A. Proofs

A.1. Proof of Lamma 1

Proof. In the following, we abbreviate j in Lemma 1 for the simplicity of the notation unless there is no confusion, and prove the lemma in slightly general case of $\mathrm{V}[\boldsymbol{y}] = \Sigma$. To prove the lemma, we first state the polyhedral lemma in Lee et al. (2016) as follows:

Lemma 7 (Polyhedral Lemma; (Lee et al., 2016))**.** *Suppose $\boldsymbol{y} \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma)$. Let $\boldsymbol{c} = \Sigma\boldsymbol{\eta}(\boldsymbol{\eta}^\top \Sigma \boldsymbol{\eta})^{-1}$ for any $\boldsymbol{\eta} \in \mathbb{R}^n$, and let $\boldsymbol{z} = (I_n - \boldsymbol{c}\boldsymbol{\eta}^\top)\boldsymbol{y}$. Then we have*

$$\begin{aligned}\mathrm{Pol}(S) &= \{\boldsymbol{y} \in \mathbb{R}^n \mid A\boldsymbol{y} \le \boldsymbol{b}\} \\ &= \left\{ \boldsymbol{y} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} L(S, \boldsymbol{z}) \le \boldsymbol{\eta}^\top \boldsymbol{y} \le U(S, \boldsymbol{z}), \\ N(S, \boldsymbol{z}) \ge 0 \end{array} \right\},\end{aligned}$$

where

$$L(S, \boldsymbol{z}) = \max_{j:(A\boldsymbol{c})_j < 0} \frac{b_j - (A\boldsymbol{z})_j}{(A\boldsymbol{c})_j}, \tag{13a}$$

$$U(S, \boldsymbol{z}) = \min_{j:(A\boldsymbol{c})_j > 0} \frac{b_j - (A\boldsymbol{z})_j}{(A\boldsymbol{c})_j} \tag{13b}$$

and $N(S, \boldsymbol{z}) = \max_{j:(A\boldsymbol{c})_j = 0} b_j - (A\boldsymbol{z})_j$. In addition, $(L(S, \boldsymbol{z}), U(S, \boldsymbol{z}), N(S, \boldsymbol{z}))$ is independent of $\boldsymbol{\eta}^\top \boldsymbol{y}$.

The polyhedral lemma allows us to construct a pivotal quantity as a truncated normal distribution, that is, for any $\boldsymbol{z}$, we have

$$[F_{0, \boldsymbol{\eta}^\top \Sigma \boldsymbol{\eta}}^{[L(S,\boldsymbol{z}), U(S,\boldsymbol{z})]}(\boldsymbol{\eta}^\top \boldsymbol{y}) | \boldsymbol{y} \in \mathrm{Pol}(S)] \sim \mathrm{Unif}(0, 1), \tag{14}$$

where $\mathrm{Unif}(0, 1)$ denotes the standard (continuous) uniform distribution. In fact, by letting $\boldsymbol{z}_0$ be an arbitrary realization of $\boldsymbol{z}$, one can see that

$$\begin{aligned}&[\boldsymbol{\eta}^\top \boldsymbol{y} \mid \boldsymbol{y} \in \mathrm{Pol}(S), \boldsymbol{z} = \boldsymbol{z}_0] \\ &\overset{\mathrm{d}}{=} [\boldsymbol{\eta}^\top \boldsymbol{y} \mid L(S, \boldsymbol{z}_0) \le \boldsymbol{\eta}^\top \boldsymbol{y} \le U(S, \boldsymbol{z}_0)] \\ &\sim \mathrm{TN}(\boldsymbol{\eta}^\top \boldsymbol{\mu}, \boldsymbol{\eta}^\top \Sigma \boldsymbol{\eta}, L(S, \boldsymbol{z}_0), U(S, \boldsymbol{z}_0)),\end{aligned}$$

where $\overset{\mathrm{d}}{=}$ denotes the equality of random variables in distribution. Therefore, probability integral transformation implies

$$[F_{\boldsymbol{\eta}^\top \boldsymbol{\mu}, \boldsymbol{\eta}^\top \Sigma \boldsymbol{\eta}}^{[L(S,\boldsymbol{z}), U(S,\boldsymbol{z})]}(\boldsymbol{\eta}^\top \boldsymbol{y}) \mid \boldsymbol{y} \in \mathrm{Pol}(S), \boldsymbol{z} = \boldsymbol{z}_0]$$

has a uniform distribution $\mathrm{Unif}(0, 1)$ for any $\boldsymbol{z}_0$. By integrating out $\boldsymbol{z}_0$, the pivotal quantity Eq.(14) holds. In addition, an lower α-percentile of the distribution can be obtained as

$$q_\alpha = (F_{\boldsymbol{\eta}^\top \boldsymbol{\mu}, \boldsymbol{\eta}^\top \Sigma \boldsymbol{\eta}}^{[L(S,\boldsymbol{z}), U(S,\boldsymbol{z})]})^{-1}(\alpha).$$

In the following, let us denote $(S, \boldsymbol{z})$ by S for shorthand. The remaining is to show that truncation points in Eqs.(13) are equivalent to

$$L(S) = \boldsymbol{\eta}^\top \boldsymbol{y} + \theta_L \boldsymbol{\eta}^\top \Sigma \boldsymbol{\eta} \quad \text{where} \quad \theta_L = \min_{\theta \in \mathbb{R}} \theta \quad \text{s.t.} \quad \boldsymbol{y} + \theta \Sigma \eta \in \mathrm{Pol}(S) \tag{15a}$$

and

$$U(S) = \boldsymbol{\eta}^\top \boldsymbol{y} + \theta_U \boldsymbol{\eta}^\top \Sigma \boldsymbol{\eta} \quad \text{where} \quad \theta_U = \max_{\theta \in \mathbb{R}} \theta \quad \text{s.t.} \quad \boldsymbol{y} + \theta \Sigma \eta \in \mathrm{Pol}(S), \tag{15b}$$

respectively. Simple calculation shows that, for any $\theta \in \mathbb{R}$, we have

$$\begin{aligned}&\boldsymbol{y} + \theta \Sigma \boldsymbol{\eta} \in \mathrm{Pol}(S) \\ &\Leftrightarrow A(\boldsymbol{y} + \theta \Sigma \eta) \le \boldsymbol{b} \\ &\Leftrightarrow \theta \cdot A \Sigma \eta \le \boldsymbol{b} - A\boldsymbol{y}. \\ &\Leftrightarrow \begin{cases} \theta \le (\boldsymbol{b} - A\boldsymbol{y})_j / (A\Sigma\boldsymbol{\eta})_j, & (A\Sigma\boldsymbol{\eta})_j > 0 \\ \theta \ge (\boldsymbol{b} - A\boldsymbol{y})_j / (A\Sigma\boldsymbol{\eta})_j, & (A\Sigma\boldsymbol{\eta})_j < 0 \\ 0 \le (\boldsymbol{b} - A\boldsymbol{y})_j, & (A\Sigma\boldsymbol{\eta})_j = 0 \end{cases}.\end{aligned}$$

On the other hand, by the definition of $\boldsymbol{c}$ and $\boldsymbol{z}$ in Lemma 7, it is easy to see that

$$L(S) = \boldsymbol{\eta}^\top \boldsymbol{y} + \boldsymbol{\eta}^\top \Sigma \boldsymbol{\eta} \max_{j:(A\Sigma\boldsymbol{\eta})_j < 0} \frac{(\boldsymbol{b} - A\boldsymbol{y})_j}{(A\Sigma\boldsymbol{\eta})_j}$$

Therefore, for each j such that $(A\Sigma\boldsymbol{\eta})_j < 0$, we have

$$\max_{j:(A\Sigma\boldsymbol{\eta})_j < 0} \frac{(\boldsymbol{b} - A\boldsymbol{y})_j}{(A\Sigma\boldsymbol{\eta})_j} \le \theta$$

and thus the minimum possible feasible θ would be

$$\begin{aligned}\theta_L &= \min\{\theta \in \mathbb{R} \mid \boldsymbol{y} + \theta \Sigma \eta \in \mathrm{Pol}(S)\} \\ &= \max_{j:(A\Sigma\boldsymbol{\eta})_j < 0} \frac{(\boldsymbol{b} - A\boldsymbol{y})_j}{(A\Sigma\boldsymbol{\eta})_j}.\end{aligned}$$

Similarly, we see that the equivalency of $U(S)$.

To complete the proof, let us consider a Gaussian random variable $\boldsymbol{y}$ with mean $X\boldsymbol{\beta}^*$ and covariance matrix $\sigma^2 I_n$ with some constant σ^2. We can choose $\boldsymbol{\eta} = (X_S^+)^\top \boldsymbol{e}_j$ for testing the null hypothesis $\mathrm{H}_{0,j} : \beta^*_{S,j} = 0$ for each $j \in S$, since $\boldsymbol{\eta}^\top \boldsymbol{y}$ reduces to the j-th element of an ordinary least square estimator for the selected model, and in this case, $\boldsymbol{\eta}^\top \Sigma \boldsymbol{\eta}$ reduces to

$$\sigma_S^2 = \sigma^2 \|\boldsymbol{\eta}\|^2 = \sigma^2 (X_S^\top X_S)^{-1}_{jj}.$$

Then the critical values are computed as

$$\ell^S_{\alpha/2} = q_{\alpha/2} = (F_{0, \sigma_S^2}^{[L(S), U(S)]})^{-1}(\alpha/2)$$

and

$$u^S_{\alpha/2} = q_{1-\alpha/2} = (F^{[L(S),U(S)]}_{0,\sigma^2_S})^{-1}(1-\alpha/2),$$

respectively. From the above argument, there are no matter to compute the truncation points in Eqs.(15) based on the observations. In this case, Eqs.(15) can be written as

$$L(S) = \boldsymbol{\eta}^\top \boldsymbol{y} + \theta_L \sigma^2 (X_S^\top X_S)^{-1}_{jj}$$
$$\text{where} \quad \theta_L = \min_{\theta \in \mathbb{R}} \theta \quad \text{s.t.} \quad \boldsymbol{y} + \theta\sigma^2 (X_S^+)^\top \boldsymbol{e}_j \in \mathrm{Pol}(S)$$

and

$$U(S) = \boldsymbol{\eta}^\top \boldsymbol{y} + \theta_U \sigma^2 (X_S^\top X_S)^{-1}_{jj}$$
$$\text{where} \quad \theta_U = \max_{\theta \in \mathbb{R}} \theta \quad \text{s.t.} \quad \boldsymbol{y} + \theta\sigma^2 (X_S^+)^\top \boldsymbol{e}_j \in \mathrm{Pol}(S),$$

respectively, but we can ignore the scaling factor σ^2 because

$$\min\{\theta \in \mathbb{R}^n \mid \boldsymbol{y} + \theta (X_S^+)^\top \top \boldsymbol{e}_j \in \mathrm{Pol}(S)\}$$
$$= \min\{\sigma^2\theta \in \mathbb{R}^n \mid \boldsymbol{y} + \theta\sigma^2 (X_S^+)^\top \boldsymbol{e}_j \in \mathrm{Pol}(S)\}$$

and

$$\max\{\theta \in \mathbb{R}^n \mid \boldsymbol{y} + \theta (X_S^+)^\top \boldsymbol{e}_j \in \mathrm{Pol}(S)\}$$
$$= \max\{\sigma^2\theta \in \mathbb{R}^n \mid \boldsymbol{y} + \theta\sigma^2 (X_S^+)^\top \boldsymbol{e}_j \in \mathrm{Pol}(S)\}.$$

■

A.2. Proof of Lamma 3

Proof. Since $x_{ij} \in [0,1]$, for any pair $(j, \tilde{j})$ such that $\tilde{j} \in Des(j)$, $x_j \geq x_{\tilde{j}}$ holds. Then,

$$|\boldsymbol{x}_{\cdot\tilde{j}}^\top \boldsymbol{y}| = |\sum_{i:y_i>0} x_{i\tilde{j}} y_i + \sum_{i:y_i<0} x_{i\tilde{j}} y_i|$$
$$\leq \max\left\{\sum_{i:y_i>0} x_{i\tilde{j}} y_i, -\sum_{i:y_i<0} x_{i\tilde{j}} y_i\right\}$$
$$\leq \max\left\{\sum_{i:y_i>0} x_{ij} y_i, -\sum_{i:y_i<0} x_{ij} y_i\right\}.$$

■

A.3. Proof of Lemma 4

Proof. In MS, from Eq.(9), the constraint $\boldsymbol{y} + \theta\boldsymbol{\eta} \in \mathrm{Pol}(S)$ is written as

$$(-s_j \boldsymbol{x}_{\cdot j} - \boldsymbol{x}_{\cdot j'})^\top (\boldsymbol{y} + \theta\boldsymbol{\eta}) \leq 0$$
$$\Leftrightarrow \frac{-(s_j \boldsymbol{x}_{\cdot j} + \boldsymbol{x}_{\cdot j'})^\top \boldsymbol{y}}{(s_j \boldsymbol{x}_{\cdot j} + \boldsymbol{x}_{\cdot j'})^\top \boldsymbol{\eta}} \leq \theta \text{ if } (s_j \boldsymbol{x}_{\cdot j} + \boldsymbol{x}_{\cdot j'})^\top \boldsymbol{\eta} > 0, \tag{16a}$$
$$\text{and } \frac{-(s_j \boldsymbol{x}_{\cdot j} + \boldsymbol{x}_{\cdot j'})^\top \boldsymbol{y}}{(s_j \boldsymbol{x}_{\cdot j} + \boldsymbol{x}_{\cdot j'})^\top \boldsymbol{\eta}} \geq \theta \text{ if } (s_j \boldsymbol{x}_{\cdot j} + \boldsymbol{x}_{\cdot j'})^\top \boldsymbol{\eta} < 0. \tag{16b}$$
$$(-s_j \boldsymbol{x}_{\cdot j} + \boldsymbol{x}_{\cdot j'})^\top (\boldsymbol{y} + \theta\boldsymbol{\eta}) \leq 0$$
$$\Leftrightarrow \frac{-(s_j \boldsymbol{x}_{\cdot j} - \boldsymbol{x}_{\cdot j'})^\top \boldsymbol{y}}{(s_j \boldsymbol{x}_{\cdot j} - \boldsymbol{x}_{\cdot j'})^\top \boldsymbol{\eta}} \leq \theta \text{ if } (s_j \boldsymbol{x}_{\cdot j} - \boldsymbol{x}_{\cdot j'})^\top \boldsymbol{\eta} > 0 \tag{16c}$$
$$\text{and } \frac{-(s_j \boldsymbol{x}_{\cdot j} - \boldsymbol{x}_{\cdot j'})^\top \boldsymbol{y}}{(s_j \boldsymbol{x}_{\cdot j} - \boldsymbol{x}_{\cdot j'})^\top \boldsymbol{\eta}} \geq \theta \text{ if } (s_j \boldsymbol{x}_{\cdot j} - \boldsymbol{x}_{\cdot j'})^\top \boldsymbol{\eta} < 0. \tag{16d}$$
$$-s_j \boldsymbol{x}_{\cdot j}^\top (\boldsymbol{y} + \theta\boldsymbol{\eta}) \leq 0$$
$$\Leftrightarrow \frac{-s_j \boldsymbol{x}_{\cdot j}^\top \boldsymbol{y}}{s_j \boldsymbol{x}_{\cdot j}^\top \boldsymbol{\eta}} \leq \theta \text{ if } s_j \boldsymbol{x}_{\cdot j}^\top \boldsymbol{\eta} > 0 \tag{16e}$$
$$\text{and } \frac{-s_j \boldsymbol{x}_{\cdot j}^\top \boldsymbol{y}}{s_j \boldsymbol{x}_{\cdot j}^\top \boldsymbol{\eta}} \geq \theta \text{ if } s_j \boldsymbol{x}_{\cdot j}^\top \boldsymbol{\eta} < 0 \tag{16f}$$

for all $(j, j') \in S \times \bar{S}$. The conditions in Eqs.(16a), (16c), and (16e) suggests that $-\theta_L$ must be at least smaller than $\theta_L^{(a)}$ in Eq.(11a), $\theta_L^{(b)}$ in Eq.(11c), and $\theta_L^{(c)}$ in the second last inequality in Eq.(11), respectively. Therefore, we have

$$\theta_L = -\min\{\theta_L^{(a)}, \theta_L^{(b)}, \theta_L^{(c)}\}.$$

Similarly, the conditions in Eqs.(16b), (16d), and (16f) imply that

$$\theta_L = -\max\{\theta_U^{(a)}, \theta_U^{(b)}, \theta_U^{(c)}\}.$$

■

A.4. Proof of Lemma 5

Proof. First, note that $0 \leq x_{i\tilde{j}'} \leq x_{ij'} \leq 1$ for any $(j, j', \tilde{j}') \in S \times \bar{S} \times Des_j(j')$. We first prove Eq.(12a).

$$(s_j \boldsymbol{x}_{\cdot j} + \boldsymbol{x}_{\cdot \tilde{j}'})^\top \boldsymbol{y} = s_j \boldsymbol{x}_{\cdot j}^\top \boldsymbol{y} + \sum_{i:y_i>0} x_{i\tilde{j}'} y_i + \sum_{i:y_i<0} x_{i\tilde{j}'} y_i$$
$$\geq s_j \boldsymbol{x}_{\cdot j}^\top \boldsymbol{y} + \sum_{i:y_i<0} x_{i\tilde{j}'} y_i.$$
$$\geq s_j \boldsymbol{x}_{\cdot j}^\top \boldsymbol{y} + \sum_{i:y_i<0} x_{ij'} y_i = L_E^{(a)},$$

which proves the first line. Next, we prove Eq.(12b).

$$
\begin{aligned}
(s_j \boldsymbol{x}_{\cdot j} + \boldsymbol{x}_{\cdot \tilde{j}'})^\top \boldsymbol{y} &= s_j \boldsymbol{x}_{\cdot j}^\top \boldsymbol{y} + \sum_{i: y_i > 0} x_{i\tilde{j}'} y_i + \sum_{i: y_i < 0} x_{i\tilde{j}'} y_i \\
&\leq s_j \boldsymbol{x}_{\cdot j}^\top \boldsymbol{y} + \sum_{i: y_i > 0} x_{i\tilde{j}'} y_i. \\
&\leq s_j \boldsymbol{x}_{\cdot j}^\top \boldsymbol{y} + \sum_{i: y_i > 0} x_{ij'} y_i = U_E^{(a)},
\end{aligned}
$$

which proves the second line. Eqs. (12c) to (12h) are proved similarly. ■

A.5. Proof of Theorem 6

Proof. First, we prove (i). For any $(j, j', \tilde{j}') \in S \times \bar{S} \times Des_j(j')$, by using Lemma 5 directly, a lower and an upper bound of $s_j \boldsymbol{x}_{\cdot j}^\top \boldsymbol{y} + \boldsymbol{x}_{\cdot \tilde{j}'}^\top \boldsymbol{y}$ can be obtained as

$$
L_E^{(a)} \leq s_j \boldsymbol{x}_{\cdot j}^\top \boldsymbol{y} + \boldsymbol{x}_{\cdot \tilde{j}'}^\top \boldsymbol{y} \leq U_E^{(a)} \tag{17}
$$

Similarly, a lower and an upper bound of $s_j \boldsymbol{x}_{\cdot j}^\top \boldsymbol{\eta} + \boldsymbol{x}_{\cdot \tilde{j}'}^\top \boldsymbol{\eta}$ can be also obtained as

$$
L_D^{(a)} \leq s_j \boldsymbol{x}_{\cdot j}^\top \boldsymbol{\eta} + \boldsymbol{x}_{\cdot \tilde{j}'}^\top \boldsymbol{\eta} \leq U_D^{(a)} \tag{18}
$$

From Eq.(18), we have

$$
U_D^{(a)} < 0 \ \Rightarrow \ (s_j \boldsymbol{x}_{\cdot j} + \boldsymbol{x}_{\cdot \tilde{j}'})^\top \boldsymbol{\eta} < 0
$$

for all $(j, \tilde{j}') \in S \times Des_j(j')$. It means that the $(j, \tilde{j}')$-th constraint does not affect the solution of the optimization problem in Eq.(11a). Now, we consider the case of $U_D^{(a)} > 0$. If $L_D^{(a)} > 0$, the value

$$
\frac{(s_j \boldsymbol{x}_{\cdot j} + \boldsymbol{x}_{\cdot j'})^\top \boldsymbol{y}}{(s_j \boldsymbol{x}_{\cdot j} + \boldsymbol{x}_{\cdot j'})^\top \boldsymbol{\eta}}
$$

can be bounded below by $L_E^{(a)}/U_D^{(a)}$ when $L_E^{(a)} > 0$, and $L_E^{(a)}/L_D^{(a)}$ when $L_E^{(a)} < 0$, while the value can take any small values if $L_D^{(a)} < 0$. As a result, for the current optimal solution $\hat{\theta}_L^{(a)}$, (j, j')-th constraint does not affect the solution of the optimization problem Eq.(11a), if

$$
L_D^{(a)} > 0, \ L_E^{(a)} > 0 \ \text{ and } \ \frac{L_E^{(a)}}{U_D^{(a)}} > \hat{\theta}_L^{(a)},
$$

or

$$
L_D^{(a)} > 0, \ L_E^{(a)} < 0 \ \text{ and } \ \frac{L_E^{(a)}}{L_D^{(a)}} > \hat{\theta}_L^{(a)},
$$

because $L_D^{(a)} > 0$ implies $U_D^{(a)} > 0$. Similarly, we can prove (ii) – (iv) by the same argument. ■

B. Selectivxe inference for OMP

Lemma 8. *Let $\boldsymbol{\eta} := (X^+)^\top \boldsymbol{e}_j$. The solutions of the optimization problems in (7) are respectively written as*

$$
\begin{aligned}
\theta_L &= -\min\{\theta_L^{(a)}, \theta_L^{(b)}, \theta_L^{(c)}\}, \\
\theta_U &= -\max\{\theta_U^{(a)}, \theta_U^{(b)}, \theta_U^{(c)}\},
\end{aligned}
$$

where

$$
\theta_L^{(a)} := \min_{\substack{h \in [k],\ j' \in \bar{S}_h, \\ (s_{(h)} \boldsymbol{x}_{\cdot(h)} + \boldsymbol{x}_{\cdot j'})^\top P_{S_h} \boldsymbol{\eta} > 0}} \frac{(s_{(h)} \boldsymbol{x}_{\cdot(h)} + \boldsymbol{x}_{\cdot j'})^\top P_{S_h} \boldsymbol{y}}{(s_{(h)} \boldsymbol{x}_{\cdot(h)} + \boldsymbol{x}_{\cdot j'})^\top P_{S_h} \boldsymbol{\eta}}, \tag{19a}
$$

$$
\theta_L^{(b)} := \min_{\substack{h \in [k],\ j' \in \bar{S}_h, \\ (s_{(h)} \boldsymbol{x}_{\cdot(h)} - \boldsymbol{x}_{\cdot j'})^\top P_{S_h} \boldsymbol{\eta} > 0}} \frac{(s_{(h)} \boldsymbol{x}_{\cdot(h)} - \boldsymbol{x}_{\cdot j'})^\top P_{S_h} \boldsymbol{y}}{(s_{(h)} \boldsymbol{x}_{\cdot(h)} - \boldsymbol{x}_{\cdot j'})^\top P_{S_h} \boldsymbol{\eta}}, \tag{19b}
$$

$$
\theta_L^{(c)} := \min_{\substack{h \in [k], \\ s_{(h)} \boldsymbol{x}_{\cdot(h)}^\top P_{S_h} \boldsymbol{\eta} > 0}} \frac{s_{(h)} \boldsymbol{x}_{\cdot(h)}^\top P_{S_h} \boldsymbol{y}}{s_{(h)} \boldsymbol{x}_{\cdot(h)}^\top P_{S_h} \boldsymbol{\eta}}, \tag{19c}
$$

$$
\theta_U^{(a)} := \max_{\substack{h \in [k],\ j' \in \bar{S}_h, \\ (s_{(h)} \boldsymbol{x}_{\cdot(h)} + \boldsymbol{x}_{\cdot j'})^\top P_{S_h} \boldsymbol{\eta} < 0}} \frac{(s_{(h)} \boldsymbol{x}_{\cdot(h)} + \boldsymbol{x}_{\cdot j'})^\top P_{S_h} \boldsymbol{y}}{(s_{(h)} \boldsymbol{x}_{\cdot(h)} + \boldsymbol{x}_{\cdot j'})^\top P_{S_h} \boldsymbol{\eta}}, \tag{19d}
$$

$$
\theta_U^{(b)} := \max_{\substack{h \in [k],\ j' \in \bar{S}_h, \\ (s_{(h)} \boldsymbol{x}_{\cdot(h)} - \boldsymbol{x}_{\cdot j'})^\top P_{S_h} \boldsymbol{\eta} < 0}} \frac{(s_{(h)} \boldsymbol{x}_{\cdot(h)} - \boldsymbol{x}_{\cdot j'})^\top P_{S_h} \boldsymbol{y}}{(s_{(h)} \boldsymbol{x}_{\cdot(h)} - \boldsymbol{x}_{\cdot j'})^\top P_{S_h} \boldsymbol{\eta}}, \tag{19e}
$$

$$
\theta_U^{(c)} := \max_{\substack{h \in [k], \\ s_{(h)} \boldsymbol{x}_{\cdot(h)}^\top P_{S_h} \boldsymbol{\eta} < 0}} \frac{s_{(h)} \boldsymbol{x}_{\cdot(h)}^\top P_{S_h} \boldsymbol{y}}{s_{(h)} \boldsymbol{x}_{\cdot(h)}^\top P_{S_h} \boldsymbol{\eta}}. \tag{19f}
$$

Lemma 9. *For any $h \in [k]$ and $(j', \tilde{j}') \in \bar{S}_h \times Des_{(h)}(j')$,*

$$
\begin{aligned}
L_E^{(a)} &:= s_{(h)} \boldsymbol{x}_{\cdot(h)}^\top P_{S_h} \boldsymbol{y} + \sum_{i:[P_{S_h}\boldsymbol{y}]_i < 0} x_{ij'} [P_{S_h}\boldsymbol{y}]_i \\
&\leq (s_{(h)} \boldsymbol{x}_{\cdot(h)} + \boldsymbol{x}_{\cdot \tilde{j}'})^\top P_{S_h} \boldsymbol{y}, \\
U_E^{(a)} &:= s_{(h)} \boldsymbol{x}_{\cdot(h)}^\top P_{S_h} \boldsymbol{y} + \sum_{i:[P_{S_h}\boldsymbol{y}]_i > 0} x_{ij'} [P_{S_h}\boldsymbol{y}]_i \\
&\geq (s_{(h)} \boldsymbol{x}_{\cdot(h)} + \boldsymbol{x}_{\cdot \tilde{j}'})^\top P_{S_h} \boldsymbol{y}, \\
L_D^{(a)} &:= s_{(h)} \boldsymbol{x}_{\cdot(h)}^\top \boldsymbol{\eta} + \sum_{i:[P_{S_h}\boldsymbol{\eta}]_i < 0} x_{ij'} [P_{S_h}\boldsymbol{\eta}]_i \\
&\leq (s_{(h)} \boldsymbol{x}_{\cdot(h)} + \boldsymbol{x}_{\cdot \tilde{j}'})^\top P_{S_h} \boldsymbol{\eta}, \\
U_D^{(a)} &:= s_{(h)} \boldsymbol{x}_{\cdot(h)}^\top \boldsymbol{\eta} + \sum_{i:[P_{S_h}\boldsymbol{\eta}]_i > 0} x_{ij'} [P_{S_h}\boldsymbol{\eta}]_i \\
&\geq (s_{(h)} \boldsymbol{x}_{\cdot(h)} + \boldsymbol{x}_{\cdot \tilde{j}'})^\top P_{S_h} \boldsymbol{\eta}, \\
L_E^{(b)} &:= s_{(h)} \boldsymbol{x}_{\cdot(h)}^\top P_{S_h} \boldsymbol{y} - \sum_{i:[P_{S_h}\boldsymbol{y}]_i > 0} x_{ij'} [P_{S_h}\boldsymbol{y}]_i \\
&\leq (s_{(h)} \boldsymbol{x}_{\cdot(h)} - \boldsymbol{x}_{\cdot \tilde{j}'})^\top P_{S_h} \boldsymbol{y}, \\
U_E^{(b)} &:= s_{(h)} \boldsymbol{x}_{\cdot(h)}^\top P_{S_h} \boldsymbol{y} - \sum_{i:[P_{S_h}\boldsymbol{y}]_i < 0} x_{ij'} [P_{S_h}\boldsymbol{y}]_i \\
&\geq (s_{(h)} \boldsymbol{x}_{\cdot(h)} - \boldsymbol{x}_{\cdot \tilde{j}'})^\top P_{S_h} \boldsymbol{y}, \\
L_D^{(b)} &:= s_{(h)} \boldsymbol{x}_{\cdot(h)}^\top \boldsymbol{\eta} - \sum_{i:[P_{S_h}\boldsymbol{\eta}]_i > 0} x_{ij'} [P_{S_h}\boldsymbol{\eta}]_i \\
&\leq (s_{(h)} \boldsymbol{x}_{\cdot(h)} - \boldsymbol{x}_{\cdot \tilde{j}'})^\top P_{S_h} \boldsymbol{\eta}, \\
U_D^{(b)} &:= s_{(h)} \boldsymbol{x}_{\cdot(h)}^\top \boldsymbol{\eta} - \sum_{i:[P_{S_h}\boldsymbol{\eta}]_i < 0} x_{ij'} [P_{S_h}\boldsymbol{\eta}]_i \\
&\geq (s_{(h)} \boldsymbol{x}_{\cdot(h)} - \boldsymbol{x}_{\cdot \tilde{j}'})^\top P_{S_h} \boldsymbol{\eta}.
\end{aligned}
$$

Theorem 10. *(i) Consider solving the optimization problem in Eq.(19a), and let $\hat{\theta}_L^{(a)}$ be the current optimal solution, i.e., we know that the optimal $\theta_L^{(a)}$ is at least no greater than $\hat{\theta}_L^{(a)}$. If*

$$
\begin{aligned}
\{U_D^{(a)} < 0\} &\cup \{L_D^{(a)} > 0, L_E^{(a)} < 0, L_E^{(a)}/L_D^{(a)} > \hat{\theta}_L^{(a)}\} \\
&\cup \{L_D^{(a)} > 0, L_E^{(a)} > 0, L_E^{(a)}/U_D^{(a)} > \hat{\theta}_L^{(a)}\}
\end{aligned}
$$

is true, then the $\tilde{j}'$-th constraint in Eq. (10a) for any $h \in [k]$ and $(j', \tilde{j}') \in \bar{S}_h \times Des_{(h)}(j')$ does not affect the optimal solution in Eq.(19a).

(ii) Next, consider solving the optimization problem in Eq.(19b), and let $\hat{\theta}_L^{(b)}$ be the current optimal solution. If

$$
\begin{aligned}
\{U_D^{(b)} < 0\} &\cup \{L_D^{(b)} > 0, L_E^{(b)} < 0, L_E^{(b)}/L_D^{(b)} < \hat{\theta}_L^{(b)}\} \\
&\cup \{L_D^{(b)} > 0, L_E^{(b)} > 0, L_E^{(b)}/U_D^{(b)} < \hat{\theta}_L^{(b)}\}
\end{aligned}
$$

is true, then the $\tilde{j}'$-th constraint in Eq. (10b) for any $h \in [k]$ and $(j', \tilde{j}') \in \bar{S}_h \times Des_{(h)}(j')$ does not affect the optimal solution in Eq.(19b).

(iii) Furthermore, consider solving the optimization problem in Eq.(19d), and let $\hat{\theta}_U^{(a)}$ be the current optimal solution. If

$$
\begin{aligned}
\{L_D^{(a)} > 0\} &\cup \{U_D^{(a)} < 0, L_E^{(a)} < 0, L_E^{(a)}/U_D^{(a)} > \hat{\theta}_U^{(a)}\} \\
&\cup \{U_D^{(a)} < 0, L_E^{(a)} > 0, L_E^{(a)}/L_D^{(a)} > \hat{\theta}_U^{(a)}\}
\end{aligned}
$$

is true, then the $\tilde{j}'$-th constraint in Eq. (10a) for any $h \in [k]$ and $(j', \tilde{j}') \in \bar{S}_h \times Des_{(h)}(j')$ does not affect the optimal solution in Eq.(19d).

(iv) Finally, consider solving the optimization problem in Eq.(19e), and let $\hat{\theta}_U^{(b)}$ be the current optimal solution. If

$$
\begin{aligned}
\{L_D^{(b)} > 0\} &\cup \{U_D^{(b)} < 0, L_E^{(b)} < 0, L_E^{(b)}/U_D^{(b)} > \hat{\theta}_U^{(b)}\} \\
&\cup \{U_D^{(b)} < 0, L_E^{(b)} > 0, L_E^{(b)}/L_D^{(b)} > \hat{\theta}_U^{(b)}\}
\end{aligned}
$$

is true, then the $(\tilde{j}'$-th constraint in Eq. (10b) for any $h \in [k]$ and $(j', \tilde{j}') \in \bar{S}_h \times Des_{(h)}(j')$ does not affect the optimal solution in Eq.(19e).