Claim 11: \((\|v\|, 1)^2 \geq 1\)

**Proof:** Reformulate as a constrained minimization problem, with \(x \in \mathbb{R}^N\):

\[
\min_x \{ \langle x, 1 \rangle \} \quad \text{such that: } \|x\|^2 = 1, \; x \geq 0
\]

\[
\therefore \quad L = \langle x, 1 \rangle - \lambda(\|x\|^2 - 1) - \sum_{n=1}^{N} \mu_n(\langle x, \delta_n \rangle - 0)
\]

\[
\text{such that: } \mu_n \geq 0 \; \forall n
\]

\[
\therefore \quad \nabla_x L = 1 - 2\lambda x - \sum_{n=1}^{N} \mu_n \delta_n
\]

\[
\therefore \quad 2\lambda x' = \sum_{n=1}^{N} (1 - \mu_n) \delta_n
\]

\[
\therefore \quad x' = \frac{\sum_{n=1}^{N} (1 - \mu_n) \delta_n}{\sqrt{\sum_{n=1}^{N} (1 - \mu_n)^2}} \quad [\mu_n \leq 1 \; \forall n]
\]

\[
\therefore \quad \langle x', 1 \rangle = \frac{\sum_{n=1}^{N} (1 - \mu_n)}{\sqrt{\sum_{n=1}^{N} (1 - \mu_n)^2}} \geq \sqrt{\frac{\sum_{n=1}^{N} (1 - \mu_n)^2}{\sum_{n=1}^{N} (1 - \mu_n)^2}} = \sqrt{\frac{\sum_{n=1}^{N} (1 - \mu_n)^2}{\sum_{n=1}^{N} (1 - \mu_n)^2}}
\]

To have unit norm, \(x\) must contain at least one non-zero element. Without loss of generality, we assume \(x_1 > 0\); and hence: \(\mu_1 = 0\)

\[
\therefore \quad \langle x', 1 \rangle \geq \sqrt{1 + \sum_{n=2}^{N} (1 - \mu_n)^2} \geq 1
\]

Q.E.D.

Claim 12: \(\max_i \{ (1 - 2\delta_i)^2 \} \geq \frac{4}{N} \) for \(N \geq 4\)

**Proof:** Reformulate as a constrained minimization problem with \(x \in \mathbb{R}^N\). Without loss of generality, assume that \(\langle x, 1 \rangle \geq 0\) and that its first element \(x_1\) is a minimal element (i.e. \(x_1 \leq x_n \; \forall n\)).

\[
\min_x \{ \langle x, 1 - 2\delta_i \rangle \} \quad \text{such that: } \|x\|^2 = 1, \; x \geq x_1\; 1
\]

\[
\therefore \quad L = \langle x, 1 - 2\delta_i \rangle - \lambda(\|x\|^2 - 1) - \sum_{n=2}^{N} \mu_n(\langle x, \delta_n \rangle - x_1)
\]

\[
\therefore \quad \nabla_x L = [1 - 2\delta_i] - 2\lambda x - \sum_{n=2}^{N} \mu_n \delta_n
\]

\[
\therefore \quad 2\lambda x' = -\delta_i + \sum_{n=2}^{N} (1 - \mu_n) \delta_n
\]

\[
\therefore \quad x' = \frac{-\delta_i + \sum_{n=2}^{N} (1 - \mu_n) \delta_n}{\sqrt{1 + \sum_{n=2}^{N} (1 - \mu_n)^2}}
\]

\[
\therefore \quad \langle x', 1 - 2\delta_i \rangle = \frac{1 + \sum_{n=2}^{N} (1 - \mu_n)}{\sqrt{1 + \sum_{n=2}^{N} (1 - \mu_n)^2}} = \frac{1 + \sum_{n=2}^{N} (1 - \mu_n)}{\sqrt{1 + \sum_{n=2}^{N} (1 - \mu_n)^2}}
\]

Note that if \(x_n > x_1\) then \(\mu_n = 0\), and if \(x_n = x_1\) then \((1 - \mu_n) = -1\). Let \(M\) be the number of unique indices \(n \geq 2\) for which \(x_n = x_1\).

\[
\therefore \quad \langle x', 1 - 2\delta_i \rangle = \frac{1 + \sum_{n=2}^{N} (1 - \mu_n)}{\sqrt{1 + \sum_{n=2}^{N} (1 - \mu_n)^2}} \geq \frac{2}{\sqrt{N}} \quad \therefore \quad \langle x', 1 - 2\delta_i \rangle^2 \geq \frac{4}{N}
\]

Q.E.D.