A. Proof of Theorem 1

In the following, for simplicity, we will drop the subscript $n$ when there is no ambiguity. Therefore, $V_n$ is denoted $V$ and so on.

To prove normality-type results of the maximum likelihood estimator $\hat{\theta}$, typically we first show the $n^{-1/2}$-consistency of $\hat{\theta}$ to $\theta^*$. Then, by using a second-order Taylor expansion or Newton-step, we can prove the desired normality of $\hat{\theta}$. More details can be found in standard textbooks such as Van der Vaart (2000).

Since $m$ is twice differentiable with $\bar{m} \geq 0$, the maximum-likelihood estimation can be written as the solution to the following equation

$$\sum_{i=1}^{n} (Y_i - \mu(X'_i\theta)) X_i = 0. \tag{15}$$

Define $G(\theta) := \sum_{i=1}^{n} (\mu(X'_i\theta) - \mu(X'_i\theta^*)) X_i$, and we have

$$G(\theta^*) = 0 \text{ and } G(\hat{\theta}) = \sum_{i=1}^{n} \epsilon_i X_i, \tag{16}$$

where the noise $\epsilon_i$ is defined in (1). For convenience, define $Z := G(\hat{\theta}) = \sum_{i=1}^{n} \epsilon_i X_i$.

**Step 1: Consistency of $\hat{\theta}$**. We first prove the consistency of $\hat{\theta}$. For any $\theta_1, \theta_2 \in \mathbb{R}^d$, mean value theorem implies that there exists some $\overline{\theta} = v\theta_1 + (1-v)\theta_2$ with $0 < v < 1$, such that

$$G(\theta_1) - G(\theta_2) = \left[\sum_{i=1}^{n} \mu(X'_i\overline{\theta})X_iX'_i\right](\theta_1 - \theta_2) := F(\overline{\theta})(\theta_1 - \theta_2). \tag{17}$$

Since $\bar{\mu} > 0$ and $\lambda_{\min}(V) > 0$, we have

$$(\theta_1 - \theta_2)'(G(\theta_1) - G(\theta_2)) \geq (\theta_1 - \theta_2)'(\kappa V)(\theta_1 - \theta_2) > 0$$

for any $\theta_1 \neq \theta_2$. Hence, $G(\theta)$ is an injection from $\mathbb{R}^d$ to $\mathbb{R}^d$, and so $G^{-1}$ is a well-defined function. Consequently, (15) has a unique solution $\hat{\theta} = G^{-1}(Z)$.

Let us consider an $\eta$-neighborhood of $\theta^*$, $\mathbb{B}_\eta := \{\theta : \|\theta - \theta^*\| \leq \eta\}$, where $\eta > 0$ is a constant that will be specified later. Note that $\mathbb{B}_\eta$ is a convex set, thus $\theta \in \mathbb{B}_\eta$ as long as $\theta_1, \theta_2 \in \mathbb{B}_\eta$. Define $\kappa_\eta := \inf_{\theta \in \mathbb{B}_\eta} \mu(x'\theta) > 0$. From (17), for any $\theta \in \mathbb{B}_\eta$,

$$\|G(\theta)\|^2_{V^{-1}} = \|G(\theta) - G(\theta^*)\|^2_{V^{-1}} \geq (\theta - \theta^*)'(F(\theta)V^{-1}F(\overline{\theta}))(\theta - \theta^*) \geq \kappa_\eta^2 \lambda_{\min}(V) \|\theta - \theta^*\|^2,$$

where the last inequality is due to the fact that $F(\overline{\theta}) \succeq \kappa_\eta V$.

On the other hand, Lemma A of Chen et al. (1999) implies that

$$\left\{ \theta : \|G(\theta)\|_{V^{-1}} \leq \kappa_\eta \sqrt{\lambda_{\min}(V)} \right\} \subset \mathbb{B}_\eta.$$

Now it remains to upper bound $\|Z\|_{V^{-1}} = \|G(\hat{\theta})\|_{V^{-1}}$ to ensure $\hat{\theta} \in \mathbb{B}_\eta$. To do so, we need the following technical lemma, whose proof is deferred to Section C.
**Lemma 7.** Recall $\sigma$ which is the constant in (2). For any $\delta > 0$, define the following event:

\[ \mathcal{E}_G := \left\{ \|Z\|_{V^{-1}} \leq 4\sigma \sqrt{d + \log(1/\delta)} \right\}. \]

Then, $\mathcal{E}_G$ holds with probability at least $1 - \delta$.

Suppose $\mathcal{E}_G$ holds for the rest of the proof. Then, $\eta \geq \frac{4\sigma}{\kappa_\eta} \sqrt{\frac{d + \log(1/\delta)}{\lambda_{\min}(V)}}$ implies $\left\| \hat{\theta} - \theta \right\| \leq \eta$. Since $\kappa = \kappa_1$, we have $\kappa_\eta \geq \kappa$ as long as $\eta \leq 1$. Thus, we have

\[ \left\| \hat{\theta} - \theta \right\| \leq \frac{4\sigma}{\kappa} \sqrt{\frac{d + \log(1/\delta)}{\lambda_{\min}(V)}} \leq 1, \quad (18) \]

when $\lambda_{\min}(V) \geq 16\sigma^2 [d + \log(1/\delta)] / \kappa^2$.

**Step 2: Normality of $\hat{\theta}$.** Now, we are ready to precede to prove the normality result. The following assumes $\mathcal{E}_G$ holds (which is high-probability event, according to Lemma 7).

Define $\Delta := \hat{\theta} - \theta^*$. It follows from (17) that there exists a $v \in [0, 1]$ such that

\[ Z = G(\hat{\theta}) - G(\theta^*) = (H + E)\Delta, \]

where $\hat{\theta} := v\theta^* + (1 - v)\hat{\theta}$, $H := F(\theta^*) = \sum_{i=1}^n \mu(X'_i \theta^*) X_i X'_i$ and $E := F(\hat{\theta}) - F(\theta^*)$. Intuitively, when $\hat{\theta}$ and $\theta^*$ are close, elements in $E$ are small. By the mean value theorem,

\[ E = \sum_{i=1}^n \left( \mu(X'_i \hat{\theta}) - \mu(X'_i \theta^*) \right) X_i X'_i = \sum_{i=1}^n \mu(r_i) X'_i X_i \Delta X_i X'_i \]

for some $r_i \in \mathbb{R}$. Since $\mu \leq M_\mu$ and $v \in [0, 1]$, for any $x \in \mathbb{R}^d \setminus \{0\}$, we have

\[ x'H^{-1/2}EH^{-1/2}x = (1 - v) \sum_{i=1}^n \mu(r_i) X'_i \Delta \left\| x'H^{-1/2}X_i \right\|^2 \]

\[ \leq \sum_{i=1}^n M_\mu \| X_i \| \| \Delta \| \left\| x'H^{-1/2}X_i \right\|^2 \]

\[ \leq M_\mu \| \Delta \| \left( x'H^{-1/2} \left( \sum_{i=1}^n X_i X'_i \right) H^{-1/2}x \right) \]

\[ \leq \frac{M_\mu}{\kappa} \| \Delta \| \| x \|^2, \]

where we have used the assumption that $\| X_i \| \leq 1$ for the second inequality. Therefore,

\[ \left\| H^{-1/2}EH^{-1/2} \right\| \leq \frac{M_\mu}{\kappa} \| \Delta \| \leq \frac{4M_\mu \sigma}{\kappa^2} \sqrt{\frac{d + \log(1/\delta)}{\lambda_{\min}(V)}}. \quad (19) \]

When $\lambda_{\min}(V) \geq 64M_\mu^2 \sigma^2 (d + \log(1/\delta)) / \kappa^4$, we have

\[ \left\| H^{-1/2}EH^{-1/2} \right\| \leq 1/2. \quad (20) \]

Now we are ready to prove the theorem. For any $x \in \mathbb{R}^d$,

\[ x'(\hat{\theta} - \theta^*) = x'(H + E)^{-1}Z = x'H^{-1}Z - x'H^{-1}E(H + E)^{-1}Z. \quad (21) \]

Note that the matrix $(H + E)$ is nonsingular, so its inversion exists.
For the first term, \{\epsilon_i\} are sub-Gaussian random variables with sub-Gaussian parameter \(\sigma\). Define
\[D := [X_1, X_2, \ldots, X_n]' \in \mathbb{R}^{n \times d}\]
to be the design matrix. Hoeffding inequality gives
\[
P\{|x'H^{-1}Z| \geq t\} \leq 2 \exp \left\{ -\frac{t^2}{2\sigma^2 \|x'H^{-1}D\|^2} \right\}.
\] (22)
Since \(H \succeq \kappa V = \kappa D'D\), we have
\[
\|x'H^{-1}D\|^2 = x'H^{-1}D'DH^{-1}x \leq \frac{1}{\kappa^2} x'V^{-1}x = \frac{1}{\kappa^2} \|x\|_{V^{-1}}^2,
\]
so (22) implies
\[
P\{|x'H^{-1}Z| \geq t\} \leq 2 \exp \left\{ -\frac{t^2\kappa^2}{2\sigma^2 \|x\|_{V^{-1}}^2} \right\}.
\]
Let the right-hand side be \(2\delta\) and solve for \(t\), we obtain that with probability at least \(1 - 2\delta\),
\[
|x'H^{-1}Z| \leq \sqrt{2\sigma\kappa} \sqrt{\log(1/\delta)} \|x\|_{V^{-1}}.
\] (23)
For the second term,
\[
|x'H^{-1}E(H + E)^{-1}Z| \leq \|x\|_{H^{-1}} \left\| H^{-1/2}E(H + E)^{-1}Z \right\| \\
\leq \|x\|_{H^{-1}} \left\| H^{-1/2}E(H + E)^{-1}H^{1/2} \right\| \|Z\|_{H^{-1}} \\
\leq \frac{1}{\kappa} \|x\|_{V^{-1}} \left\| H^{-1/2}E(H + E)^{-1}H^{1/2} \right\| \|Z\|_{V^{-1}},
\] (24)
where the last inequality is due to the fact that \(H \succeq \kappa V\). Since \((H + E)^{-1} = H^{-1} - H^{-1}E(H + E)^{-1}\), we have
\[
\left\| H^{-1/2}E(H + E)^{-1}H^{1/2} \right\| = \left\| H^{-1/2}E \left( H^{-1} - H^{-1}E(H + E)^{-1} \right) H^{1/2} \right\| \\
= \left\| H^{-1/2}EH^{-1/2} + H^{-1/2}EH^{-1}E(H + E)^{-1}H^{1/2} \right\| \\
\leq \left\| H^{-1/2}EH^{-1/2} \right\| + \left\| H^{-1/2}EH^{-1/2} \right\| \left\| H^{-1/2}E(H + E)^{-1}H^{1/2} \right\|.
\]
By solving this inequality, we get
\[
\left\| H^{-1/2}E(H + E)^{-1}H^{1/2} \right\| \leq \left\| H^{-1/2}EH^{-1/2} \right\| \left|1 - \frac{\left\| H^{-1/2}EH^{-1/2} \right\|}{\left\| H^{-1/2}EH^{-1/2} \right\|}\right| \leq 2\left\| H^{-1/2}EH^{-1/2} \right\| \leq \frac{8M_s \sigma}{\kappa^2} \sqrt{\frac{d + \log(1/\delta)}{\lambda_{\min}(V)}},
\]
where we have used (20) and (19) in the second and third inequalities, respectively. Combining it with (24) and the bound in \(\mathcal{E}_G\), we have
\[
|x'H^{-1}E(H + E)^{-1}Z| \leq \frac{32M_s \sigma^2}{\kappa^3} \sqrt{\frac{d + \log(1/\delta)}{\lambda_{\min}(V)}} \|x\|_{V^{-1}}.
\] (25)
From (21), (23) and (25), one can see that (5) holds as long as the lower bound (4) for \(\lambda_{\min}(V)\) holds. Finally, an application of a union bound on two small-probability events (given in Lemma 7 and (23), respectively) asserts that (5) holds with probability at least \(1 - 3\delta\).
B. Proof of Proposition 1

In the following, for simplicity, we will drop the subscript $n$ when there is no ambiguity. Therefore, $V_n$ is denoted $V$ and so on.

Let $X$ be a random vector drawn from the distribution $\nu$. Define $Z := \Sigma^{-1/2}X$. Then $Z$ is isotropic, namely, $\mathbb{E}[ZZ'] = I_d$. Define $U = \sum_{i=1}^n Z_i Z_i' = \Sigma^{-1/2}V\Sigma^{-1/2}$. From Lemma 1, we have that, for any $t$, with probability at least $1 - 2\exp(-C_2t^2)$,

$$\lambda_{\min}(U) \geq n - C_1\sigma^2 \sqrt{nd} - \sigma^2 t\sqrt{n}.$$  

where $\sigma$ is the sub-Gaussian parameter of $Z$, which is upper-bounded by $\|\Sigma^{-1/2}\| = \lambda_{\min}^{-1/2}(\Sigma)$ (see, e.g., Vershynin (2012)). We thus can rewrite the above inequality (which holds with probability $1 - \delta$) as

$$\lambda_{\min}(U) \geq n - \lambda_{\min}^{-1}(\Sigma) \left( C_1\sigma^2 \sqrt{nd} + t\sqrt{n} \right).$$

We now bound the minimum eigenvalue of $V$, as follows:

$$\lambda_{\min}(V) = \min_{x \in \mathbb{R}^d} x'Vx = \min_{x \in \mathbb{R}^d} x'\Sigma^{1/2}U\Sigma^{1/2}x \geq \lambda_{\min}(U) \min_{x \in \mathbb{R}^d} x'\Sigma x = \lambda_{\min}(U)\lambda_{\min}(\Sigma) \geq \lambda_{\min}(\Sigma)(n - \lambda_{\min}^{-1}(\Sigma)(C_1\sigma^2 \sqrt{nd} + t\sqrt{n})) = \lambda_{\min}(\Sigma)n - C_1\sqrt{nd} - C_2\sqrt{n}\log(1/\delta).$$

Finally, it can be verified (Lemma 9) that the last expression above is no less than $B$ as long as

$$n \geq \left( \frac{C_1\sqrt{d} + C_2\sqrt{\log(1/\delta)}}{\lambda_{\min}(\Sigma)} \right)^2 + \frac{2B}{\lambda_{\min}(\Sigma)},$$

finishing the proof.

C. Technical Lemmas and Proofs

C.1. Proof of Lemma 7

Noting that

$$\|Z\|_{V^{-1}} = \|V^{-1/2}Z\|_2 = \sup_{\|a\|_2 \leq 1} \langle a, V^{-1/2}Z \rangle,$$

let $\hat{B}$ be a $1/2$-net of the unit ball $\mathbb{B}^d$. Then $|\hat{B}| \leq 6^d$ (Pollard, 1990, Lemma 4.1), and for any $x \in \mathbb{B}^d$, there is a $\hat{x} \in \hat{B}$ such that $\|x - \hat{x}\| \leq 1/2$. Consequently,

$$\langle x, V^{-1/2}Z \rangle = \langle \hat{x}, V^{-1/2}Z \rangle + \langle x - \hat{x}, V^{-1/2}Z \rangle = \langle \hat{x}, V^{-1/2}Z \rangle + \|x - \hat{x}\| \langle \frac{x - \hat{x}}{\|x - \hat{x}\|}, V^{-1/2}Z \rangle \leq \langle \hat{x}, V^{-1/2}Z \rangle + \frac{1}{2} \sup_{z \in \mathbb{B}^d} \langle z, V^{-1/2}Z \rangle.$$  

Taking supremum on both sides, we get

$$\sup_{x \in \mathbb{B}^d} \langle x, V^{-1/2}Z \rangle \leq 2 \max_{\hat{x} \in \hat{B}} \langle \hat{x}, V^{-1/2}Z \rangle.$$
Then a union bound argument implies
\[
P \left \{ \| Z \|_{V^{-1}} > t \right \} \leq \sum_{x \in \mathbb{B}} P \left \{ \max(\hat{x}, V^{-1/2}Z) > t/2 \right \} \leq \sum_{x \in \mathbb{B}} \exp \left \{ -\frac{t^2}{8\sigma^2 \| x' V^{-1/2} X \|^2} \right \} \leq \exp \left \{ -t^2/(8\sigma^2) + d \log 6 \right \},
\]

where we have used Hoeffding’s inequality for the third inequality and \(|\mathbb{B}| \leq 6^d\) for the last inequality. A choice of \(t = 4\sigma \sqrt{d + \log(1/\delta)}\) completes the proof.

### C.2. Proof of Lemma 2

By Abbasi-Yadkori et al. (2011, Lemma 11), we have
\[
\sum_{t=m+1}^{m+n} \| X_t \|_{V_t^{-1}}^2 \leq 2 \log \frac{\det V_{m+n+1}}{\det V_{m+1}} \leq 2d \log \left( \frac{\text{tr}(V_{m+1}) + n}{d} \right) - 2 \log \det V_{m+1}.
\]

Note that \(\text{tr}(V_{m+1}) = \sum_{i=1}^{m} \text{tr}(X_i X_i') = \sum_{i=1}^{m} \| X_i \|^2 \leq m\) and that \(\det V_{m+1} = \prod_{i=1}^{d} \lambda_i \geq \lambda_{\min}^d(V_{m+1}) \geq 1\), where \(\{\lambda_i\}\) are the eigenvalues of \(V_{m+1}\). Applying Cauchy-Schwartz inequality yields
\[
\sum_{t=m+1}^{m+n} \| X_t \|_{V_t^{-1}}^2 \leq \sqrt{n \sum_{t=m+1}^{m+n} \| X_t \|_{V_t^{-1}}^2} \leq \sqrt{2nd \log \left( \frac{n + m}{d} \right)}.
\]

### C.3. Proof of Lemma 3

Define \(G_t(\theta) = \sum_{i=1}^{t-1} (\mu(X_i') - \mu(X_i')) X_i\) and \(Z_t = \sum_{i=1}^{t-1} \epsilon_i X_i\). Following the same argument as in the proof of Theorem 1, we have \(G_t(\hat{\theta}_t) = Z_t\) and
\[
\| G_t(\theta) \|_{V_t^{-1}}^2 \geq \kappa^2 \| \theta - \theta^* \|_{V_t}^2
\]

for any \(\theta \in \{ \theta : \| \theta - \theta^* \| \leq 1 \}\). Combining (26) with the following lemma and the equality \(Z_t = G_t(\hat{\theta}_t)\) completes the proof.

**Lemma 8.** Suppose there is an integer \(m\) such that \(\lambda_{\min}(V_m) \geq 1\), then for any \(\delta \in (0, 1)\), with probability at least \(1 - \delta\), for all \(t > m\),
\[
\| Z_t \|_{V_t^{-1}}^2 \leq 4\sigma^2 \left( \frac{d}{2} \log(1 + 2t/d) + \log(1/\delta) \right).
\]

**Proof.** For convenience, fix \(t\) such that \(t > m\), and denote \(V_t\) and \(Z_t\) by \(V\) and \(Z\), respectively. Furthermore, define \(\bar{V} := V + \lambda I\) and let \(1\) be the vector of all 1s. It is easy to observe that
\[
\| Z \|_{V^{-1}}^2 = \| Z \|_{\bar{V}^{-1}}^2 + Z'(V^{-1} - \bar{V}^{-1})Z
\]

We start with bounding the second term. The Sherman-Morrison formula gives
\[
\bar{V}^{-1} = V^{-1} - \frac{\lambda V^{-2}}{1 + \lambda V^{-1}}.
\]

Since \(V^{-1} 1 1 \geq 0\), the above implies that
\[
0 \leq Z'(V^{-1} - \bar{V}^{-1})Z \leq \lambda Z' V^{-2} Z \leq \lambda \| V^{-1} \| \| Z \|_{V^{-1}}^2 = \frac{\lambda}{\lambda_{\min}(V)} \| Z \|_{V^{-1}}^2.
\]
Since $\lambda_{\min}(V) \geq \lambda_{\min}(V_m) \geq 1$, we now have

$$0 \leq Z'(V^{-1} - \tilde{V}^{-1})Z \leq \lambda \|Z\|^2_{V^{-1}}.$$ 

The above inequality together with (27) implies that

$$\|Z\|^2_{V^{-1}} \leq (1 - \lambda)^{-1} \|Z\|^2_{\tilde{V}^{-1}}.$$ 

The proof can be finished by applying Theorem 1 and Lemma 10 from Abbasi-Yadkori et al. (2011) to bound $\|Z\|^2_{\tilde{V}^{-1}}$, using $\lambda = 1/2$.

C.4. Proof of Lemma 6

We will prove the first part of the lemma by induction. It is easy to check the lemma holds for $s = 1$. Suppose we have $a_t^* \in A_s$ and we want to prove $a_t^* \in A_{s+1}$. Since the algorithm proceeds to stage $s + 1$, we know from step 2b that

$$|m_{t,a}^{(s)} - x_{t,a}^*| \leq w_{t,a}^{(s)} \leq 2^{-s}$$

for all $a \in A_s$. Specially, it holds for $a = a_t^*$ because $a_t^* \in A_s$ by our induction step. Then the optimality of $a_t^*$ implies

$$m_{t,a_t^*}^{(s)} \geq x_{t,a_t^*}^* - 2^{-s} \geq x_{t,a_t^*}^* - 2^{-s} \geq m_{t,a_t^*}^{(s)} - 2 \cdot 2^{-s}$$

for all $a \in A_s$. Thus we have $a_t^* \in A_{s+1}$ according to step 2d.

Suppose $a_t$ is selected at stage $s_t$ in step 2b. If $s_t = 1$, obviously the lemma holds because $0 \leq \mu(x) \leq 1$ for all $x$. If $s_t > 1$, since we have proved $a_t^* \in A_{s_t}$, again step 2b at stage $s_t - 1$ implies

$$|m_{t,a}^{(s_t-1)} - x_{t,a}^*| \leq 2^{-s_t+1}$$

for $a = a_t$ and $a = a_t^*$. Step 2d at stage $s_t - 1$ implies

$$m_{t,a_t^*}^{(s_t-1)} - m_{t,a_t}^{(s_t-1)} \leq 2 \cdot 2^{-s_t+1}.$$ 

Combining above two inequalities, we get

$$x_{t,a_t^*}^* \geq m_{t,a_t^*}^{(s_t-1)} - 2^{-s_t+1} \geq m_{t,a_t}^{(s_t-1)} - 3 \cdot 2^{-s_t+1} \geq x_{t,a_t^*}^* - 4 \cdot 2^{-s_t+1}.$$ 

When $a_t$ is selected in step 2c, since $m_{t,a_t}^{(s_t)} \geq m_{t,a_t^*}^{(s_t)}$, we have

$$x_{t,a_t^*}^* \geq m_{t,a_t^*}^{(s_t)} - 1/\sqrt{T} \geq m_{t,a_t}^{(s_t)} - 1/\sqrt{T} \geq x_{t,a_t^*}^* - 2/\sqrt{T}.$$ 

Using the fact that $\mu(x_1) - \mu(x_2) \leq L_\mu(x_1 - x_2)$ for $x_1 \geq x_2$, we will get the desired result.

C.5. Proof of Lemma 9

**Lemma 9.** Let $a$ and $b$ be two positive constants. If $m \geq a^2 + 2b$, then $m - a\sqrt{m} - b \geq 0$.

**Proof.** The function $t \mapsto t^2 - at - b$ is monotonically increasing for $t \geq a/2$. Since $m \geq a^2 + 2b$, we have $\sqrt{m} \geq a/2$, so

$$m - a\sqrt{m} - b \geq a^2 + 2b - a\sqrt{a^2 + 2b} - b \geq a^2 + 2b - a\sqrt{a^2 + 2b + b/a^2} = a^2 + b - a\sqrt{(a + b/a)^2} = a^2 + b - a(a + b/a) = 0.$$