k u_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, u_k \in O\} \subseteq \text{span}(O)$ , where  $\mathbb{R}_{>0}$  is the set of all (strictly) positive real numbers. Furthermore, for any  $\lambda$  in  $\mathbb{R}_{>0}$  and  $u$  in  $\mathcal{V}$ , we let  $\lambda O + \{v\} := \{\lambda u + v : u \in O\}$ . A subset  $O$  of  $\mathcal{V}$  is called a *convex cone* if it is closed under positive finite linear combinations, i.e. if  $\text{posi}(O) = O$ . A convex cone  $\mathcal{K}$  is called *proper* if  $\mathcal{K} \cap -\mathcal{K} = \{0\}$ . With any proper convex cone  $\mathcal{K} \subseteq \mathcal{V}$ , we associate an ordering  $\leq_{\mathcal{K}}$  on  $\mathcal{V}$  as follows:  $u \leq_{\mathcal{K}} v \Leftrightarrow v - u \in \mathcal{K}$  for any  $u$  and  $v$  in  $\mathcal{V}$ . For any  $u$  and  $v$  in  $\mathcal{V}$ , we write  $u <_{\mathcal{K}} v$  if  $u \leq_{\mathcal{K}} v$  and  $u \neq v$ . We collect all the options  $u$  for which  $0 <_{\mathcal{K}} u$  in  $\mathcal{V}_{>0}$ . When we work with gambles, then  $\mathcal{V} = \mathcal{L}(\mathcal{X})$  and the ordering will be the standard one  $\leq$ , given by  $f \leq g \Leftrightarrow (\forall x \in \mathcal{X}) f(x) \leq g(x)$ . We collect the positive gambles—gambles  $f$  for which  $0 < f$ —in  $\mathcal{L}(\mathcal{X})_{>0}$ . Then  $\leq$  corresponds to  $\leq_{\mathcal{K}}$  where we let  $\mathcal{K} := \mathcal{L}(\mathcal{X})_{>0} \cup \{0\}$ .

We denote by  $\mathcal{Q}(\mathcal{V})$  the set of all non-empty *finite* subsets of  $\mathcal{V}$ . Elements of  $\mathcal{Q}(\mathcal{V})$  are the option sets amongst which a subject can choose his preferred options.

A choice function  $C$  on  $\mathcal{V}$  is a map  $C: \mathcal{Q} \rightarrow \mathcal{Q} \cup \{\emptyset\}: O \mapsto C(O)$  such that  $C(O) \subseteq O$ . Not every such map represents rational beliefs; only the coherent ones are considered to do so. We call a choice function  $C$  on  $\mathcal{V}$  *coherent*<sup>3</sup> if for all  $O, O_1$  and  $O_2$  in  $\mathcal{Q}(\mathcal{V})$ ,  $u$  and  $v$  in  $\mathcal{V}$ , and  $\lambda$  in  $\mathbb{R}_{>0}$ :

- C<sub>1</sub>.  $C(O) \neq \emptyset$ ;
- C<sub>2</sub>. if  $u < v$  then  $\{v\} = C(\{u, v\})$ ;
- C<sub>3</sub>. a. if  $C(O_2) \subseteq O_2 \setminus O_1$  and  $O_1 \subseteq O_2 \subseteq O$  then  $C(O) \subseteq O \setminus O_1$ ;  
b. if  $C(O_2) \subseteq O_1$  and  $O \subseteq O_2 \setminus O_1$  then  $C(O_2 \setminus O) \subseteq O_1$ ;
- C<sub>4</sub>. a. if  $O_1 \subseteq C(O_2)$  then  $\lambda O_1 \subseteq C(\lambda O_2)$ ;  
b. if  $O_1 \subseteq C(O_2)$  then  $O_1 + \{u\} \subseteq C(O_2 + \{u\})$ .

Consider two isomorphic vector spaces  $\mathcal{V}$  and  $\mathcal{W}$ , a linear order isomorphism  $\phi$  between  $\mathcal{V}$  and  $\mathcal{W}$ , and a choice function  $C$  on  $\mathcal{V}$ . Define the choice function  $C'$  on  $\mathcal{W}$  as  $u \in C(O) \Leftrightarrow \phi(u) \in C'(\phi(O))$  for all  $O$  in  $\mathcal{Q}(\mathcal{V})$  and  $u$  in  $O$ . Then, because  $\phi$  is a bijection,  $C$  satisfies Axioms **C<sub>1</sub>** and **C<sub>3</sub>** if and only if  $C'$  does; furthermore, because  $\phi$  is order preserving,  $C$  satisfies Axiom **C<sub>2</sub>** if and only if  $C'$  does; and finally, because  $\phi$  is linear,  $C$  satisfies Axiom **C<sub>4</sub>** if and only if  $C'$  does: such isomorphisms preserve coherence.

1. Due to page constraints, the proofs are not included in the paper. Readers interested in verifying the main proofs can access them through [arXiv:0801.0980](https://arxiv.org/abs/0801.0980).

2. This also allows us to connect our approach with the theory of coherent choice functions by Seidenfeld et al. (2010), where the authors define their choice function on *horse lotteries* instead of gambles. We intend to report on this later.

3. Our rationality axioms are based on those by Seidenfeld et al. (2010), slightly modified for use with sets of desirable options.

A set of desirable options (or gambles)  $D \subseteq \mathcal{V}$  is essentially the restriction to pairwise comparison of a choice function:  $D = \{u \in \mathcal{V} \setminus \{0\} : \{u\} = C(\{0, u\})\}$ . We call  $D$  coherent if  $0 \notin D$ ,  $\mathcal{V}_{>0} \subseteq D$ ,  $u \in D \Rightarrow \lambda u \in D$ , and  $u, v \in D \Rightarrow u + v \in D$  for all  $u$  and  $v$  in  $\mathcal{V}$  and  $\lambda$  in  $\mathbb{R}_{>0}$ .  $D$  is coherent if the choice function  $C$  it is based on, is coherent.

Since, as we will see, an exchangeability assessment amounts to a specific indifference assessment, we recall how to model such assessments (Van Camp et al., 2017, Sec. 5). Next to  $C(O)$ —the options that the agent *strictly* prefers from  $O$ —or  $D$ —the options that he *strictly* prefers to  $0$ —we consider the options in  $I \subseteq \mathcal{V}$ , which the agent considers to be *equivalent to the zero option*. We call a set of indifferent options  $I$  coherent if, for all  $u$  and  $v$  in  $\mathcal{V}$  and  $\lambda$  in  $\mathbb{R}$ :

- I<sub>1</sub>.  $0 \in I$ ;
- I<sub>2</sub>. if  $u \in \mathcal{V}_{>0} \cup \mathcal{V}_{<0}$  then  $u \notin I$ ;
- I<sub>3</sub>. if  $u \in I$  then  $\lambda u \in I$ ;
- I<sub>4</sub>. if  $u, v \in I$  then  $u + v \in I$ .

We collect all options that are indifferent to an option  $u$  in  $\mathcal{V}$  into the *equivalence class*  $[u] := \{v \in \mathcal{V} : v - u \in I\} = \{u\} + I$ . The set of all these equivalence classes is the *quotient space*  $\mathcal{V}/I := \{[u] : u \in \mathcal{V}\}$ , a linear space itself. We provide it with the natural ordering inherited from  $\mathcal{V}$ :  $\tilde{u} \leq \tilde{v} \Leftrightarrow (\exists u \in \tilde{u}, v \in \tilde{v}) u \leq v$ , for all  $\tilde{u}$  and  $\tilde{v}$  in  $\mathcal{V}/I$ .

In the remainder of this section, we will recall some of the results by Van Camp et al. (2017), needed for this paper. Consider any coherent set of indifferent options  $I$ . A choice function  $C$  is called *compatible* with  $I$  if there is some *representing* choice function  $C'$  on  $\mathcal{V}/I$  such that  $C(O) = \{u \in O : [u] \in C'(O/I)\}$  for all  $O$  in  $\mathcal{Q}(\mathcal{V})$ . In that case,  $C'$  is uniquely determined by  $C'(O/I) = C(O)/I$  for all  $O$  in  $\mathcal{Q}(\mathcal{V})$ , and, moreover,  $C$  is coherent if and only if  $C'$  is. Equivalently, we find the following useful characterisation:  $C$  is compatible with  $I$  if and only if  $0 \in C(O) \Leftrightarrow u \in C(O)$  for all  $u$  in  $I$  and  $O \supseteq \{0, u\}$  in  $\mathcal{Q}(\mathcal{V})$ , which corresponds to the definition of indifference given by Seidenfeld (1988).

For desirability, compatibility with a coherent set of indifferent options  $I$  is defined as follows. We call a set of desirable gambles  $D$  *compatible* with  $I$  if  $D + I \subseteq D$ , and this is equivalent to  $D = \bigcup D'$  where  $D' \subseteq \mathcal{V}/I$  is the *representing* set of desirable options. In that case,  $D'$  is uniquely given by  $D' = D/I$ —so  $D = \bigcup_{u \in D} [u]$ —and, moreover,  $D$  is coherent if and only if  $D'$  is.

### 3. Finite Exchangeability

Consider  $n \in \mathbb{N}$  uncertain variables  $X_1, \dots, X_n$  taking values in a non-empty set  $\mathcal{X}$ . The possibility space of the uncertain sequence  $(X_1, \dots, X_n)$  is  $\mathcal{X}^n$ .

We denote by  $x = (x_1, \dots, x_n)$  an arbitrary element of  $\mathcal{X}^n$ . For any  $n$  in  $\mathbb{N}$  we call  $\mathcal{P}_n$  the group of all permutations  $\pi$  of the index set  $\{1, \dots, n\}$ . There are  $|\mathcal{P}_n| = n!$  such permutations. With any such permutation  $\pi$ , we associate a permutation of  $\mathcal{X}^n$ , also denoted by  $\pi$ , and defined by  $(\pi x)_k := x_{\pi(k)}$  for every  $k$  in  $\{1, \dots, n\}$ , or in other words,  $\pi(x_1, \dots, x_n) = (x_{\pi(1)}, \dots, x_{\pi(n)})$ . Similarly, we lift  $\pi$  to a permutation  $\pi^t$  on  $\mathcal{L}(\mathcal{X}^n)$  by letting  $\pi^t f := f \circ \pi$ , so  $(\pi^t f)(x) = f(\pi x)$  for all  $x$  in  $\mathcal{X}^n$ . Observe that  $\pi^t$  is a linear permutation of the vector space  $\mathcal{L}(\mathcal{X}^n)$  of all gambles on  $\mathcal{X}^n$ .

If a subject assesses that the sequence of variables  $X$  in  $\mathcal{X}^n$  is exchangeable, this means that he is indifferent between any gamble  $f$  on  $\mathcal{X}^n$  and its permuted variant  $\pi^t f$ , for all  $\pi$  in  $\mathcal{P}_n$ . This leads us to the following proposal for the corresponding set of indifferent gambles:

$$I_{\mathcal{P}_n} := \text{span}\{f - \pi^t f : f \in \mathcal{L}(\mathcal{X}^n), \pi \in \mathcal{P}_n\}. \quad (1)$$

**Definition 1** A choice function  $C$  on  $\mathcal{L}(\mathcal{X}^n)$  is called (finitely) exchangeable if it is compatible with  $I_{\mathcal{P}_n}$ . Similarly, a set of desirable gambles  $D \subseteq \mathcal{L}(\mathcal{X}^n)$  is called (finitely) exchangeable if it is compatible with  $I_{\mathcal{P}_n}$ .

Of course, so far, we do not yet know whether this notion of exchangeability is well-defined: indeed, we do not know yet whether  $I_{\mathcal{P}_n}$  is a coherent set of indifferent gambles. In the next section, we will show that this is indeed the case.

### 3.1 Count Vectors

Let us now provide the tools necessary to prove that  $I_{\mathcal{P}_n}$  is a coherent set of indifferent gambles, as introduced by [De Cooman et al. \(2009\)](#) and [De Cooman and Quaeghebeur \(2012\)](#).

The permutation invariant atoms  $[x] := \{\pi x : x \in \mathcal{X}^n\}$ ,  $x$  in  $\mathcal{X}^n$  are the smallest permutation invariant subsets of  $\mathcal{X}^n$ . We introduce the counting map  $T: \mathcal{X}^n \rightarrow \mathcal{N}^n: x \mapsto T(x)$  where  $T(x)$  is called the count vector of  $x$ . It is the  $\mathcal{X}$ -tuple with components  $T_z(x) := |\{k \in \{1, \dots, n\} : x_k = z\}|$  for all  $z$  in  $\mathcal{X}$ , so  $T_z(x)$  is the number of times that  $z$  occurs in the sequence  $x_1, \dots, x_n$ . The range of  $T$ —the set  $\mathcal{N}^n$ —is called the set of possible count vectors and is given by  $\mathcal{N}^n := \{m \in \mathbb{Z}_{\geq 0}^{\mathcal{X}} : \sum_{x \in \mathcal{X}} m_x = n\}$ . Applying any permutation to  $x$  leaves its result under the counting map unchanged. For any  $x$  in  $\mathcal{X}^n$ , if  $m = T(x)$  then  $[x] = \{y \in \mathcal{X}^n : T(y) = m\}$ , so the permutation invariant atom  $[x]$  is completely determined by the count vector  $m$  of all its elements, and is therefore also denoted by  $[T(x)] = [m]$ . Remark that  $\{[m] : m \in \mathcal{N}^n\}$  partitions  $\mathcal{X}^n$  into disjoint parts with constant count vectors, and that  $|[m]| = \binom{n}{m} := \frac{n!}{\prod_{z \in \mathcal{X}} m_z!}$ .

In order to extend the idea of the count vectors for use with gambles, let us define the set of all permutation invariant gambles as  $\mathcal{L}_{\mathcal{P}_n}(\mathcal{X}^n) := \{f \in \mathcal{L}(\mathcal{X}^n) : (\forall \pi \in \mathcal{P}_n) \pi^t f = f\} \subseteq \mathcal{L}(\mathcal{X}^n)$ , and a special transformation  $\text{inv}_{\mathcal{P}_n}$  of the linear space  $\mathcal{L}(\mathcal{X}^n)$

$$\text{inv}_{\mathcal{P}_n}: \mathcal{L}(\mathcal{X}^n) \rightarrow \mathcal{L}(\mathcal{X}^n): f \mapsto \text{inv}_{\mathcal{P}_n}(f) := \frac{1}{n!} \sum_{\pi \in \mathcal{P}_n} \pi^t f,$$

which, as we will see, is closely linked with  $\mathcal{L}_{\mathcal{P}_n}(\mathcal{X}^n)$  ([De Cooman and Quaeghebeur, 2012](#); [Van Camp et al., 2017](#)).

**Proposition 2**  $\text{inv}_{\mathcal{P}_n}$  is a linear transformation of  $\mathcal{L}(\mathcal{X}^n)$ , and

- (i)  $\text{inv}_{\mathcal{P}_n} \circ \pi^t = \text{inv}_{\mathcal{P}_n} = \pi^t \circ \text{inv}_{\mathcal{P}_n}$  for all  $\pi$  in  $\mathcal{P}$ ;
- (ii)  $\text{inv}_{\mathcal{P}_n} \circ \text{inv}_{\mathcal{P}_n} = \text{inv}_{\mathcal{P}_n}$ ;
- (iii)  $\text{kern}(\text{inv}_{\mathcal{P}_n}) = I_{\mathcal{P}_n}$ ;
- (iv)  $\text{rng}(\text{inv}_{\mathcal{P}_n}) = \mathcal{L}_{\mathcal{P}_n}(\mathcal{X}^n)$ .

So we see that  $\text{inv}_{\mathcal{P}_n}$  is a linear projection operator that maps any gamble to a permutation invariant counterpart.

As shown by [De Cooman and Quaeghebeur \(2012\)](#), the linear projection operator  $\text{inv}_{\mathcal{P}_n}$  renders a gamble insensitive to permutation by replacing it with the uniform average of all its permutations. As a result, it assumes the same value for all gambles that can be related to each other through some permutation:  $\text{inv}_{\mathcal{P}_n}(f) = \text{inv}_{\mathcal{P}_n}(g)$  if  $f = \pi^t g$  for some  $\pi$  in  $\mathcal{P}_n$ , for all  $f$  and  $g$  in  $\mathcal{L}(\mathcal{X}^n)$ . Furthermore, for any  $f$  in  $\mathcal{L}(\mathcal{X}^n)$ , its transformation  $\text{inv}_{\mathcal{P}_n}(f)$  is permutation invariant and therefore constant on the permutation invariant atoms  $[m]$ :  $(\text{inv}_{\mathcal{P}_n}(f))(x) = (\text{inv}_{\mathcal{P}_n}(f))(y)$  if  $[x] = [y]$ , for all  $x$  and  $y$  in  $\mathcal{X}^n$ . We can use the properties of  $\text{inv}_{\mathcal{P}_n}$  to prove that  $I_{\mathcal{P}_n}$  is suitable for the definition of exchangeability.

**Proposition 3** For any  $n$  in  $\mathbb{N}$ , the set  $I_{\mathcal{P}_n}$ , defined in Eq. (1), is a coherent set of indifferent gambles.

Since  $I_{\mathcal{P}_n}$  is coherent, exchangeability is well-defined, and by the discussion in Sec. 2, the representing choice function  $C'$  is defined on  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$ , and, similarly, the representing set of desirable gambles  $D' \subseteq \mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$ . So we can focus on the quotient space and its elements, exchangeable equivalent classes of gambles.

But before we do that, it will pay to further explore the notions we have introduced thus far.

Consider any  $f$  in  $\mathcal{L}(\mathcal{X}^n)$ . What is the constant value that  $\text{inv}_{\mathcal{P}_n}(f)$  assumes on a permutation invariant atom  $[m]$ ? To answer this question, consider any  $x$  in  $[m]$ , then  $(\text{inv}_{\mathcal{P}_n}(f))(x) = \frac{1}{n!} \sum_{\pi \in \mathcal{P}_n} f(\pi x) = \frac{1}{n!} \frac{|\mathcal{P}_n|}{|[m]|} \sum_{y \in \{\pi x : \pi \in \mathcal{P}_n\}} f(y) = \frac{1}{\binom{n}{m}} \sum_{y \in [x]} f(y) = \frac{1}{\binom{n}{m}} \sum_{y \in [m]} f(y)$  where we used the fact that  $|\mathcal{P}_n| = n!$  and  $|[m]| = \binom{n}{m}$ , whence  $\text{inv}_{\mathcal{P}_n} = \sum_{m \in \mathcal{N}^n} \mathbb{H}_n(\cdot|m) \mathbb{I}_{[m]}$ , where  $\mathbb{H}_n(\cdot|m)$  is the linear expectation operator associated with the uniform distribution on the invariant atom  $[m]$ :

$$\mathbb{H}_n(f|m) := \frac{1}{\binom{n}{m}} \sum_{y \in [m]} f(y) \text{ for all } f \text{ in } \mathcal{L}(\mathcal{X}^n) \text{ and } m \text{ in } \mathcal{N}^n.$$

It characterises a (multivariate) hyper-geometric distribution (Johnson et al., 1997), associated with random sampling without replacement from an urn with  $n$  balls of types  $\mathcal{X}$ , whose composition is characterised by the count vector  $m$ .

The result of subjecting a gamble  $f$  on  $\mathcal{X}^n$  to the map

$$\mathbb{H}_n: \mathcal{L}(\mathcal{X}^n) \rightarrow \mathcal{L}(\mathcal{N}^n): f \mapsto \mathbb{H}_n(f) := \mathbb{H}_n(f|\cdot)$$

is the gamble  $\mathbb{H}_n(f)$  on  $\mathcal{N}^n$  that assumes the value  $\frac{1}{\binom{n}{m}} \sum_{y \in [m]} f(y)$  in every  $m$  in  $\mathcal{N}^n$ .

### 3.2 Exchangeable Equivalent Classes of Gambles

We already know that exchangeable choice functions are represented by choice functions on the quotient space  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$ , and similarly for sets of desirable gambles. In the quest for an elegant representation theorem, we thus need to focus on the quotient space  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$  and its elements, which are exchangeable equivalent classes of gambles.

In this section we investigate how the representation of permutation invariant gambles helps us find a representation for exchangeable choice functions. To that end, the representation will use equivalence classes  $[f] := \{f\} + I_{\mathcal{P}_n}$  of gambles, for any  $f$  in  $\mathcal{L}(\mathcal{X}^n)$ . Recall that the quotient space  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n} := \{[f] : f \in \mathcal{L}(\mathcal{X}^n)\}$  is a linear space itself, with additive identity  $[0] = I_{\mathcal{P}_n}$ , and therefore any element  $\tilde{f}$  of  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$  is invariant under addition of  $I_{\mathcal{P}_n}$ :  $\tilde{f} + I_{\mathcal{P}_n} = \tilde{f}$ . Elements of  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$  will be generically denoted by  $\tilde{f}$  or  $\tilde{g}$ .

**Proposition 4** Consider any  $f$  and  $g$  in  $\mathcal{L}(\mathcal{X}^n)$ . Then  $[f] = [g]$  if and only if  $\mathbb{H}_n(f) = \mathbb{H}_n(g)$ .

Therefore, it makes sense to introduce the map  $\tilde{\mathbb{H}}_n$ :

$$\tilde{\mathbb{H}}_n: \mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n} \rightarrow \mathcal{L}(\mathcal{N}^n): \tilde{f} \mapsto \mathbb{H}_n(f) \text{ for any } f \text{ in } \tilde{f}. \quad (2)$$

Then Proposition 4 guarantees that elements of  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$  are characterised using  $\tilde{\mathbb{H}}_n$ , in the sense that  $\tilde{f} = \{f \in \mathcal{L}(\mathcal{X}^n) : \mathbb{H}_n(f) = \tilde{\mathbb{H}}_n(\tilde{f})\}$  for all  $\tilde{f}$  in  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$ .

The map  $\tilde{H}_n$  takes as an argument an equivalence class of gambles, and maps it to some representing gamble on the count vectors. It will be useful later on to consider the inverse map  $\tilde{H}_n^{-1}$ :

$$\tilde{H}_n^{-1}: \mathcal{L}(\mathcal{N}^n) \rightarrow \mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}: f \mapsto \left[ \sum_{m \in \mathcal{N}^n} f(m) \mathbb{I}_{[m]} \right]. \quad (3)$$

**Proposition 5** *The maps  $\tilde{H}_n$  as defined in Eq. (2) and  $\tilde{H}_n^{-1}$  as defined in Eq. (3) are each other's inverses.*

The importance of Prop. 5 lies in the fact that now,  $\tilde{H}_n$  is a bijection between  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$  and  $\mathcal{L}(\mathcal{N}^n)$ , and therefore, exchangeable equivalence classes of gambles are in a one-to-one correspondence with gambles on count vectors.

$$\begin{array}{ccc} \mathcal{L}(\mathcal{X}^n) & \xrightarrow{H_n} & \mathcal{L}(\mathcal{N}^n) \\ \left[ \cdot \right] \downarrow & \swarrow \tilde{H}_n & \\ \mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n} & & \end{array}$$

The commuting diagram shows the surjections  $[\cdot]: \mathcal{L}(\mathcal{X}^n) \rightarrow \mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}: f \mapsto [f]$  and  $H_n$  (indicated with a single arrow), and the bijection  $\tilde{H}_n$  (indicated with a double arrow). Since the representing choice function  $C'$  is defined from  $C$  through  $[\cdot]$ —working point-wise on sets—this already suggests that  $C'$  can be transformed into a choice function on  $\mathcal{L}(\mathcal{N}^n)$ . To prove that they preserve coherence, there is only one missing link: the map  $\tilde{H}_n$  should be linear and preserve the ordering between  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$  and  $\mathcal{L}(\mathcal{N}^n)$ . Therefore, to define the ordering  $\leq$  on  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$ , as usual, we let  $\leq$  be inherited by the ordering  $\leq$  on  $\mathcal{L}(\mathcal{X}^n)$ :

$$\tilde{f} \leq \tilde{g} \Leftrightarrow (\exists f \in \tilde{f}, \exists g \in \tilde{g}) f \leq g$$

for all  $\tilde{f}$  and  $\tilde{g}$  in  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$ , turning  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$  into an ordered linear space. It turns out that this vector ordering on  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$  can be represented elegantly using  $\tilde{H}_n$ :

**Proposition 6** *Consider any  $\tilde{f}$  and  $\tilde{g}$  in  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$ , then  $\tilde{f} \leq \tilde{g}$  if and only if  $\tilde{H}_n(\tilde{f}) \leq \tilde{H}_n(\tilde{g})$ .*

Props. 5 and 6 imply that  $H_n$  is a linear order isomorphism.

### 3.3 A Representation Theorem

Now that we have found a linear order isomorphism  $\tilde{H}_n$  between  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$  and  $\mathcal{L}(\mathcal{N}^n)$ , we are ready to represent coherent and exchangeable choice functions.

**Theorem 7 (Finite Representation)** *Consider any choice function  $C$  on  $\mathcal{L}(\mathcal{X}^n)$ . Then  $C$  is exchangeable if and only if there is a unique representing choice function  $\tilde{C}$  on  $\mathcal{L}(\mathcal{N}^n)$  such that*

$$C(O) = \{f \in O : H_n(f) \in \tilde{C}(H_n(O))\} \text{ for all } O \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^n)).$$

*Furthermore, in that case,  $\tilde{C}$  is given by  $\tilde{C}(H_n(O)) = H_n(C(O))$  for all  $O$  in  $\mathcal{Q}(\mathcal{L}(\mathcal{X}^n))$ . Finally,  $C$  is coherent if and only if  $\tilde{C}$  is.*

*Similarly, consider any set of desirable gambles  $D \subseteq \mathcal{L}(\mathcal{X}^n)$ . Then  $D$  is exchangeable if and only if there is a unique representing set of desirable gambles  $\tilde{D} \subseteq \mathcal{L}(\mathcal{N}^n)$  such that  $D = \cup \tilde{H}_n^{-1}(\tilde{D})$ . Furthermore, in that case,  $\tilde{D}$  is given by  $\tilde{D} = H_n(D)$ . Finally,  $D$  is coherent if and only if  $\tilde{D}$  is.*

The number of occurrences of any outcome in a sequence  $(x_1, \dots, x_n)$  is fixed by its count vector  $m$  in  $\mathcal{N}^n$ . If we impose an exchangeability assessment on it, then we see, using Theorem 7, that the joint model on  $\mathcal{X}^n$  is characterised by a model on  $\mathcal{L}(\mathcal{N}^n)$ . So an exchangeable choice function  $C$  essentially represents preferences between urns with  $n$  balls of types  $\mathcal{X}$  with different compositions  $m$ : the choice  $C(O)$  between the gambles in  $O$  is based upon the composition  $m$ .

### 3.4 Finite Representation in Terms of Polynomials

In Sec. 4, we will prove a similar representation theorem for infinite sequences. Since it no longer makes sense to *count* in such sequences, we first need to find a equivalent representation theorem in terms of something that does not depend on counts. More specifically, we need, for every  $n$  in  $\mathbb{N}$  another order-isomorphic linear space to  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$ , that allows for embedding: the linear space for  $n_1 < n_2$  must be a subspace of the one for  $n_2$ .

All the maps in this section have been introduced by De Cooman et al. (2009) and De Cooman and Quaeghebeur (2012). We use their ideas and work with polynomials on the  $\mathcal{X}$ -simplex  $\Sigma_{\mathcal{X}} := \{\theta \in \mathbb{R}^{\mathcal{X}} : \theta \geq 0, \sum_{x \in \mathcal{X}} \theta_x = 1\}$ . We consider the special subset  $\mathcal{V}(\Sigma_{\mathcal{X}})$  of  $\mathcal{L}(\Sigma_{\mathcal{X}})$ :  $\mathcal{V}(\Sigma_{\mathcal{X}})$  are the *polynomial gambles*  $h$  on  $\Sigma_{\mathcal{X}}$ , which are those gambles that are the restriction to  $\Sigma_{\mathcal{X}}$  of a multivariate polynomial  $p$  on  $\mathbb{R}^{\mathcal{X}}$ , in the sense that  $h(\theta) = p(\theta)$  for all  $\theta$  in  $\Sigma_{\mathcal{X}}$ . We call  $p$  then a representation of  $h$ . It will be useful to introduce a notation for polynomial gambles with fixed degree  $n$  in  $\mathbb{N}$ :  $\mathcal{V}^n(\Sigma_{\mathcal{X}})$  is the collection of all polynomial gambles that have at least one representation whose degree is not higher than  $n$ . Both  $\mathcal{V}(\Sigma_{\mathcal{X}})$  and  $\mathcal{V}^n(\Sigma_{\mathcal{X}})$  are linear subspaces of  $\mathcal{L}(\Sigma_{\mathcal{X}})$ , and, as wanted, for  $n_1 \leq n_2$ ,  $\mathcal{V}^{n_1}(\Sigma_{\mathcal{X}})$  is a subspace of  $\mathcal{V}^{n_2}(\Sigma_{\mathcal{X}})$ .

Some special polynomial gambles are the *Bernstein gambles*:

**Definition 8 (Bernstein gambles)** Consider any  $n$  in  $\mathbb{N}$  and any  $m$  in  $\mathcal{N}^n$ . Define the Bernstein basis polynomial  $B_m$  on  $\mathbb{R}^{\mathcal{X}}$  as  $B_m(\theta) := \binom{n}{m} \prod_{x \in \mathcal{X}} \theta_x^{m_x}$  for all  $\theta$  in  $\mathbb{R}^{\mathcal{X}}$ . The restriction to  $\Sigma_{\mathcal{X}}$  is called a Bernstein gamble, which we also denote as  $B_m$ .

As shown by De Cooman and Quaeghebeur (2012) and also by De Bock et al. (2016), the set of all Bernstein gambles constitutes a basis for the linear space  $\mathcal{V}^n(\Sigma_{\mathcal{X}})$ :

**Proposition 9** Consider any  $n$  in  $\mathbb{N}$ . The set of Bernstein gambles  $\{B_m : m \in \mathcal{N}^n\}$  constitutes a basis for the linear space  $\mathcal{V}^n(\Sigma_{\mathcal{X}})$ .

As we have seen, to preserve coherence between two ordered linear spaces, we need a linear order isomorphism. So we wonder whether there is one between  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$  and  $\mathcal{V}^n(\Sigma_{\mathcal{X}})$ . In Sec. 3.2 we have seen that there is one between  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$  and  $\mathcal{L}(\mathcal{N}^n)$ , namely  $\tilde{H}_n$ . Therefore, it suffices to find one between  $\mathcal{L}(\mathcal{N}^n)$  and  $\mathcal{V}^n(\Sigma_{\mathcal{X}})$ . Consider the map

$$\text{CoM}_n: \mathcal{L}(\mathcal{N}^n) \rightarrow \mathcal{V}^n(\Sigma_{\mathcal{X}}): r \mapsto \sum_{m \in \mathcal{N}^n} r(m) B_m.$$

Before we can establish that  $\text{CoM}_n$  is a linear order isomorphism, we need to provide the linear space  $\mathcal{V}^n(\Sigma_{\mathcal{X}})$  with an order  $\leq_B^n$ . We use the proper cone  $\{0\} \cup \text{posi}(\{B_m : m \in \mathcal{N}^n\})$  to define the order  $\leq_B^n$ :

$$h_1 \leq_B^n h_2 \Leftrightarrow h_2 - h_1 \in \{0\} \cup \text{posi}(\{B_m : m \in \mathcal{N}^n\}) \text{ for all } h_1 \text{ and } h_2 \text{ in } \mathcal{V}^n(\Sigma_{\mathcal{X}}).$$

The following proposition is proved by De Cooman and Quaeghebeur (2012).

**Proposition 10** Consider any  $n$  in  $\mathbb{N}$ . Then the map  $\text{CoM}_n$  is a linear order isomorphism between the ordered linear spaces  $\mathcal{L}(\mathcal{N}^n)$  and  $\mathcal{V}^n(\Sigma_{\mathcal{X}})$ .

The linear order isomorphism  $\text{CoM}_n$  helps us to define a linear order isomorphism between the linear spaces  $\mathcal{L}(\mathcal{X}^n)$  and  $\mathcal{V}^n(\Sigma_{\mathcal{X}})$ , a final tool needed for a representation theorem in terms of polynomial gambles. Indeed, consider for the map  $M_n := \text{CoM}_n \circ H_n$ :

$$M_n: \mathcal{L}(\mathcal{X}^n) \rightarrow \mathcal{V}^n(\Sigma_{\mathcal{X}}): f \mapsto M_n(f|\theta),$$

where  $M_n(f|\theta) := \sum_{m \in \mathcal{N}^n} \sum_{y \in [m]} f(y) \prod_{x \in \mathcal{X}} \theta_x^{m_x}$  is the expectation of  $f$  associated with the multinomial distribution whose parameters are  $n$  and  $\theta$ . We introduce its version

$$\tilde{M}_n := \text{CoM}_n \circ \tilde{H}_n, \quad (4)$$

mapping  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$  to  $\mathcal{V}^n(\Sigma_{\mathcal{X}})$ . There is an immediate connection between  $M_n$  and  $\tilde{M}_n$ : they are both compositions of two linear order isomorphisms, and are therefore linear order isomorphisms themselves. Due to Prop. 4, considering any  $\tilde{f}$  in  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$ ,  $M_n$  is constant on  $\tilde{f}$ , and the value it takes on any element of  $\tilde{f}$  is exactly  $\tilde{M}_n(\tilde{f})$ .

$$\begin{array}{ccc}
 \mathcal{L}(\mathcal{N}^n) & \xleftarrow{\text{CoM}_n} & \mathcal{V}^n(\Sigma_{\mathcal{X}}) \\
 \uparrow H_n & & \downarrow M_n \\
 & \mathcal{L}(\mathcal{X}^n) & \\
 \downarrow \tilde{H}_n & \downarrow [\cdot] & \uparrow \tilde{M}_n \\
 & \mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n} & 
 \end{array}$$

The commuting diagram shows the surjections  $[\cdot]$ ,  $H_n$  and  $M_n$ , and the bijections  $\tilde{H}_n$ ,  $\tilde{M}_n$  and  $\text{CoM}_n$ . It shows that both  $\mathcal{L}(\mathcal{N}^n)$  and  $\mathcal{V}^n(\Sigma_{\mathcal{X}})$  are order-isomorphic to  $\mathcal{L}(\mathcal{X}^n)/I_{\mathcal{P}_n}$ , so they are both suitable to define a representing choice function on. In Theorem 7, we used the space  $\mathcal{L}(\mathcal{N}^n)$ . Here, we will use the other equivalent space  $\mathcal{V}^n(\Sigma_{\mathcal{X}})$ .

**Theorem 11 (Finite Representation)** Consider any choice function  $C$  on  $\mathcal{L}(\mathcal{X}^n)$ . Then  $C$  is exchangeable if and only if there is a unique representing choice function  $\tilde{C}$  on  $\mathcal{V}^n(\Sigma_{\mathcal{X}})$  such that

$$C(O) = \{f \in O : M_n(f) \in \tilde{C}(M_n(O))\} \text{ for all } O \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^n)).$$

Furthermore, in that case,  $\tilde{C}$  is given by  $\tilde{C}(M_n(O)) = M_n(C(O))$  for all  $O$  in  $\mathcal{Q}(\mathcal{L}(\mathcal{X}^n))$ . Finally,  $C$  is coherent if and only if  $\tilde{C}$  is.

Similarly, consider any set of desirable gambles  $D \subseteq \mathcal{L}(\mathcal{X}^n)$ . Then  $D$  is exchangeable if and only if there is a unique representing set of desirable gambles  $\tilde{D} \subseteq \mathcal{V}^n(\Sigma_{\mathcal{X}})$  such that  $D = \bigcup \tilde{M}_n^{-1}(\tilde{D})$ . Furthermore, in that case,  $\tilde{D}$  is given by  $\tilde{D} = M_n(D)$ . Finally,  $D$  is coherent if and only if  $\tilde{D}$  is.

#### 4. Countable Exchangeability

In the previous section, we assumed a finite sequence  $X_1, \dots, X_n$  to be exchangeable, and inferred representation theorems. In this section, we will consider the countable sequence  $X_1, \dots, X_n, \dots$  to be exchangeable, and derive representation theorems for such assessments. We will call  $\mathcal{X}^{\mathbb{N}} := \times_{j \in \mathbb{N}} \mathcal{X}$ , the set of all possible countable sequences where each variable takes values in  $\mathcal{X}$ .



First, we will need a way to relate gambles on different domains. Let  $f$  be some gamble on  $\mathcal{X}^n$ , and let  $f^*$  be its *cylindrical extension*, defined as

$$f^*(x_1, \dots, x_n, \dots) := f(x_1, \dots, x_n) \text{ for all } (x_1, \dots, x_n, \dots) \text{ in } \mathcal{X}^{\mathbb{N}}.$$

Formally,  $f^*$  belongs to  $\mathcal{L}(\mathcal{X}^{\mathbb{N}})$  while  $f$  belongs to  $\mathcal{L}(\mathcal{X}^n)$ . However, they contain the same information, and therefore, are indistinguishable from a behavioural point of view. In this paper, we will identify  $f$  with its cylindrical extension  $f^*$ . Using this convention, we can for instance identify  $\mathcal{L}(\mathcal{X}^n)$  with a subset of  $\mathcal{L}(\mathcal{X}^{\mathbb{N}})$ , and, as an other example, for any  $\mathcal{A} \subseteq \mathcal{L}(\mathcal{X}^{\mathbb{N}})$ , regard  $\mathcal{A} \cap \mathcal{L}(\mathcal{X}^n)$  as those gambles in  $\mathcal{A}$  that depend upon the first  $n$  variables only.

#### 4.1 Marginalisation

Using the notational convention we have just discussed, we can very easily define what marginalisation means for choice functions. Given any choice function  $C$  on  $\mathcal{L}(\mathcal{X}^{\mathbb{N}})$  and any  $n$  in  $\mathbb{N}$ , its  $\mathcal{X}^n$ -marginal  $C_n$  is determined by  $C_n(O) := C(O)$  for all  $O$  in  $\mathcal{Q}(\mathcal{L}(\mathcal{X}^n))$ .

Similarly, given any set of desirable gambles  $D \subseteq \mathcal{L}(\mathcal{X}^{\mathbb{N}})$  and any  $n$  in  $\mathbb{N}$ , its  $\mathcal{X}^n$ -marginal  $D_n$  is defined by  $D_n := D \cap \mathcal{L}(\mathcal{X}^n)$ .

Coherence is preserved under marginalisation [it is an immediate consequence of the definition; see, amongst others, (De Cooman and Miranda, 2012, Proposition 6) for sets of desirable gambles].

**Proposition 12** *Consider any coherent choice function  $C$  on  $\mathcal{L}(\mathcal{X}^{\mathbb{N}})$  and any coherent set of desirable gambles  $D \subseteq \mathcal{L}(\mathcal{X}^{\mathbb{N}})$ . Then for every  $n$  in  $\mathbb{N}$ , their  $\mathcal{X}^n$ -marginals  $C_n$  and  $D_n$  are coherent.*

#### 4.2 Gambles of Finite Structure

Before we can explain what it means to assess a countable sequence to be exchangeable, we need to realise that now there are infinitely many variables. From an operational point of view, it will be impossible to describe choosing between gambles that depend upon an infinite number of variables. Indeed, since we can never observe the actual outcome in a finite time, gambles will never be actually paid off, and hence every assessment is essentially without any risk. But, it does make operational and behavioural sense to consider choices between gambles of *finite structure*: gambles that each depend on a finite number of variables only. See (De Bock et al., 2016, Sec. 3.2) for a discussion.

**Definition 13 (Gambles of finite structure)** *We will call any gamble that depends only on a finite number of variables a gamble of finite structure. We collect all such gambles in  $\bar{\mathcal{L}}(\mathcal{X}^{\mathbb{N}})$ :*

$$\bar{\mathcal{L}}(\mathcal{X}^{\mathbb{N}}) := \{f \in \mathcal{L}(\mathcal{X}^{\mathbb{N}}) : (\exists n \in \mathbb{N}) f \in \mathcal{L}(\mathcal{X}^n)\} = \bigcup_{n \in \mathbb{N}} \mathcal{L}(\mathcal{X}^n).$$

$\bar{\mathcal{L}}(\mathcal{X}^{\mathbb{N}})$  is a linear space, with the usual ordering  $\leq$ : for any  $f$  and  $g$  in  $\bar{\mathcal{L}}(\mathcal{X}^{\mathbb{N}})$ ,  $f \leq g \Leftrightarrow f(x) \leq g(x)$  for all  $x$  in  $\mathcal{X}^{\mathbb{N}}$ .

Due to our finitary context, we can even establish a converse result to Prop. 12, whose proof for the part about sets of desirable gambles can be found in (De Bock et al., 2016, Proposition 4), and for the part about choice functions is a straightforward verification of all the axioms.

**Proposition 14** *Consider any choice function  $C$  on  $\bar{\mathcal{L}}(\mathcal{X}^{\mathbb{N}})$ , and any set of desirable gambles  $D \subseteq \bar{\mathcal{L}}(\mathcal{X}^{\mathbb{N}})$ . If for every  $n$  in  $\mathbb{N}$ , its  $\mathcal{X}^n$ -marginal  $C_n$  on  $\mathcal{L}(\mathcal{X}^n)$  is coherent, then  $C$  is coherent. Similarly, if for every  $n$  in  $\mathbb{N}$ , its  $\mathcal{X}^n$ -marginal  $D_n \subseteq \mathcal{L}(\mathcal{X}^n)$  is coherent, then  $D$  is coherent.*

### 4.3 Set of indifferent gambles

If a subject assesses the sequence of variables  $X_1, \dots, X_n, \dots$  to be exchangeable, this means that he is indifferent between any gamble  $f$  in  $\tilde{\mathcal{L}}(\mathcal{X}^{\mathbb{N}})$  and its permuted variant  $\pi^t f$ , for any  $\pi$  in  $\mathcal{P}_n$ , where  $n$  now is the (finite) number of variables that  $f$  depends upon: his set of indifferent gambles is

$$I_{\mathcal{P}} := \{f \in \tilde{\mathcal{L}}(\mathcal{X}^{\mathbb{N}}) : (\exists n \in \mathbb{N}) f \in I_{\mathcal{P}_n}\} = \bigcup_{n \in \mathbb{N}} I_{\mathcal{P}_n}.$$

If we want to use  $I_{\mathcal{P}}$  to define countable exchangeability, it must be a coherent set of indifferent gambles.

**Proposition 15** *The set  $I_{\mathcal{P}}$  is a coherent set of indifferent gambles.*

Countable exchangeability is now easily defined, similar to the definition for the finite case.

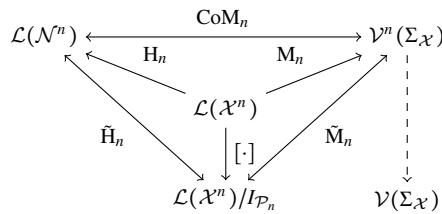
**Definition 16** *A choice function  $C$  on  $\tilde{\mathcal{L}}(\mathcal{X}^{\mathbb{N}})$  is called (countably) exchangeable if  $C$  is compatible with  $I_{\mathcal{P}}$ . Similarly, a set of desirable gambles  $D \subseteq \tilde{\mathcal{L}}(\mathcal{X}^{\mathbb{N}})$  is called (countably) exchangeable if it is compatible with  $I_{\mathcal{P}}$ .*

This definition is closely related to its finite counterpart.

**Proposition 17** *Consider any coherent choice function  $C$  on  $\tilde{\mathcal{L}}(\mathcal{X}^{\mathbb{N}})$ . Then  $C$  is exchangeable if and only if for every choice of  $n$  in  $\mathbb{N}$ , the  $\mathcal{X}^n$ -marginal  $C_n$  of  $C$  is exchangeable. Similarly, consider any coherent set of desirable gambles  $D \subseteq \tilde{\mathcal{L}}(\mathcal{X}^{\mathbb{N}})$ . Then  $D$  is exchangeable if and only if for every choice of  $n$  in  $\mathbb{N}$ , the  $\mathcal{X}^n$ -marginal  $D_n$  of  $D$  is exchangeable.*

### 4.4 A Representation Theorem for Countable Sequences

We will look for a similar representation result. However, since we no longer deal with finite sequences of length  $n$ , now the representing choice function won't be defined on  $\mathcal{V}^n(\Sigma_{\mathcal{X}})$ , but instead on  $\mathcal{V}(\Sigma_{\mathcal{X}})$ .



In the commuting diagram, a dashed line represents an embedding: indeed, for every  $n$  in  $\mathbb{N}$ ,  $\mathcal{V}^n(\Sigma_{\mathcal{X}})$  is a subspace of  $\mathcal{V}(\Sigma_{\mathcal{X}})$ . That shows the importance of the polynomial representation.

As we have seen, in order to define coherent choice functions on some linear space, we need to provide it with a vector ordering. Similar to what we did before, we use the proper cone  $\{0\} \cup \text{posi}(\{B_m : m \in \mathcal{N}^n, n \in \mathbb{N}\})$  to define the order  $\leq_B$  on  $\mathcal{V}(\Sigma_{\mathcal{X}})$ :

$$h_1 \leq_B h_2 \Leftrightarrow h_2 - h_1 \in \{0\} \cup \text{posi}(\{B_m : m \in \mathcal{N}^n, n \in \mathbb{N}\})$$

for all  $h_1$  and  $h_2$  in  $\mathcal{V}(\Sigma_{\mathcal{X}})$ .

Keeping Props. 12 and 14 in mind, the following result is not surprising.

**Proposition 18** Consider any choice function  $C'$  on  $\mathcal{V}(\Sigma_{\mathcal{X}})$ . Then  $C'$  is coherent if and only if for every  $n$  in  $\mathbb{N}$  the choice function  $C'_n$ , given by  $C'_n(O) := C'(O)$  for all  $O$  in  $\mathcal{Q}(\mathcal{V}^n(\Sigma_{\mathcal{X}}))$  is coherent.

**Theorem 19 (Countable Representation)** Consider any choice function  $C$  on  $\bar{\mathcal{L}}(\mathcal{X}^{\mathbb{N}})$ . Then  $C$  is exchangeable if and only if there is a unique representing choice function  $\tilde{C}$  on  $\mathcal{V}(\Sigma_{\mathcal{X}})$  such that, for every  $n$  in  $\mathbb{N}$ , the  $\mathcal{X}^n$ -marginal  $C_n$  of  $C$  is determined by

$$C_n(O) = \{f \in O : M_n(f) \in \tilde{C}(M_n(O))\} \text{ for all } O \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^n)).$$

Furthermore, in that case,  $\tilde{C}$  is given by  $\tilde{C}(O) := \bigcup_{n \in \mathbb{N}} \tilde{C}_n(O \cap \mathcal{V}^n(\Sigma_{\mathcal{X}}))$  for all  $O$  in  $\mathcal{Q}(\mathcal{V}(\Sigma_{\mathcal{X}}))$ , with  $\tilde{C}_n(M_n(O)) := M_n(C_n(O))$  for every  $O$  in  $\mathcal{Q}(\mathcal{L}(\mathcal{X}^n))$ , and where we let  $\tilde{C}_n(\emptyset) := \emptyset$  for notational convenience. Finally,  $C$  is coherent if and only if  $\tilde{C}$  is.

Similarly, consider any set of desirable gambles  $D \subseteq \bar{\mathcal{L}}(\mathcal{X}^{\mathbb{N}})$ . Then  $D$  is exchangeable if and only if there is a unique representing  $\tilde{D} \subseteq \mathcal{V}(\Sigma_{\mathcal{X}})$  such that, for every  $n$  in  $\mathbb{N}$ , the  $\mathcal{X}^n$ -marginal  $D_n$  is given by  $D_n = \bigcup \tilde{M}_n^{-1}(\tilde{D} \cap \mathcal{V}^n(\Sigma_{\mathcal{X}}))$ . Furthermore, in that case,  $\tilde{D}$  is given by  $\tilde{D} = \bigcup_{n \in \mathbb{N}} M_n(D_n)$ . Finally,  $D$  is coherent if and only if  $\tilde{D}$  is.

## 5. Conclusion

We have studied exchangeability and we have found counterparts to de Finetti's finite and countable representation results, in the general setting of choice functions. We have shown that an exchangeability assessment is a particular indifference assessment, where we identified the set of indifferent options. The main idea that made (finite) representation possible is the linear order isomorphism  $\tilde{H}_n^{-1}$  between the quotient space and the set of gambles on count vectors, indicating that (finitely) exchangeable choice functions can be represented by a choice function that essentially represents preferences between urns with  $n$  balls of types  $\mathcal{X}$  with different compositions  $m$ . Alternatively, for the countable case, we have shown that there is a polynomial representation.

Choice functions form a belief structure (Van Camp et al., 2017). Therefore, any infimum of coherent choice functions is a coherent choice function itself. Since any infimum of choice functions compatible with some fixed set of indifferent options  $I$ , is compatible with  $I$  as well (Van Camp et al., 2017), our results indicate that, using choice functions, it is conceptually easy to reason about exchangeable sequences: infima of exchangeable and coherent choice functions will be exchangeable and coherent as well.

A possible future goal is to investigate how exchangeability behaves under updating. In (De Cooman and Quaeghebeur, 2012) it is shown that, for exchangeable sets of desirable gambles, updating can be done directly for the representing set of desirable gambles in the count space. We expect this to be the case for choice functions as well. Other possible extensions are to develop a framework for partial exchangeability, and to model other structural judgements, such as an irrelevance assessment.

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