Abstract

This document includes additional proofs for the estimation error of spectral initializers, discusses the real-valued LSPE, and provides detailed derivations for each of the proposed LSPE.

D. Proof of Proposition 1

Our goal is to first evaluate the S-MSE of the unnormalized spectral initializer in (2)

\[ US-MSE_{SI} = E \left[ \left\| \beta \sum_{m=1}^{M} T(y_m) a_m a_m^H - xx^H \right\|_F^2 \right] \]

and then minimize the resulting expression over the parameter \( \beta \). The unnormalized spectral MSE can be expanded into the following form:

\[
|\beta|^2 \sum_{m=1}^{M} \sum_{m'=1}^{M} E[\{T(y_m)T(y_{m'})\}] \text{tr}(a_m a_m^H a_{m'} a_{m'}^H) \\
- \beta^* \sum_{m=1}^{M} \text{tr}(a_m a_m^H E[\{T(y_m)xx^H\}]) \\
- \beta \sum_{m=1}^{M} \text{tr}(E[xx^H T(y_m)]a_m a_m^H) \\
+ E[\|xx^H\|_F^2].
\]

By using the definitions

\[
\tilde{V}_m = E[\{T(y_m)xx^H\}], m = 1, \ldots, M, \\
\tilde{T} = E[\{T(y)y^T\}],
\]

we can simplify the above expression into

\[
|\beta|^2 \sum_{m=1}^{M} \sum_{m'=1}^{M} \tilde{T}_{m,m'} |a_m^H a_{m'}|^2 + E[\|xx^H\|_F^2] \\
- \beta^* \sum_{m=1}^{M} \text{tr}(a_m a_m^H \tilde{V}_m) - \beta \sum_{m=1}^{M} \text{tr}(\tilde{V}_m a_m a_m^H).
\]

We can now find the optimal parameter for \( \beta \) by taking the derivative with respect to \( \beta^* \) and setting the expression to zero. The resulting optimal scaling parameter is given by

\[
\hat{\beta} = \frac{\sum_{m=1}^{M} \text{tr}\left( a_m a_m^H \tilde{V}_m \right)}{\sum_{m=1}^{M} \sum_{m'=1}^{M} T_{m,m'} |a_m^H a_{m'}|^2}.
\]

We now plug in \( \hat{\beta} \) into the expression (24), which yields

\[
S-MSE_{SI} = \frac{\sum_{m=1}^{M} \text{tr}\left( a_m a_m^H \tilde{V}_m \right)}{\sum_{m=1}^{M} \sum_{m'=1}^{M} T_{m,m'} |a_m^H a_{m'}|^2} \\
	imes \sum_{m=1}^{M} \sum_{m'=1}^{M} \tilde{T}_{m,m'} |a_m^H a_{m'}|^2 \\
- \frac{\sum_{m=1}^{M} \text{tr}\left( \tilde{V}_m a_m a_m^H \right)}{\sum_{m=1}^{M} \sum_{m'=1}^{M} T_{m,m'} |a_m^H a_{m'}|^2} \sum_{m=1}^{M} \text{tr}\left( a_m a_m^H \tilde{V}_m \right) \\
- \frac{\sum_{m=1}^{M} \text{tr}\left( a_m a_m^H \tilde{V}_m \right)}{\sum_{m=1}^{M} \sum_{m'=1}^{M} T_{m,m'} |a_m^H a_{m'}|^2} \sum_{m=1}^{M} \text{tr}\left( \tilde{V}_m a_m a_m^H \right) \\
+ E[\|xx^H\|_F^2].
\]

This expression can be simplified further to obtain:

\[
S-MSE_{SI} = R_{xx'Hy} - \frac{\sum_{m=1}^{M} \text{tr}\left( a_m a_m^H \tilde{V}_m a_m^H \right)}{\sum_{m=1}^{M} \sum_{m'=1}^{M} T_{m,m'} |a_m^H a_{m'}|^2},
\]

which is what we wanted to show in (12).
E. Real-Valued Phase Retrieval

We now focus on the case where the signal vector \( x \) to be recovered and the measurement matrix \( A \) are both real-valued. We derive the LSPE by using the following assumptions, which are reasonable for phase retrieval problems.

**Assumptions 3.** Let \( \mathcal{H} = \mathbb{R} \). Assume square measurements \( f(z) = z^2 \) and the identity preprocessing function \( T(y) = y \). Assume that the signal vector \( x \in \mathbb{R}^N \) is i.i.d. zero-mean Gaussian distributed with covariance matrix \( C_x = \sigma_x^2 I_N \), i.e., \( x \sim \mathcal{N}(0_{N \times 1}, \sigma_x^2 I_N) \); the parameter \( \sigma_x^2 \) denotes the signal variance. Assume that the signal noise vector \( e^x \) is zero-mean Gaussian with covariance matrix \( C_{e^x} \), i.e., \( e^x \sim \mathcal{N}(0_{M \times 1}, C_{e^x}) \), and the measurement noise vector \( e^y \) is Gaussian with mean \( \bar{e}^y \) and covariance matrix \( C_{e^y} \), i.e., \( e^y \sim \mathcal{N}(\bar{e}^y, C_{e^y}) \). Furthermore assume that \( x, e^x, \) and \( e^y \) are independent.

Under these assumptions, we can derive the following LSPE which we call LSPE-R; the detailed derivations of this spectral estimator are given in Appendix F.

**Estimator 3 (LSPE-R).** Let Assumptions 3 hold. Then, the spectral estimation matrix is given by

\[
D_y^R = K_x + \sum_{m=1}^{M} t_m V_m, \tag{26}
\]

where \( K_x = \sigma_x^2 I_N \), the vector \( t \in \mathbb{R}^M \) is given by the solution to the linear system \( Tt = y - \bar{y} \)

\[
\bar{y} = \text{diag}(C_x) + \bar{e}^y
\]

\[
C_x = \sigma_x^2 A A^T + C_{e^x}
\]

\[
T = 2C_x \odot C_x + C_{e^y}
\]

and \( V_m = 2\sigma_x^2 A_m A_m^T \), \( m = 1, \ldots, M \). The spectral estimate \( \hat{x} \) is given by the (scaled) leading eigenvector of \( D_y^R \) in (26). Furthermore, the S-MSE is given by Theorem 2.

F. Derivation of Estimator 3

We now use Theorem 1 to derive Estimator 3 under Assumptions 3. To this end, we require the three quantities: \( \mathcal{T}(y) \), \( T \), and \( V_m, m = 1, \ldots, M \), which we derive separately.

Computing \( \mathcal{T}(y) \) To compute the real-valued vector \( \mathcal{T}(y) = \mathbb{E}[T(y)] \),

we need the following result on the bivariate folded normal distribution developed in (Kan & Robotti, 2017, Sec. 3.1).

**Lemma 1.** Let \( [u_1, u_2] \sim \mathcal{N}(\mu, \Sigma) \) be a pair of real-valued jointly Gaussian random variables with covariance matrix

\[
\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{1,2}^2 \\ \sigma_{1,2} & \sigma_2^2 \end{bmatrix}.
\]

Then, for \( m = 1, 2 \), the pair of random variables \( (\nu_1, \nu_2) \) with \( \nu_1 = u_1^2 \) and \( \nu_2 = u_2^2 \) follows the bivariate folded normal distribution with the following (centered) moments:

\[
\bar{\nu}_m = \mathbb{E}[\nu_m^2] = \sigma_m^2 + \mu_m^2
\]

\[
[C_{\nu}]_{1,2} = \mathbb{E}[(\nu_1 - \bar{\nu}_1)(\nu_2 - \bar{\nu}_2)] = 4\mu_1\mu_2\sigma_{1,2}^2 + 2\sigma_{1,2}^4,
\]

\[
[C_{\nu}]_{1,1} = \mathbb{E}[(\nu_1 - \bar{\nu}_1)^2] = 2\sigma_1^4 + 4\mu_1^2\sigma_1^2.
\]

Let \( \bar{z} = \mathbb{E}[z] \) denote the mean vector and \( C_z = AC_x A^T + C_{e^x} = \sigma_x^2 AA^T + C_{e^x} \) the covariance matrix of the “phased” measurements \( z = Ax + e^x \). Then, by defining \( \sigma_m^2 = [C_z]_{m,m} \), we can compute the \( m \)th entry \( \mathcal{T}(y_m) \) using Lemma 1 as follows:

\[
\mathcal{T}(y_m) = \bar{y}_m = \mathbb{E}[z_m^2 + n_m^2] = \sigma_m^2 + \bar{e}_m^y.
\]

Hence, in compact vector notation we have

\[
\mathcal{T}(y) = \bar{y} = \text{diag}(C_z) + \bar{e}^y. \tag{29}
\]

Computing \( T \) To compute the real-valued matrix

\[
T = \mathbb{E}[(\mathcal{T}(y) - \mathcal{T}(y))(\mathcal{T}(y) - \mathcal{T}(y))^T]
\]

\[
= \mathbb{E}[(\mathcal{T}(y))^2] - \mathcal{T}(y)\mathcal{T}(y)^T, \tag{30}
\]

we only need to compute the matrix \( \mathbb{E}[(\mathcal{T}(y))^2] \) as the vector \( \mathcal{T}(y) \) was computed in (29). We compute this matrix entry-wise as

\[
T_{m,m'} = \mathbb{E}[(\mathcal{T}(y_m) - \mathcal{T}(y_{m'}))(\mathcal{T}(y_{m'}) - \mathcal{T}(y_{m'}))] = \mathbb{E}[y_m y_{m'}^* - \bar{y}_m \bar{y}_{m'}^* + \bar{e}_m \bar{e}_{m'}^*]
\]

\[
= \mathbb{E}[|z_m|^2 |z_{m'}|^2 - 2 \sigma_{m,m'}^2 \sigma_m^2 + [C_{e^y}]_{m,m'}]
\]

where (a) follows from (28). The only unknown term in the above expression is \( \mathbb{E}[|z_m|^2 |z_{m'}|^2] \). This term is the second moment of the random vector \( |z_m|^2, |z_{m'}|^2 \), which follows a bivariate folded normal distribution. For \( m \neq m' \), Lemma 1 yields

\[
\mathbb{E}[|z_m|^2 |z_{m'}|^2] = \sigma_m^2 \sigma_{m'}^2 + 2\sigma_{m,m'}^4,
\]

with \( \sigma_{m,m'}^2 = [C_z]_{m,m'} \). For \( m = m' \), Lemma 1 yields

\[
\mathbb{E}[y_m^2] = \mathbb{E}[|z_m|^4] = 3\sigma_m^4.
\]

Hence, we have

\[
T_{m,m'} = [C_{e^y}]_{m,m'} + \begin{cases} 
2\sigma_{m,m'}^4 & \text{if } m \neq m' \\
2\sigma_m^4 & \text{if } m = m',
\end{cases}
\]

which can be written in compact matrix form as

\[
T = 2C_x \odot C_x + C_{e^y}.
\]
Computing $V_m$ To compute the matrices

$$V_m = E[(T(y_m) - T(y_m)x^H - K_x)]$$

$$= E[T(y_m)x^H] - T(y_m)K_x$$  \hspace{1cm} (31)

for $m = 1, \ldots, M$, we only need to compute the complex-valued matrix $E[T(y_m)x^H]$ as the two other quantities $K_x = E[x^H]$ and $T(y_m)$ are known. We compute this matrix entry-wise as

$$[V_m]_{n,n'} = E[(T(y_m) - T(y_m))x_n x_{n'}^*] = E[y_m x_n x_{n'}^*] - \bar{y}_m[C_x]_{n,n'}.$$  \hspace{1cm}

Since $\bar{y}_m$ is known from (28), we focus on computing

$$E[y_m x_n x_{n'}^*]$$

$$= E\left[\left(\sum_{j=1}^{N} A^*_j x_j^* + e_m^*\right) \times \left(\sum_{j'=1}^{N} A_{j'} x_{j'} + e_{m,j'}^*\right) x_n x_{n'}^*\right]$$

$$= E\left[\left(\sum_{j=1}^{N} A^*_j x_j^* \sum_{j'=1}^{N} A_{j'} x_{j'}\right) x_n x_{n'}^*\right] + E[e_m^* x_n x_{n'}^*] + E[e_{m,j'}^* x_n x_{n'}^*]$$

$$= \sum_{j=1}^{N} A^*_j \sum_{j'=1}^{N} A_{j'} E[x_j^* x_{j'} x_n x_{n'}^*]$$

where $\bar{y}_m = E[y_m x_n x_{n'}^*] - \bar{y}_m(C_x)_{n,n'}$.  \hspace{1cm} (32)

The only unknown in the above expression is the double summation in (32). Since we assumed that the entries of the signal vector $x$ are i.i.d., most of the terms in this summation are zero. For $n \neq n'$, there are only two nonzero terms, corresponding to the cases of $(j, j') = (n, n')$ and $(j, j') = (n', n)$. Thus, for $n \neq n'$ we have

$$= 2A^*_m A_n x_n x_{n'}^* E[x_j^* x_{j'} x_n x_{n'}^*]$$

where (b) follows from Lemma 1. For $n = n'$, we have

$$= |A_{m,n}|^2 E[|x_n|^2] + \sum_{j \neq n=1}^{N} |A_{m,j}|^2 E[|x_j|^2]$$

where (c) follows from Lemma 1. For $n = n'$, we have

$$= 2|A_{m,n}|^2 |C_x|^2 + \sum_{j \neq n=1}^{N} |A_{m,j}|^2 |C_x|_{j,j}$$

As for (33), (c) follows from Lemma 1. By combining the above results, we have

$$V_m = 2C^H A_m A_n C_x + (a^H_m C_x a_n)(C_x^H \odot I) + (C_x)_{m,m} - \sigma^{2}_m C_x = 2\sigma^{2}_m A_m A_n,$$

where $a^H_m$ denotes the $m$th row of the matrix $A$.  \hspace{1cm}

G. Derivation of Estimator 1

We now use Theorem 1 to derive Estimator 1 under Assumptions 1. To this end, we require the three quantities: $T(y)$, $T(y)^T$, and $V_m$, $m = 1, \ldots, M$, which we derive separately.

Computing $T(y)$ To compute the real-valued vector $T(y) = \hat{y}$ in (27), we need the following definitions. Let $\bar{z} = E[z]$ denote the mean vector and $C_z = A^H C_x A$ denote the covariance matrix of the "phased" measurements $z = Ax + e$. Then, using Lemma 1, we have

$$\bar{y}_m = E[|z_m|^2 + \bar{e}_m^2] = E[|z_m|^2 + |z_m|^2 + \bar{e}_m^2]$$

$$= \sigma^2_m + \bar{e}_m^2.$$  \hspace{1cm} (34)

where we have used the definition $\sigma^2_m = |C_x|_{m,m}$. Hence, in compact vector notation we have

$$T(y) = \hat{y} = \text{diag}(C_z) + \bar{e}.$$  \hspace{1cm}

Computing $T(y)^T$ To compute the real-valued $T(y)^T$ in (30), we will frequently use the following result. Since the vector $z$ is a complex circularly-symmetric jointly Gaussian vector, we can extract the covariances of the real and imaginary parts separately as:

$$E[zz^H] = E[z_R z^H_R] = \frac{1}{2} E[C_z] = \frac{1}{2} C_z$$  \hspace{1cm} (35)

$$E[z_R z^H] = -E[z_I z^H_I] = \frac{1}{2} \text{Im}(C_z) = \frac{1}{2} C_z$$  \hspace{1cm} (36)

The quantity $\bar{y}_m$ is given by (34). Hence, we now compute

$$E[y_m y_m^*]$$

where $\bar{y}_m$ denotes the $m$th row of the matrix $A$.  \hspace{1cm}

Analogously, we can compute
\[ E[(z_m^2 + e_m^2)(z_m'^2 + e_m'^2)] = E[(z_m^2 + |z_m|^2)(z_m'^2 + |z_m'|^2)] + [C_{ve}]_{m,m}. \]

The first two terms above are a second moment of the variables \([|z_m|^2, |z_m'|^2]\) and \([|z_m|^2, |z_m'|^2]\), which follow a bivariate folded normal distribution. We first focus on \([|z_m|^2, |z_m'|^2]\). With Lemma 1, we can calculate the moments using the covariance \( E[z_R^T z_R^H] \) given in (35). To this end, define \( \sigma_{m,m',\mathcal{R}}^2 = |C_z\mathcal{R}|_{m,m'} \) and \( \sigma_{m,\mathcal{R}}^2 = |C_z\mathcal{R}|_{m,m} \). Thus, we have
\[
E[|z_m|^2|z_m'|^2] = \begin{bmatrix} \sigma_{m,m',\mathcal{R}}^2 \sigma_{m',\mathcal{R}}^2 - \sigma_{m,m',\mathcal{R}}^4 & \sigma_{m,m',\mathcal{R}}^2 \sigma_{m',\mathcal{R}}^4 \\ \sigma_{m,m',\mathcal{R}}^2 \sigma_{m',\mathcal{R}}^4 & \sigma_{m,m',\mathcal{R}}^4 \end{bmatrix},
\]
where \( \sigma_{m,m',\mathcal{R}}^2 = |C_z\mathcal{R}|_{m,m'} \).

By combining the above results, we have
\[
T_{m,m'} = \begin{bmatrix} \sigma_{m,m',\mathcal{R}}^2 \sigma_{m',\mathcal{R}}^2 - \sigma_{m,m',\mathcal{R}}^4 & \sigma_{m,m',\mathcal{R}}^2 \sigma_{m',\mathcal{R}}^4 \\ \sigma_{m,m',\mathcal{R}}^2 \sigma_{m',\mathcal{R}}^4 & \sigma_{m,m',\mathcal{R}}^4 \end{bmatrix}, \quad m \neq m'.
\]

Computing \( V_m \) To compute the matrices \( V_m, m = 1, \ldots, M \), in (31), we need the complex-valued matrix \( E[\mathcal{T}(y_m) xx^H] \). We compute this matrix entry-wise as
\[
[V_m]_{m,n'} = E[(\mathcal{T}(y_m) - \mathcal{T}(y_m)) x_n x_n^*] = E[y_m x_n x_n^*] - \bar{y}_m |C_x|_{m,n'}.
\]

Since \( \bar{y}_m \) is given by (34), we only need to compute
\[
E[y_m x_n x_n^*] = E\left[\left(\sum_{j=1}^N A_{m,j}^* x_j + e_m^*\right) x_n x_n^*\right] = \sum_{j=1}^N A_{m,j}^* \sum_{j'=1}^N A_{m,j'} E[x_j x_{j'} x_n x_n^*] + \sum_{j=1}^N A_{m,j}^* \sum_{j'=1}^N A_{m,j'} E[x_j x_{j'} x_n x_n^*] + \sum_{j=1}^N A_{m,j}^* \sum_{j'=1}^N A_{m,j'} E[x_j^* x_{j'} x_n x_n^*] + \sum_{j=1}^N A_{m,j}^* \sum_{j'=1}^N A_{m,j'} E[x_j^* x_{j'} x_n x_n^*]
\]

We will first simplify the term
\[
\sum_{j=1}^N A_{m,j}^* \sum_{j'=1}^N A_{m,j'} E[x_j x_{j'} x_n x_n^*] = A_{m,n}^* A_{m,n'} |C_x|_{m,n} |C_x|_{n,n'},
\]

since we assumed that the signal vector \( x \) has i.i.d. zero-mean entries, most of the terms in this summation are zero. For \( n \neq n' \), there is only one non-zero term for \((j, j') = (n', n)\). Thus, for \( n \neq n' \) we have
\[
\sum_{j=1}^N A_{m,j}^* \sum_{j'=1}^N A_{m,j'} E[x_j x_{j'} x_n x_n^*] = A_{m,n}^* A_{m,n'} |C_x|_{m,n} |C_x|_{n,n'},
\]

Next, for \( n = n' \), we have
\[
\sum_{j=1}^N A_{m,j}^* \sum_{j'=1}^N A_{m,j'} E[x_j x_{j'} x_n x_n^*] = A_{m,n}^* A_{m,n} E[x_n x_n^*] |C_x|_{n,n},
\]

which is zero.

where (a) follows from circular symmetry of \( x \) and (b) from Lemma 1. By combining the above results, we have
\[
V_m = C_x^H a_m a_m^H C_x + (a_m^H C_x a_m)(C_x^H \otimes I)
\]
H. Derivation of Estimator 2

We now use Theorem 1 to derive Estimator 2 under Assumptions 2. To this end, we require the three quantities: $T(y)$, $T$, and $V$, which we derive separately.

Computing $T(y)$ To derive an expression for $T(y)$ in (27), we need the following two results.

**Lemma 2.** Let $u \sim CN(0_{M \times 1}, \Sigma)$ be a complex-valued circularly-symmetric jointly Gaussian random vector with positive definite covariance matrix $\Sigma \in \mathbb{C}^{M \times M}$. Then, for the random variable $\nu = \exp(-u^HGu)$ with positive definite $G \in \mathbb{C}^{M \times M}$ and $\Sigma^{-1}$ positive definite, we have the following result:

$$
E[\nu] = \frac{1}{|\Sigma + \frac{1}{\Sigma}|}.
$$

**Proof.** We first expand the expected value into

$$
E[\nu] = E[\exp(-u^HGu)] = \int_{C_M} \exp(-u^H(G + \Sigma^{-1})u)du,
$$

where $|\Sigma| > 0$ is the determinant of $\Sigma$. We can now simplify the above expression as follows:

$$
= \frac{1}{\pi^M |\Sigma|} \frac{1}{\pi^M |(G + \Sigma^{-1})^{-1}|} \int_{C_M} \exp(-u^H(G + \Sigma^{-1})u)du
$$

$$
= \frac{|G + \Sigma^{-1}|^{-1}}{|\Sigma|} = \frac{1}{|\Sigma + \frac{1}{\Sigma}|}.
$$

where we also required that $G + \Sigma^{-1}$ is positive definite.

**Lemma 3.** Let $u \sim N(\bar{u}, \Sigma)$ be a real-valued Gaussian random vector with mean $\bar{u}$ and covariance $\Sigma$, and $\gamma \in \mathbb{R}^N$ be a given vector. Then, we have

$$
E[\exp(-\gamma^Tu)] = \exp(-\gamma^T\bar{u} + \frac{1}{2}\gamma^T\Sigma\gamma).
$$

**Proof.** The proof is an immediate consequence of the moment generating function of a Gaussian random vector.

By considering Lemma 2 and Lemma 3 for scalar random variables, the $m$th entry of the preprocessed phaseless measurement is given by

$$
T(y_m) = E[T(y_m)] = E[\exp(-\gamma|z_m|^2 - \gamma|e^y_m|)]
$$

$$
= \frac{1}{|\Sigma|} \exp(-\gamma|e^y_m| + \frac{1}{2} \gamma^2 |\Sigma|_{m,m}).
$$

We define the following auxiliary vectors

$$
q_\gamma = \gamma \text{ diag}(C_z) + 1_{M \times 1}, \quad p_\gamma = \exp(-\gamma e^y + \frac{1}{2} \gamma^2 \text{ diag}(C_e^y)),
$$

which enable us to rewrite the above expression in compact vector form as

$$
T(y) = p_\gamma \odot q_\gamma.
$$

Computing $T$ To compute the matrix $T$ in (30), we only need to compute $E[T(y)T(y)^T]$, which we will compute entry-wise and in two separate steps. Concretely, we have

$$
E[T(y_m)T(y_{m'})] = E[\exp(-\gamma(|z_m|^2 + |z_{m'}|^2))] \times E[\exp(-\gamma(|e^y_m + |e^y_{m'}|))],
$$

where we compute both expected values separately. In the first step, we compute

$$
E[\exp(-\gamma(|z_m|^2 + |z_{m'}|^2))] = E[\exp(-u^HGu)],
$$

with $u = [z_m, z_{m'}]^T$ and $G = I_2$. By invoking Lemma 2 with $|\Sigma|_{m,m'} = |C_z|_{m,m'}$, we obtain

$$
E[\exp(-\gamma(|z_m|^2 + |z_{m'}|^2))] = \frac{1}{|\gamma \Sigma + I_2|} = \frac{1}{(\gamma|C_z|_{m,m} + 1)(\gamma|C_z|_{m',m'} + 1) - \gamma^2|C_z|_{m,m'}^2}.
$$

With the definition of $q_\gamma$ in (38), we can rewrite the above expression in vector form as

$$
E[\exp(-\gamma|z|^2) \exp(-\gamma|z|^2)^T] = 1_{M \times M} \odot (q_\gamma q_\gamma^T - \gamma^2 C_z \odot C_z^*).
$$

In the second step, we compute

$$
E[\exp(-\gamma(|e^y|^2 + |e^y_{m'}|^2))] = E[\exp(-\gamma^T u)]
$$

with $u = [e^y_m, e^y_{m'}]^T$ and $\gamma^T = [\gamma, \gamma]$. By invoking Lemma 3 with mean $\bar{u} = [e^y_m, [e^y_{m'}]]$ and covariance $\Sigma$ given by the entries of the covariance matrix $C_e^y$ associated to the indices $m$ and $m'$, we obtain

$$
E[\exp(-\gamma(|e^y_m + |e^y_{m'}|))] = \exp(-\gamma(|e^y_m + |e^y_{m'}|)) \times \exp(-\gamma^2 |C_e^y - 2|C_e^y|_{m,m'}))
$$

With the definition of $p_\gamma$ in (39), we can rewrite the above expression in vector form as

$$
E[\exp(-\gamma e^y) \exp(-\gamma e^y)^T] = (p_\gamma p_\gamma^T) \odot \exp(\gamma^2 C_e^y)
$$

We furthermore have

$$
T(y)T(y)^T = (p_\gamma p_\gamma^T) \odot (q_\gamma q_\gamma^T).
$$

By combining the two steps with the above results, we have

$$
T = (p_\gamma p_\gamma^T) \odot (\exp(\gamma^2 C_e^y)) \odot (q_\gamma q_\gamma^T - \gamma^2 C_z \odot C_z^*) - 1_{M \times M} \odot (q_\gamma q_\gamma^T).
$$
Computing $V_m$ To compute the matrices $V_m$, $m = 1, \ldots, M$, in (31), we only need $\mathbb{E}\left[ T(y_m)x_m^Hx_m \right]$ which we will compute entry-wise and in two steps. We have

$$
\mathbb{E}\left[ T(y_m)x_n^*x_{n'}^* \right] = \mathbb{E}\left[ \exp\left( -\gamma |a^H_m x + [e^*]_m|^2 \right) x_n^*x_{n'}^* \right] \\
\times \mathbb{E}\left[ \exp\left( -\gamma |e^{yp}|_m \right) \right],
$$

where we next compute both expected values separately. As a first step, we use direct integration to compute the following expected value:

$$
\mathbb{E}\left[ \exp\left( -\gamma |a^H_m x + [e^*]_m|^2 \right) x_n^*x_{n'}^* \right] = \int_{\mathbb{C}^{N+1}} \exp\left( -\gamma |a^H_m x + [e^*]_m|^2 \right) \\
\times \frac{1}{(\pi \sigma^2)^N} \exp\left( -\frac{\|x\|^2}{\sigma^2} \right) \\
\times \frac{1}{\pi \sigma^2} \exp\left( -\frac{|[e^*]_m|^2}{\sigma_n^2} \right) x_n^*x_{n'}^* \, dx [e^*]_m.
$$

We define the following auxiliary quantities:

$$
\hat{a}_m^H = [a^H_m, 1] \\
\hat{x}^T = [x^T, [e^*]_m] \\
C_x = \begin{bmatrix} \sigma^2_1 I_N & 0_{N \times 1} \\ 0_{1 \times N} & \sigma^2_m \end{bmatrix} \\
\tilde{K}^{-1} = \hat{a}_m \hat{a}_m^H + C_x^{-1},
$$

where $\sigma^2_m = \mathbb{E}[|e^*]|^2 = \|a_m\|_m$. We now derive the above expectation in compact form as

$$
\mathbb{E}\left[ \exp\left( -\gamma |\hat{a}_m^H \hat{x}|^2 \right) \hat{x}_n \hat{x}_{n'}^* \right] = \frac{1}{(\pi \sigma^2)^N \pi \sigma_n^2} \int_{\mathbb{C}^{N+1}} \exp\left( -\gamma \hat{x}_n^H C_x^{-1} \hat{x}_n \right) \hat{x}_n \hat{x}_{n'}^* \, d\hat{x} \\
= \frac{1}{|\pi C_x|} \int_{\mathbb{C}^{N+1}} \exp\left( -\hat{x}_n^H (\hat{a}_m \hat{a}_m^H + C_x^{-1}) \hat{x}_n \right) \hat{x}_n \hat{x}_{n'}^* \, d\hat{x} \\
= \frac{1}{|\pi C_x|} \int_{\mathbb{C}^{N+1}} \exp\left( -\hat{x}_n^H \tilde{K}^{-1} \hat{x}_n \right) \hat{x}_n \hat{x}_{n'}^* \, d\hat{x},
$$

where $n = 1, \ldots, N+1, n' = 1, \ldots, N+1$. We can further rewrite this expression as

$$
\frac{1}{|\pi C_x|} \int_{\mathbb{C}^{N+1}} \exp\left( -\hat{x}_n^H \tilde{K}^{-1} \hat{x}_n \right) \hat{x}_n \hat{x}_{n'}^* \, d\hat{x} \\
= \frac{|\pi \tilde{K}|}{|\pi \tilde{K}| |\pi C_x|} \int_{\mathbb{C}^{N+1}} \exp\left( -\hat{x}_n^H \tilde{K}^{-1} \hat{x}_n \right) \hat{x}_n \hat{x}_{n'}^* \, d\hat{x}.
$$

It is now key to realize that

$$
\frac{1}{|\pi \tilde{K}|} \int_{\mathbb{C}^{N+1}} \exp\left( -\hat{x}_n^H \tilde{K}^{-1} \hat{x}_n \right) \hat{x}_n \hat{x}_{n'}^* \, d\hat{x} \\
= \mathbb{E}\left[ \hat{x}_n \hat{x}_{n'}^* \right] = |\tilde{K}|_{n,n'},
$$

and hence we have

$$
\mathbb{E}\left[ \exp\left( -\gamma |\hat{a}_m^H \hat{x}|^2 \right) \hat{x}_n \hat{x}_{n'}^* \right] = |\tilde{K}|_{n,n'}.
$$

We can now use the matrix-determinant lemma to simplify

$$
|\gamma \hat{a}_m \hat{a}_m^H C_x + I_{N+1}| = \gamma \hat{a}_m \hat{a}_m^H C_x + 1 \\
= \gamma (\|a_m\|^2 + \sigma^2_m) + 1
$$

and the matrix inversion lemma to simplify

$$
\tilde{K} = \left( \gamma \hat{a}_m \hat{a}_m^H + C_x^{-1} \right)^{-1} \\
= C_x - \gamma C_x \hat{a}_m \hat{a}_m^H C_x \\
= C_x - \gamma C_x \hat{a}_m \hat{a}_m^H C_x \gamma (\|a_m\|^2 + \sigma^2_m) + 1.
$$

By using these two simplifications, we have

$$
\mathbb{E}\left[ \exp\left( -\gamma |a_m^H x|^2 \right) x_n x_{n'}^* \right] = \frac{1}{\gamma (\|a_m\|^2 + \sigma^2_m) + 1} \\
\times \left[ C_x - \gamma C_x \hat{a}_m \hat{a}_m^H C_x \right]_{n,n'}
$$

and since we are only interested in the upper $N \times N$ part of the matrix $\tilde{K}$, we have

$$
\mathbb{E}\left[ \exp\left( -\gamma |a_m^H x + [e^*]|_m|^2 \right) x_n x_{n'}^* \right] \\
= \frac{1}{\gamma (\|a_m\|^2 + \sigma^2_m) + 1} \\
\times \left[ \sigma^2_1 I_N - \frac{\gamma \sigma^2_m a_m a_m^H}{\gamma (\|a_m\|^2 + \sigma^2_m) + 1} \right]_{n,n'}
$$

since for our assumptions

$$
\sigma^2_1 \|a_m\|^2 + \sigma^2_m = \|C_x\|_{m,m}.
$$

In compact matrix form, we have

$$
\mathbb{E}\left[ \exp\left( -\gamma |a_m^H x + [e^*]|_m|^2 \right) x x^H \right] = \frac{1}{\gamma |C_x|_{m,m} + 1} \left( \sigma^2_1 I_N \gamma (\|a_m\|^2 + \sigma^2_m) + 1 \right).
$$

As a second step, we use definition (39) and obtain

$$
\mathbb{E}\left[ \exp\left( -\gamma |e^{yp}|_m \right) \right] = |p_m|.
$$
By combining both steps, we obtain

\[
V_m = \frac{[p_{\gamma}]_m}{\gamma [C_z]_{m,m} + 1} \left( \sigma_z^2 I_N - \frac{\gamma \sigma_z^4 a_m a_m^H}{\gamma [C_z]_{m,m} + 1} \right) - \frac{[p_{\gamma}]_m}{\gamma [C_z]_{m,m} + 1} \sigma_z^2 I_N
\]

which is what we desperately wanted to show.