Supplementary Material:
Linear Spectral Estimators and an Application to Phase Retrieval

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Abstract

This document includes additional proofs for the estimation error of spectral initializers, discusses the real-valued LSPE, and provides detailed derivations for each of the proposed LSPE.

D. Proof of Proposition 1

Our goal is to first evaluate the S-MSE of the unnormalized spectral initializer in (2)

$$\textit{US-MSE}_{\text{SI}} = \mathbb{E}\left[\left\|\beta\sum_{m=1}^{M}\mathcal{T}(y_m)\mathbf{a}_m\mathbf{a}_m^H - \mathbf{x}\mathbf{x}^H\right\|_F^2\right]$$

and then minimize the resulting expression over the parameter β. The unnormalized spectral MSE can be expanded into the following form:

$$\begin{aligned}&|\beta|^2\sum_{m=1}^{M}\sum_{m'=1}^{M}\mathbb{E}[\mathcal{T}(y_m)\mathcal{T}(y_{m'})]\,\mathrm{tr}(\mathbf{a}_m\mathbf{a}_m^H\mathbf{a}_{m'}\mathbf{a}_{m'}^H)\\&-\beta^*\sum_{m=1}^{M}\mathrm{tr}\left(\mathbf{a}_m\mathbf{a}_m^H\,\mathbb{E}\left[\mathcal{T}(y_m)\mathbf{x}\mathbf{x}^H\right]\right)\\&-\beta\sum_{m=1}^{M}\mathrm{tr}\left(\mathbb{E}\left[\mathbf{x}\mathbf{x}^H\mathcal{T}(y_m)\right]\mathbf{a}_m\mathbf{a}_m^H\right)\\&+\mathbb{E}\left[\|\mathbf{x}\mathbf{x}^H\|_F^2\right].\end{aligned}$$

By using the definitions

$$\begin{aligned}\widetilde{\mathbf{V}}_m &= \mathbb{E}\left[\mathcal{T}(y_m)\mathbf{x}\mathbf{x}^H\right], m=1,\ldots,M,\\ \widetilde{\mathbf{T}} &= \mathbb{E}\left[\mathcal{T}(\mathbf{y})\mathcal{T}(\mathbf{y})^T\right],\end{aligned}$$

we can simplify the above expression into

$$\begin{aligned}&|\beta|^2\sum_{m=1}^{M}\sum_{m'=1}^{M}\widetilde{T}_{m,m'}|\mathbf{a}_m^H\mathbf{a}_{m'}|^2+\mathbb{E}\left[\|\mathbf{x}\mathbf{x}^H\|_F^2\right]\\&-\beta^*\sum_{m=1}^{M}\mathrm{tr}\left(\mathbf{a}_m\mathbf{a}_m^H\widetilde{\mathbf{V}}_m\right)-\beta\sum_{m=1}^{M}\mathrm{tr}\left(\widetilde{\mathbf{V}}_m^H\mathbf{a}_m\mathbf{a}_m^H\right).\end{aligned}\quad(24)$$

We can now find the optimal parameter for β by taking the derivative with respect to β^* and setting the expression to zero. The resulting optimal scaling parameter is given by

$$\hat{\beta} = \frac{\sum_{m=1}^{M}\mathrm{tr}\left(\mathbf{a}_m\mathbf{a}_m^H\widetilde{\mathbf{V}}_m\right)}{\sum_{m=1}^{M}\sum_{m'=1}^{M}\widetilde{T}_{m,m'}|\mathbf{a}_m^H\mathbf{a}_{m'}|^2}.$$

We now plug in $\hat{\beta}$ into the expression (24), which yields

$$\begin{aligned}\textit{S-MSE}_{\text{SI}} =& \left|\frac{\sum_{m=1}^{M}\mathrm{tr}\left(\mathbf{a}_m\mathbf{a}_m^H\widetilde{\mathbf{V}}_m\right)}{\sum_{m=1}^{M}\sum_{m'=1}^{M}\widetilde{T}_{m,m'}|\mathbf{a}_m^H\mathbf{a}_{m'}|^2}\right|^2 \quad (25)\\&\times\sum_{m=1}^{M}\sum_{m'=1}^{M}\widetilde{T}_{m,m'}|\mathbf{a}_m^H\mathbf{a}_{m'}|^2\\&-\frac{\sum_{m=1}^{M}\mathrm{tr}\left(\widetilde{\mathbf{V}}_m^H\mathbf{a}_m\mathbf{a}_m^H\right)}{\sum_{m=1}^{M}\sum_{m'=1}^{M}\widetilde{T}^*_{m,m'}|\mathbf{a}_m^H\mathbf{a}_{m'}|^2}\sum_{m=1}^{M}\mathrm{tr}\left(\mathbf{a}_m\mathbf{a}_m^H\widetilde{\mathbf{V}}_m\right)\\&-\frac{\sum_{m=1}^{M}\mathrm{tr}\left(\mathbf{a}_m\mathbf{a}_m^H\widetilde{\mathbf{V}}_m\right)}{\sum_{m=1}^{M}\sum_{m'=1}^{M}\widetilde{T}_{m,m'}|\mathbf{a}_m^H\mathbf{a}_{m'}|^2}\sum_{m=1}^{M}\mathrm{tr}\left(\widetilde{\mathbf{V}}_m^H\mathbf{a}_m\mathbf{a}_m^H\right)\\&+\mathbb{E}\left[\|\mathbf{x}\mathbf{x}^H\|_F^2\right].\end{aligned}$$

This expression can be simplified further to obtain:

$$\begin{aligned}\textit{S-MSE}_{\text{SI}} =& \frac{\left|\sum_{m=1}^{M}\mathrm{tr}\left(\mathbf{a}_m\mathbf{a}_m^H\widetilde{\mathbf{V}}_m\right)\right|^2}{\sum_{m=1}^{M}\sum_{m'=1}^{M}\widetilde{T}^*_{m,m'}|\mathbf{a}_m^H\mathbf{a}_{m'}|^2}\\&-\frac{\left|\sum_{m=1}^{M}\mathrm{tr}\left(\mathbf{a}_m\mathbf{a}_m^H\widetilde{\mathbf{V}}_m\right)\right|^2}{\sum_{m=1}^{M}\sum_{m'=1}^{M}\widetilde{T}^*_{m,m'}|\mathbf{a}_m^H\mathbf{a}_{m'}|^2}\\&-\frac{\left|\sum_{m=1}^{M}\mathrm{tr}\left(\mathbf{a}_m\mathbf{a}_m^H\widetilde{\mathbf{V}}_m\right)\right|^2}{\sum_{m=1}^{M}\sum_{m'=1}^{M}\widetilde{T}_{m,m'}|\mathbf{a}_m^H\mathbf{a}_{m'}|^2}\\&+\mathbb{E}\left[\|\mathbf{x}\mathbf{x}^H\|_F^2\right],\\=& R_{\mathbf{x}\mathbf{x}^H}-\frac{\left|\sum_{m=1}^{M}\mathbf{a}_m^H\widetilde{\mathbf{V}}_m\mathbf{a}_m\right|^2}{\sum_{m=1}^{M}\sum_{m'=1}^{M}\widetilde{T}_{m,m'}|\mathbf{a}_m^H\mathbf{a}_{m'}|^2},\end{aligned}$$

which is what we wanted to show in (12).

E. Real-Valued Phase Retrieval

We now focus on the case where the signal vector $\mathbf{x}$ to be recovered and the measurement matrix $\mathbf{A}$ are both real-valued. We derive the LSPE by using the following assumptions, which are reasonable for phase retrieval problems.

Assumptions 3. *Let $\mathcal{H} = \mathbb{R}$. Assume square measurements $f(z) = z^2$ and the identity preprocessing function $\mathcal{T}(y) = y$. Assume that the signal vector $\mathbf{x} \in \mathbb{R}^N$ is i.i.d. zero-mean Gaussian distributed with covariance matrix $\mathbf{C}_\mathbf{x} = \sigma_x^2 \mathbf{I}_N$, i.e., $\mathbf{x} \sim \mathcal{N}(\mathbf{0}_{N\times 1}, \sigma_x^2 \mathbf{I}_N)$; the parameter σ_x^2 denotes the signal variance. Assume that the signal noise vector $\mathbf{e}^z$ is zero-mean Gaussian with covariance matrix $\mathbf{C}_{\mathbf{e}^z}$, i.e., $\mathbf{e}^z \sim \mathcal{N}(\mathbf{0}_{M\times 1}, \mathbf{C}_{\mathbf{e}^z})$, and the measurement noise vector $\mathbf{e}^y$ is Gaussian with mean $\bar{\mathbf{e}}^y$ and covariance matrix $\mathbf{C}_{\mathbf{e}^y}$, i.e., $\mathbf{e}^y \sim \mathcal{N}(\bar{\mathbf{e}}^y, \mathbf{C}_{\mathbf{e}^y})$. Furthermore assume that $\mathbf{x}$, $\mathbf{e}^z$, and $\mathbf{e}^y$ are independent.*

Under these assumptions, we can derive the following LSPE which we call LSPE-$\mathbb{R}$; the detailed derivations of this spectral estimator are given in Appendix F.

Estimator 3 (LSPE-$\mathbb{R}$)**.** *Let Assumptions 3 hold. Then, the spectral estimation matrix is given by*

$$\mathbf{D}_\mathbf{y}^\mathbb{R} = \mathbf{K}_\mathbf{x} + \sum_{m=1}^{M} t_m \mathbf{V}_m, \qquad (26)$$

where $\mathbf{K}_\mathbf{x} = \sigma_x^2 \mathbf{I}_N$, the vector $\mathbf{t} \in \mathbb{R}^M$ is given by the solution to the linear system $\mathbf{T}\mathbf{t} = \mathbf{y} - \bar{\mathbf{y}}$ with

$$\begin{aligned} \bar{\mathbf{y}} &= \operatorname{diag}(\mathbf{C}_\mathbf{z}) + \bar{\mathbf{e}}^y \\ \mathbf{C}_\mathbf{z} &= \sigma_x^2 \mathbf{A}\mathbf{A}^T + \mathbf{C}_{\mathbf{e}^z} \\ \mathbf{T} &= 2\mathbf{C}_\mathbf{z} \odot \mathbf{C}_\mathbf{z} + \mathbf{C}_{\mathbf{e}^y} \end{aligned}$$

and $\mathbf{V}_m = 2\sigma_x^4 \mathbf{a}_m \mathbf{a}_m^T$, $m = 1, \ldots, M$. The spectral estimate $\hat{\mathbf{x}}$ is given by the (scaled) leading eigenvector of $\mathbf{D}_\mathbf{y}^\mathbb{R}$ in (26). Furthermore, the S-MSE is given by Theorem 2.

F. Derivation of Estimator 3

We now use Theorem 1 to derive Estimator 3 under Assumptions 3. To this end, we require the three quantities: $\overline{\mathcal{T}}(\mathbf{y})$, $\mathbf{T}$, and $\mathbf{V}_m$, $m = 1, \ldots, M$, which we derive separately.

Computing $\overline{\mathcal{T}}(\mathbf{y})$ To compute the real-valued vector

$$\overline{\mathcal{T}}(\mathbf{y}) = \mathbb{E}[\mathcal{T}(\mathbf{y})], \qquad (27)$$

we need the following result on the bivariate folded normal distribution developed in (Kan & Robotti, 2017, Sec. 3.1).

Lemma 1. *Let $[u_1, u_2] \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ be a pair of real-valued jointly Gaussian random variables with covariance matrix*

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{1,2}^2 \\ \sigma_{1,2}^2 & \sigma_2^2 \end{bmatrix}.$$

Then, for $m = 1, 2$, the pair of random variables (ν_1, ν_2) with $\nu_1 = u_1^2$ and $\nu_2 = u_2^2$ follows the bivariate folded normal distribution with the following (centered) moments:

$$\begin{aligned} \bar{\nu}_m &= \mathbb{E}\left[u_m^2\right] = \sigma_m^2 + \mu_m^2 \\ [\mathbf{C}_{\boldsymbol{\nu}}]_{1,2} &= \mathbb{E}[(\nu_1 - \bar{\nu_1})(\nu_2 - \bar{\nu_2})] \\ &= 4\mu_1\mu_2\sigma_{1,2}^2 + 2\sigma_{1,2}^4 \\ [\mathbf{C}_{\boldsymbol{\nu}}]_{1,1} &= \mathbb{E}\left[(\nu_1 - \bar{\nu_1})^2\right] = 2\sigma_1^4 + 4\mu_1^2\sigma_1^2. \end{aligned}$$

Let $\bar{\mathbf{z}} = \mathbb{E}[\mathbf{z}]$ denote the mean vector and $\mathbf{C}_\mathbf{z} = \mathbf{A}\mathbf{C}_\mathbf{x}\mathbf{A}^H + \mathbf{C}_{\mathbf{e}^z} = \sigma_x^2 \mathbf{A}\mathbf{A}^H + \mathbf{C}_{\mathbf{e}^z}$ the covariance matrix of the "phased" measurements $\mathbf{z} = \mathbf{A}\mathbf{x} + \mathbf{e}^z$. Then, by defining $\sigma_m^2 = [\mathbf{C}_\mathbf{z}]_{m,m}$, we can compute the mth entry $\overline{\mathcal{T}}(y_m)$ using Lemma 1 as follows:

$$\overline{\mathcal{T}}(y_m) = \bar{y}_m = \mathbb{E}\left[|z_m|^2 + n_m^y\right] = \sigma_m^2 + \bar{e}_m^y. \qquad (28)$$

Hence, in compact vector notation we have

$$\overline{\mathcal{T}}(\mathbf{y}) = \bar{\mathbf{y}} = \operatorname{diag}(\mathbf{C}_\mathbf{z}) + \bar{\mathbf{e}}^y. \qquad (29)$$

Computing T To compute the real-valued matrix

$$\begin{aligned} \mathbf{T} &= \mathbb{E}\left[(\mathcal{T}(\mathbf{y}) - \overline{\mathcal{T}}(\mathbf{y}))(\mathcal{T}(\mathbf{y}) - \overline{\mathcal{T}}(\mathbf{y}))^T\right] \\ &= \mathbb{E}\left[\mathcal{T}(\mathbf{y})\mathcal{T}(\mathbf{y})^T\right] - \overline{\mathcal{T}}(\mathbf{y})\overline{\mathcal{T}}(\mathbf{y})^T, \end{aligned} \qquad (30)$$

we only need to compute the matrix $\mathbb{E}\left[\mathcal{T}(\mathbf{y})\mathcal{T}(\mathbf{y})^T\right]$ as the vector $\overline{\mathcal{T}}(\mathbf{y})$ was computed in (29). We compute this matrix entry-wise as

$$\begin{aligned} T_{m,m'} &= \mathbb{E}\left[(\mathcal{T}(y_m) - \overline{\mathcal{T}}(y_m))(\mathcal{T}(y_{m'}) - \overline{\mathcal{T}}(y_{m'}))\right] \\ &= \mathbb{E}[y_m y_{m'}^*] - \bar{y}_m \bar{y}_{m'}^* \\ &\overset{(a)}{=} \mathbb{E}\left[(|z_m|^2 + e_m^y)(|z_{m'}|^2 + e_{m'}^y)\right] \\ &\quad - (\sigma_m^2 + \bar{e}_m^y)(\sigma_{m'}^2 + \bar{e}_{m'}^y) \\ &= \mathbb{E}\left[|z_m|^2 |z_{m'}|^2\right] - \sigma_{m'}^2\sigma_m^2 + [\mathbf{C}_{\mathbf{e}^y}]_{m,m'}, \end{aligned}$$

where (a) follows from (28). The only unknown term in the above expression is $\mathbb{E}\left[|z_m|^2|z_{m'}|^2\right]$. This term is the second moment of the random vector $[|z_m|^2, |z_{m'}|^2]$, which follows a bivariate folded normal distribution. For $m \neq m'$, Lemma 1 yields

$$\mathbb{E}\left[|z_m|^2|z_{m'}|^2\right] = \sigma_m^2\sigma_{m'}^2 + 2\sigma_{m,m'}^4$$

with $\sigma_{m,m'}^2 = [\mathbf{C}_\mathbf{z}]_{m,m'}$. For $m = m'$, Lemma 1 yields

$$\mathbb{E}\left[|y_m|^2\right] = \mathbb{E}\left[|z_m|^4\right] = 3\sigma_m^4.$$

Hence, we have

$$T_{m,m'} = [\mathbf{C}_{\mathbf{e}^y}]_{m,m'} + \begin{cases} 2\sigma_{m,m'}^4 & \text{if } m \neq m' \\ 2\sigma_m^4 & \text{if } m = m', \end{cases}$$

which can be written in compact matrix form as

$$\mathbf{T} = 2\mathbf{C}_\mathbf{z} \odot \mathbf{C}_\mathbf{z} + \mathbf{C}_{\mathbf{e}^y}.$$

Computing $\mathbf{V}_m$ To compute the matrices

$$\begin{aligned}\mathbf{V}_m &= \mathbb{E}\left[(\mathcal{T}(y_m)-\overline{\mathcal{T}}(y_m))(\mathbf{x}\mathbf{x}^H-\mathbf{K}_\mathbf{x})\right]\\ &= \mathbb{E}\left[\mathcal{T}(y_m)\mathbf{x}\mathbf{x}^H\right]-\overline{\mathcal{T}}(y_m)\mathbf{K}_\mathbf{x} \qquad (31)\end{aligned}$$

for $m=1,\dots,M$, we only need to compute the complex-valued matrix $\mathbb{E}\left[\mathcal{T}(y_m)\mathbf{x}\mathbf{x}^H\right]$ as the two other quantities $\mathbf{K}_\mathbf{x}=\mathbb{E}\left[\mathbf{x}\mathbf{x}^H\right]$ and $\overline{\mathcal{T}}(y_m)$ are known. We compute this matrix entry-wise as

$$\begin{aligned}[\mathbf{V}_m]_{n,n'} &= \mathbb{E}\left[(\mathcal{T}(y_m)-\overline{\mathcal{T}}(y_m))x_n x_{n'}^*\right]\\ &= \mathbb{E}[y_m x_n x_{n'}^*]-\bar{y}_m[\mathbf{C}_\mathbf{x}]_{n,n'}.\end{aligned}$$

Since $\bar{y}_m$ is known from (28), we focus on computing

$$\begin{aligned}&\mathbb{E}[y_m x_n x_{n'}^*]\\ &=\mathbb{E}\left[\left(\left(\sum_{j=1}^N A_{m,j}^* x_j^* + e_m^z\right)\right.\right.\\ &\quad\left.\left.\times\left(\sum_{j'=1}^N A_{m,j'}x_{j'}+e_m^z\right)+e_m^y\right)x_n x_{n'}^*\right]\\ &=\mathbb{E}\left[\left(\sum_{j=1}^N A_{m,j}^* x_j^* \sum_{j'=1}^N A_{m,j'}x_{j'}\right)x_n x_{n'}^*\right]\\ &\quad+\mathbb{E}\left[|e_m^z|^2 x_n x_{n'}^*\right]+\mathbb{E}[e_m^y x_n x_{n'}^*]\\ &=\sum_{j=1}^N A_{m,j}^*\sum_{j'=1}^N A_{m,j'}\,\mathbb{E}\left[x_j^* x_{j'} x_n x_{n'}^*\right] \qquad (32)\\ &\quad+([\mathbf{C}_{\mathbf{e}^z}]_{m,m}+\bar{e}_m^y)[\mathbf{C}_\mathbf{x}]_{n,n'}.\end{aligned}$$

The only unknown in the above expression is the double summation in (32). Since we assumed that the entries of the signal vector $\mathbf{x}$ are i.i.d., most of the terms in this summation are zero. For $n\neq n'$, there are only two nonzero terms, corresponding to the cases of $(j,j')=(n,n')$ and $(j,j')=(n',n)$. Thus, for $n\neq n'$ we have

$$\begin{aligned}&\sum_{j=1}^N A_{m,j}^*\sum_{j'=1}^N A_{m,j'}\,\mathbb{E}\left[x_j^* x_{j'} x_n x_{n'}^*\right]\\ &\qquad=2A_{m,n}^*A_{m,n'}\,\mathbb{E}\left[|x_n|^2|x_{n'}|^2\right]\\ &\qquad\overset{\text{(b)}}{=}2A_{m,n}^*A_{m,n'}[\mathbf{C}_\mathbf{x}]_{n,n}[\mathbf{C}_\mathbf{x}]_{n',n'}, \qquad (33)\end{aligned}$$

where (b) follows from Lemma 1. For $n=n'$, we have

$$\begin{aligned}&\sum_{j=1}^N A_{m,j}^*\sum_{j'=1}^N A_{m,j'}\,\mathbb{E}\left[x_j^* x_{j'} x_n x_n^*\right]\\ &=|A_{m,n}|^2\,\mathbb{E}\left[|x_n|^4\right]+\sum_{j\neq n,j=1}^N |A_{m,j}|^2\,\mathbb{E}\left[|x_j|^2|x_n|^2\right]\\ &\overset{\text{(c)}}{=}3|A_{m,n}|^2[\mathbf{C}_\mathbf{x}]_{n,n}^2+\sum_{j\neq n,j=1}^N |A_{m,j}|^2[\mathbf{C}_\mathbf{x}]_{j,j}[\mathbf{C}_\mathbf{x}]_{n,n}\\ &=2|A_{m,n}|^2[\mathbf{C}_\mathbf{x}]_{n,n}^2+\sum_{j=1}^N |A_{m,j}|^2[\mathbf{C}_\mathbf{x}]_{j,j}[\mathbf{C}_\mathbf{x}]_{n,n}.\end{aligned}$$

As for (33), (c) follows from Lemma 1. By combining the above results, we have

$$\begin{aligned}\mathbf{V}_m =& 2\mathbf{C}_\mathbf{x}^H\mathbf{a}_m\mathbf{a}_m^H\mathbf{C}_\mathbf{x}+(\mathbf{a}_m^H\mathbf{C}_\mathbf{x}\mathbf{a}_m)(\mathbf{C}_\mathbf{x}^H\odot\mathbf{I})\\ &+([\mathbf{C}_{\mathbf{e}^z}]_{m,m}-\sigma_m^2)\mathbf{C}_\mathbf{x}=2\sigma_x^4\mathbf{a}_m\mathbf{a}_m^H,\end{aligned}$$

where $\mathbf{a}_m^H$ denotes the mth row of the matrix $\mathbf{A}$.

G. Derivation of Estimator 1

We now use Theorem 1 to derive Estimator 1 under Assumptions 1. To this end, we require the three quantities: $\overline{\mathcal{T}}(\mathbf{y})$, $\mathbf{T}$, and $\mathbf{V}_m$, $m=1,\dots,M$, which we derive separately.

Computing $\overline{\mathcal{T}}(\mathbf{y})$ To compute the real-valued vector $\overline{\mathcal{T}}(\mathbf{y})=\bar{\mathbf{y}}$ in (27), we need the following definitions. Let $\bar{\mathbf{z}}=\mathbb{E}[\mathbf{z}]$ denote the mean vector and $\mathbf{C}_\mathbf{z}=\mathbf{A}\mathbf{C}_\mathbf{x}\mathbf{A}^H+\mathbf{C}_{\mathbf{e}^z}=\sigma_x^2\mathbf{A}\mathbf{A}^H+\mathbf{C}_{\mathbf{e}^z}$ the covariance matrix of the "phased" measurements $\mathbf{z}=\mathbf{A}\mathbf{x}+\mathbf{e}^z$. Then, using Lemma 1 with the definitions $\bar{\mathbf{z}}$ and $\mathbf{C}_\mathbf{z}$, we have

$$\begin{aligned}\bar{y}_m &= \mathbb{E}\left[|z_m|^2+\bar{e}_m^y\right]=\mathbb{E}\left[|z_{m,\mathcal{R}}|^2+|z_{m,\mathcal{I}}|^2+\bar{e}_m^y\right]\\ &=\sigma_m^2+\bar{e}_m^y, \qquad (34)\end{aligned}$$

where we have used the definition $\sigma_m^2=[\mathbf{C}_\mathbf{z}]_{m,m}$. Hence, in compact vector notation we have

$$\overline{\mathcal{T}}(\mathbf{y})=\bar{\mathbf{y}}=\operatorname{diag}(\mathbf{C}_\mathbf{z})+\bar{\mathbf{e}}^y.$$

Computing $\mathbf{T}$ To compute the real-valued matrix $\mathbf{T}$ in (30), we will frequently use the following result. Since the vector $\mathbf{z}$ is a complex circularly-symmetric jointly Gaussian vector, we can extract the covariance matrices of the real and imaginary parts separately as:

$$\mathbb{E}\left[\mathbf{z}_\mathcal{I}\mathbf{z}_\mathcal{I}^H\right]\overset{\text{(a)}}{=}\mathbb{E}\left[\mathbf{z}_\mathcal{R}\mathbf{z}_\mathcal{R}^H\right]=\frac{1}{2}\Re\{\mathbf{C}_\mathbf{z}\}=\frac{1}{2}\mathbf{C}_{\mathbf{z},\mathcal{R}} \qquad (35)$$

$$\mathbb{E}\left[\mathbf{z}_\mathcal{R}\mathbf{z}_\mathcal{I}^H\right]=-\mathbb{E}\left[\mathbf{z}_\mathcal{I}\mathbf{z}_\mathcal{R}^H\right]=\frac{1}{2}\Im\{\mathbf{C}_\mathbf{z}\}=\frac{1}{2}\mathbf{C}_{\mathbf{z},\mathcal{I}}, \qquad (36)$$

where (a) follows from circular symmetry of the random vector $\mathbf{x}$. We are now ready to compute the individual entries of $\mathbb{E}\left[\mathcal{T}(\mathbf{y})\mathcal{T}(\mathbf{y})^T\right]$ as

$$\begin{aligned}T_{m,m'} &= \mathbb{E}\left[(\mathcal{T}(y_m)-\overline{\mathcal{T}}(y_m))(\mathcal{T}(y_{m'})-\overline{\mathcal{T}}(y_{m'}))\right]\\ &=\mathbb{E}[(y_m-\bar{y}_m)(y_{m'}-\bar{y}_{m'})^*]\\ &=\mathbb{E}[y_m y_{m'}^*]-\bar{y}_m\bar{y}_{m'}^*.\end{aligned}$$

The quantity $\bar{y}_m$ is given by (34). Hence, we now compute

$$\mathbb{E}[y_m y_{m'}^*]$$

$$
\begin{aligned}
&= \mathbb{E}\left[(|z_m|^2 + e^y_m)(|z'_m|^2 + e^y_{m'})^*\right] \\
&= \mathbb{E}\left[\left(|z_{m,\mathcal{R}}|^2 + |z_{m,\mathcal{I}}|^2\right)\left(|z_{m',\mathcal{R}}|^2 + |z_{m',\mathcal{I}}|^2\right)\right] \\
&\quad + [\mathbf{C}_{\mathbf{e}^y}]_{m,m} \\
&= 2\,\mathbb{E}\left[|z_{m,\mathcal{R}}|^2|z_{m',\mathcal{R}}|^2\right] + 2\,\mathbb{E}\left[|z_{m,\mathcal{R}}|^2|z_{m',\mathcal{I}}|^2\right] \\
&\quad + [\mathbf{C}_{\mathbf{e}^y}]_{m,m}.
\end{aligned}
$$

The first two terms above are a second moment of the variables $[|z_{m,\mathcal{R}}|^2, |z_{m',\mathcal{R}}|^2]$ and $[|z_{m,\mathcal{R}}|^2, |z_{m',\mathcal{I}}|^2]$, which follow a bivariate folded normal distributions. We first focus on $[|z_{m,\mathcal{R}}|^2, |z_{m',\mathcal{R}}|^2]$. With Lemma 1, we can calculate the moments using the covariance $\mathbb{E}\left[\mathbf{z}_\mathcal{R}\mathbf{z}_\mathcal{R}^H\right]$ given in (35). To this end, define $\sigma^2_{m,m',\mathcal{R}} = [\mathbf{C}_{\mathbf{z},\mathcal{R}}]_{m,m'}$ and $\sigma^2_{m,\mathcal{R}} = [\mathbf{C}_{\mathbf{z},\mathcal{R}}]_{m,m}$. Thus, we have

$$
\mathbb{E}\left[|z_{m,\mathcal{R}}|^2|z_{m',\mathcal{R}}|^2\right] = \begin{cases} \frac{\sigma^2_{m,\mathcal{R}}}{2}\frac{\sigma^2_{m',\mathcal{R}}}{2} + \frac{\sigma^4_{m,m',\mathcal{R}}}{2}, & m \neq m' \\ 3\frac{\sigma^4_{m,\mathcal{R}}}{4}, & m = m'. \end{cases}
$$

Analogously, we can compute $\mathbb{E}\left[\mathbf{z}_\mathcal{R}\mathbf{z}_\mathcal{I}^H\right]$ in (36) from the covariance matrix of $[|z_{m,\mathcal{R}}|^2, |z_{m',\mathcal{I}}|^2]$, with $\sigma^2_{m,m',\mathcal{I}} = [\mathbf{C}_{\mathbf{z},\mathcal{I}}]_{m,m'}$ and noting that $\sigma^2_{m,\mathcal{I}} = [\mathbf{C}_{\mathbf{z},\mathcal{I}}]_{m,m} = 0$ as

$$
\mathbb{E}\left[|z_{m,\mathcal{R}}|^2|z_{m',\mathcal{I}}|^2\right] = \begin{cases} \frac{\sigma^2_{m,\mathcal{R}}}{2}\frac{\sigma^2_{m',\mathcal{R}}}{2} + 2\frac{\sigma^4_{m,m',\mathcal{I}}}{4}, & m \neq m' \\ 3\frac{\sigma^4_{m,\mathcal{R}}}{4}, & m = m'. \end{cases}
$$

By combining the above results, we have

$$
\begin{aligned}
T_{m,m'} &= \begin{cases} \sigma^2_{m,\mathcal{R}}\sigma^2_{m',\mathcal{R}} + \sigma^4_{m,m',\mathcal{R}} + \sigma^4_{m,m',\mathcal{I}}, & m \neq m' \\ 2\,\sigma^4_{m,\mathcal{R}}, & m = m', \end{cases} \\
&\quad + [\mathbf{C}_{\mathbf{e}^y}]_{m,m'} - \bar{y}_m\bar{y}^*_{m'} \\
&= [\mathbf{C}_{\mathbf{e}^y}]_{m,m'} + \begin{cases} \sigma^4_{m,m',\mathcal{R}} + \sigma^4_{m,m',\mathcal{I}}, & m \neq m' \\ \sigma^4_{m,\mathcal{R}}, & m = m', \end{cases}
\end{aligned}
$$

which can be written in matrix form as

$$
\mathbf{T} = \mathbf{C}_\mathbf{z} \odot \mathbf{C}^*_\mathbf{z} + \mathbf{C}_{\mathbf{e}^y}.
$$

Computing $\mathbf{V}_m$ To compute the matrices $\mathbf{V}_m$, $m = 1,\ldots,M$, in (31), we need the complex-valued matrix $\mathbb{E}\left[\mathcal{T}(y_m)\mathbf{x}\mathbf{x}^H\right]$. We compute this matrix entry-wise as

$$
\begin{aligned}
[\mathbf{V}_m]_{n,n'} &= \mathbb{E}\left[(\mathcal{T}(y_m) - \overline{\mathcal{T}}(y_m))x_nx^*_{n'}\right] \\
&= \mathbb{E}[y_mx_nx^*_{n'}] - \bar{y}_m[\mathbf{C}_\mathbf{x}]_{n,n'}.
\end{aligned}
$$

Since $\bar{y}_m$ is given by (34), we only need to compute

$$
\begin{aligned}
&\mathbb{E}[y_mx_nx^*_{n'}] \\
&= \mathbb{E}\Bigg[\Bigg(\Big(\sum_{j=1}^N A^*_{m,j}x^*_j + e^{z*}_m\Big) \\
&\qquad \times \Big(\sum_{j'=1}^N A_{m,j'}x_{j'} + e^z_m\Big) + e^y_m\Bigg)x_nx^*_{n'}\Bigg] \\
&= \sum_{j=1}^N A^*_{m,j}\sum_{j'=1}^N A_{m,j'}\,\mathbb{E}\left[x^*_jx_{j'}x_nx^*_{n'}\right] \\
&\quad + \mathbb{E}\left[|e^z_m|^2x_nx^*_{n'}\right] + \mathbb{E}[e^y_mx_nx^*_{n'}] \\
&= \sum_{j=1}^N A^*_{m,j}\sum_{j'=1}^N A_{m,j'}\,\mathbb{E}\left[x^*_jx_{j'}x_nx^*_{n'}\right] \\
&\quad + ([\mathbf{C}_{\mathbf{e}^z}]_{m,m} + \bar{e}^y_m)[\mathbf{C}_\mathbf{x}]_{n,n'}.
\end{aligned}
\tag{37}
$$

We will first simplify the term

$$
\sum_{j=1}^N A^*_{m,j}\sum_{j'=1}^N A_{m,j'}\,\mathbb{E}\left[x^*_jx_{j'}x_nx^*_{n'}\right].
$$

Since we assumed that the signal vector $\mathbf{x}$ has i.i.d. zero-mean entries, most of the terms in this summation are zero. For $n \neq n'$, there is only one non-zero term for $(j,j') = (n,n')$. Thus, for $n \neq n'$ we have

$$
\begin{aligned}
&\sum_{j=1}^N A^*_{m,j}\sum_{j'=1}^N A_{m,j'}\,\mathbb{E}\left[x^*_jx_{j'}x_nx^*_{n'}\right] \\
&\qquad = A^*_{m,n}A_{m,n'}[\mathbf{C}_\mathbf{x}]_{n,n}[\mathbf{C}_\mathbf{x}]_{n',n'},
\end{aligned}
$$

since the term that corresponds to $(j,j') = (n',n)$, i.e. $A^*_{m,n'}A_{m,n}\,\mathbb{E}[x^*_{n'}x^*_{n'}]\,\mathbb{E}[x_nx_n]$, is zero.

Next, for $n = n'$, we have

$$
\begin{aligned}
&\sum_{j=1}^N A^*_{m,j}\sum_{j'=1}^N A_{m,j'}\,\mathbb{E}\left[x^*_jx_{j'}x_nx^*_{n'}\right] \\
&= |A_{m,n}|^2\,\mathbb{E}\left[|x_n|^4\right] + \sum_{j\neq k,j=1}^N |A_{m,j}|^2\,\mathbb{E}\left[|x_j|^2|x_n|^2\right] \\
&= |A_{m,n}|^2\,\mathbb{E}\left[|x_{n,\mathcal{R}}|^4\right] + |A_{m,n}|^2\,\mathbb{E}\left[|x_{n,\mathcal{I}}|^4\right] \\
&\quad + 2|A_{m,n}|^2\,\mathbb{E}\left[|x_{n,\mathcal{R}}|^2|x_{n,\mathcal{I}}|^2\right] \\
&\quad + \sum_{j\neq n,j=1}^N |A_{m,j}|^2 \\
&\qquad \times \mathbb{E}\left[(|x_{j,\mathcal{R}}|^2 + |x_{j,\mathcal{I}}|^2)(|x_{n,\mathcal{R}}|^2 + |x_{n,\mathcal{I}}|^2)\right] \\
&\overset{\text{(a)}}{=} 2|A_{m,n}|^2\,\mathbb{E}\left[|x_{n,\mathcal{R}}|^4\right] \\
&\quad + 2\sum_{j=1}^N |A_{m,j}|^2\,\mathbb{E}\left[|x_{j,\mathcal{R}}|^2|x_{n,\mathcal{I}}|^2\right] \\
&\quad + 2\sum_{j\neq n,j=1}^N |A_{m,j}|^2\,\mathbb{E}\left[|x_{j,\mathcal{R}}|^2|x_{n,\mathcal{R}}|^2\right] \\
&\overset{\text{(b)}}{=} |A_{m,n}|^2[\mathbf{C}_\mathbf{x}]^2_{n,n} + \sum_{j=1}^N |A_{m,j}|^2[\mathbf{C}_\mathbf{x}]_{j,j}[\mathbf{C}_\mathbf{x}]_{n,n},
\end{aligned}
$$

where (a) follows from circular symmetry of $\mathbf{x}$ and (b) from Lemma 1. By combining the above results, we have

$$
\mathbf{V}_m = \mathbf{C}^H_\mathbf{x}\mathbf{a}_m\mathbf{a}^H_m\mathbf{C}_\mathbf{x} + (\mathbf{a}^H_m\mathbf{C}_\mathbf{x}\mathbf{a}_m)(\mathbf{C}^H_\mathbf{x} \odot \mathbf{I})
$$

$$+ ([\mathbf{C}_{\mathbf{e}^z}]_{m,m} - \sigma_m^2)\mathbf{C}_\mathbf{x} = \sigma_x^4 \mathbf{a}_m \mathbf{a}_m^H.$$

H. Derivation of Estimator 2

We now use Theorem 1 to derive Estimator 2 under Assumptions 2. To this end, we require the three quantities: $\overline{\mathcal{T}}(\mathbf{y})$, $\mathbf{T}$, and $\mathbf{V}_m$, $m = 1,\ldots,M$, which we derive separately.

Computing $\overline{\mathcal{T}}(\mathbf{y})$ To derive an expression for $\overline{\mathcal{T}}(\mathbf{y})$ in (27), we need the following two results.

Lemma 2. *Let $\mathbf{u} \sim \mathcal{CN}(\mathbf{0}_{M\times 1}, \Sigma)$ be a complex-valued circularly-symmetric jointly Gaussian random vector with positive definite covariance matrix $\Sigma \in \mathbb{C}^{M\times M}$. Then, for the random variable $\nu = \exp(-\mathbf{u}^H\mathbf{G}\mathbf{u})$ with positive definite $\mathbf{G} \in \mathbb{C}^{M\times M}$ and $\mathbf{G} + \Sigma^{-1}$ positive definite, we have the following result:*

$$\mathbb{E}[\nu] = \frac{1}{|\mathbf{G}\Sigma + \mathbf{I}_M|}.$$

Proof. We first expand the expected value into

$$\mathbb{E}[\nu] = \mathbb{E}\big[\exp(-\mathbf{u}^H\mathbf{G}\mathbf{u})\big] = \int_{\mathbb{C}^M} \exp(-\mathbf{u}^H\mathbf{G}\mathbf{u}) \frac{1}{\pi^M|\Sigma|} \exp(-\mathbf{u}^H\Sigma^{-1}\mathbf{u})d\mathbf{u},$$

where $|\Sigma| > 0$ is the determinant of Σ. We can now simplify the above expression as follows:

$$\begin{aligned}
&\int_{\mathbb{C}^M} \exp(-\mathbf{u}^H\mathbf{G}\mathbf{u}) \frac{1}{\pi^M|\Sigma|} \exp(-\mathbf{u}^H\Sigma^{-1}\mathbf{u})d\mathbf{u} \\
&= \int_{\mathbb{C}^M} \frac{1}{\pi^M|\Sigma|} \exp\big(-\mathbf{u}^H(\mathbf{G}+\Sigma^{-1})\mathbf{u}\big)d\mathbf{u} \\
&= \frac{\pi^M|(\mathbf{G}+\Sigma^{-1})^{-1}|}{\pi^M|\Sigma|} \frac{1}{\pi^M|(\mathbf{G}+\Sigma^{-1})^{-1}|} \\
&\quad \times \int_{\mathbb{C}^M} \exp\big(-\mathbf{u}^H(\mathbf{G}+\Sigma^{-1})\mathbf{u}\big)d\mathbf{u} \\
&= \frac{|(\mathbf{G}+\Sigma^{-1})^{-1}|}{|\Sigma|} = \frac{1}{|\mathbf{G}+\Sigma^{-1}||\Sigma|} = \frac{1}{|\mathbf{G}\Sigma+\mathbf{I}|},
\end{aligned}$$

where we also required that $\mathbf{G}+\Sigma^{-1}$ is positive definite. ■

Lemma 3. *Let $\mathbf{u} \sim \mathcal{N}(\bar{\mathbf{u}}, \Sigma)$ be a real-valued Gaussian random vector with mean $\bar{\mathbf{u}}$ and covariance Σ, and $\boldsymbol{\gamma} \in \mathbb{R}^N$ be a given vector. Then, we have*

$$\mathbb{E}\big[\exp(-\boldsymbol{\gamma}^T\mathbf{u})\big] = \exp\big(-\boldsymbol{\gamma}^T\bar{\mathbf{u}} + \tfrac{1}{2}\boldsymbol{\gamma}^T\Sigma\boldsymbol{\gamma}\big).$$

Proof. The proof is an immediate consequence of the moment generating function of a Gaussian random vector. ■

By considering Lemma 2 and Lemma 3 for scalar random variables, the mth entry of the preprocessed phaseless measurement is given by

$$\begin{aligned}
\overline{\mathcal{T}}(y_m) &= \mathbb{E}[\mathcal{T}(y_m)] = \mathbb{E}\big[\exp(-\gamma|z_m|^2 - \gamma[\mathbf{e}^y]_m)\big] \\
&= \frac{1}{\gamma[\mathbf{C}_\mathbf{z}]_{m,m}+1} \exp\big(-\gamma[\bar{\mathbf{e}}^y]_m + \tfrac{1}{2}\gamma^2[\mathbf{C}_{\mathbf{e}^y}]_{m,m}\big).
\end{aligned}$$

We define the following auxiliary vectors

$$\mathbf{q}_\gamma = \gamma \operatorname{diag}(\mathbf{C}_\mathbf{z}) + \mathbf{1}_{M\times 1} \quad (38)$$

$$\mathbf{p}_\gamma = \exp\big(-\gamma\bar{\mathbf{e}}^y + \tfrac{1}{2}\gamma^2 \operatorname{diag}(\mathbf{C}_{\mathbf{e}^y})\big), \quad (39)$$

which enable us to rewrite the above expression in compact vector form as

$$\overline{\mathcal{T}}(\mathbf{y}) = \mathbf{p}_\gamma \oslash \mathbf{q}_\gamma.$$

Computing T To compute the matrix $\mathbf{T}$ in (30), we only need to compute $\mathbb{E}\big[\mathcal{T}(\mathbf{y})\mathcal{T}(\mathbf{y})^T\big]$, which we will compute entry-wise and in two separate steps. Concretely, we have

$$\begin{aligned}
\mathbb{E}[\mathcal{T}(y_m)\mathcal{T}(y_{m'})] &= \mathbb{E}\big[\exp(-\gamma(|z_m|^2+|z_{m'}|^2))\big] \\
&\quad \times \mathbb{E}[\exp(-\gamma([\mathbf{e}^y]_m + [\mathbf{e}^y]_{m'}))],
\end{aligned}$$

where we compute both expected values separately. In the first step, we compute

$$\mathbb{E}\big[\exp(-\gamma(|z_m|^2+|z_{m'}|^2))\big] = \mathbb{E}\big[\exp(-\mathbf{u}^H\mathbf{G}\mathbf{u})\big],$$

with $\mathbf{u} = [z_m, z_{m'}]^T$ and $\mathbf{G} = \mathbf{I}_2\gamma$. By invoking Lemma 2 with $[\Sigma]_{m,m'} = [\mathbf{C}_\mathbf{z}]_{m,m'}$, we obtain

$$\begin{aligned}
&\mathbb{E}\big[\exp(-\gamma(|z_m|^2+|z_{m'}|^2))\big] = \frac{1}{|\gamma\Sigma+\mathbf{I}_2|} \\
&= \frac{1}{(\gamma[\mathbf{C}_\mathbf{z}]_{m,m}+1)(\gamma[\mathbf{C}_\mathbf{z}]_{m',m'}+1) - \gamma^2|[\mathbf{C}_\mathbf{z}]_{m,m'}|^2}.
\end{aligned}$$

With the definition of $\mathbf{q}_\gamma$ in (38), we can rewrite the above expression in vector form as

$$\begin{aligned}
&\mathbb{E}\big[\exp(-\gamma|\mathbf{z}|^2)\exp(-\gamma|\mathbf{z}|^2)^T\big] \\
&\qquad = \mathbf{1}_{M\times M} \oslash (\mathbf{q}_\gamma\mathbf{q}_\gamma^T - \gamma^2\mathbf{C}_\mathbf{z}\odot\mathbf{C}_\mathbf{z}^*).
\end{aligned}$$

In the second step, we compute

$$\mathbb{E}[\exp(-\gamma([\mathbf{e}^y]_m + [\mathbf{e}^y]_{m'}))] = \mathbb{E}\big[\exp(-\boldsymbol{\gamma}^T\mathbf{u})\big]$$

with $\mathbf{u} = [[\mathbf{e}^y]_m, [\mathbf{e}^y]_{m'}]^T$ and $\boldsymbol{\gamma}^T = [\gamma,\gamma]$. By invoking Lemma 3 with mean $\bar{\mathbf{u}} = [[\bar{\mathbf{e}}^y]_m, [\bar{\mathbf{e}}^y]_{m'}]$ and covariance Σ given by the entries of the covariance matrix $\mathbf{C}_{\mathbf{e}^y}$ associated to the indices m and m', we obtain

$$\begin{aligned}
&\mathbb{E}[\exp(-\gamma([\mathbf{e}^y]_m + [\mathbf{e}^y]_{m'}))] = \exp(-\gamma([\bar{\mathbf{e}}^y]_m + [\bar{\mathbf{e}}^y]_{m'})) \\
&\quad \times \exp(\tfrac{1}{2}\gamma^2([\mathbf{C}_{\mathbf{e}^y}]_{m,m} + [\mathbf{C}_{\mathbf{e}^y}]_{m',m'} + 2[\mathbf{C}_{\mathbf{e}^y}]_{m,m'})).
\end{aligned}$$

With the definition of $\mathbf{p}_\gamma$ in (39), we can rewrite the above expression in vector form as

$$\mathbb{E}\big[\exp(-\gamma\mathbf{e}^y)\exp(-\gamma\mathbf{e}^y)^T\big] = (\mathbf{p}_\gamma\mathbf{p}_\gamma^T) \odot \exp(\gamma^2\mathbf{C}_{\mathbf{e}^y})$$

We furthermore have

$$\overline{\mathcal{T}}(\mathbf{y})\overline{\mathcal{T}}(\mathbf{y})^T = (\mathbf{p}_\gamma\mathbf{p}_\gamma^T) \oslash (\mathbf{q}_\gamma\mathbf{q}_\gamma^T).$$

By combining the two steps with the above results, we have

$$\begin{aligned}
\mathbf{T} = (\mathbf{p}_\gamma\mathbf{p}_\gamma^T) \odot \big(&\exp(\gamma^2\mathbf{C}_{\mathbf{e}^y}) \oslash (\mathbf{q}_\gamma\mathbf{q}_\gamma^T - \gamma^2\mathbf{C}_\mathbf{z}\odot\mathbf{C}_\mathbf{z}^*) \\
&- \mathbf{1}_{M\times M} \oslash (\mathbf{q}_\gamma\mathbf{q}_\gamma^T)\big).
\end{aligned}$$

Computing $\mathbf{V}_m$ To compute the matrices $\mathbf{V}_m$, $m = 1,\ldots,M$, in (31), we only need $\mathbb{E}\left[\mathcal{T}(y_m)\mathbf{x}\mathbf{x}^H\right]$ which we will compute entry-wise and in two steps. We have

$$\mathbb{E}[\mathcal{T}(y_m)x_n x_{n'}^*] = \mathbb{E}\left[\exp(-\gamma|\mathbf{a}_m^H\mathbf{x} + [\mathbf{e}^z]_m|^2)x_n x_{n'}^*\right] \times \mathbb{E}[\exp(-\gamma[\mathbf{e}^y]_m)]\,,$$

where we next compute both expected values separately. As a first step, we use direct integration to compute the following expected value:

$$\mathbb{E}\left[\exp(-\gamma|\mathbf{a}_m^H\mathbf{x} + [\mathbf{e}^z]_m|^2)x_n x_{n'}^*\right] = \int_{\mathbb{C}^{N+1}} \exp(-\gamma|\mathbf{a}_m^H\mathbf{x} + [\mathbf{e}^z]_m|^2) \times \frac{1}{(\pi\sigma_x^2)^N}\exp\left(-\frac{\|\mathbf{x}\|^2}{\sigma_x^2}\right) \times \frac{1}{\pi\sigma_n^2}\exp\left(-\frac{|[\mathbf{e}^z]_m|^2}{\sigma_n^2}\right)x_n x_{n'}^* \mathbf{dx}\mathbf{d}[\mathbf{e}^z]_m.$$

We define the following auxiliary quantities:

$$\begin{aligned}\tilde{\mathbf{a}}_m^H &= [\,\mathbf{a}_m^H, 1\,]\\ \tilde{\mathbf{x}}^T &= [\,\mathbf{x}^T, [\mathbf{e}^z]_m\,]\\ \mathbf{C}_{\tilde{\mathbf{x}}} &= \begin{bmatrix} \sigma_x^2\mathbf{I}_N & \mathbf{0}_{N\times 1}\\ \mathbf{0}_{1\times N} & \sigma_m^2\end{bmatrix}\\ \widetilde{\mathbf{K}}^{-1} &= \gamma\tilde{\mathbf{a}}_m\tilde{\mathbf{a}}_m^H + \mathbf{C}_{\tilde{\mathbf{x}}}^{-1},\end{aligned}$$

where $\sigma_m^2 = \mathbb{E}\left[|[\mathbf{e}^z]_m|^2\right] = [\mathbf{C}_{\mathbf{n}^z}]_{m,m}$. We now derive the above expectation in compact form as

$$\begin{aligned}&\mathbb{E}\left[\exp(-\gamma|\tilde{\mathbf{a}}_m^H\tilde{\mathbf{x}}|^2)\tilde{x}_n\tilde{x}_{n'}^*\right] =\\ &= \frac{1}{(\pi\sigma_x^2)^N}\frac{1}{\pi\sigma_n^2}\int_{\mathbb{C}^{N+1}}\exp(-\gamma|\tilde{\mathbf{a}}^H\tilde{\mathbf{x}}|^2 - \tilde{\mathbf{x}}^H\mathbf{C}_{\tilde{\mathbf{x}}}^{-1}\tilde{\mathbf{x}})\tilde{x}_n\tilde{x}_{n'}^*\mathbf{d}\tilde{\mathbf{x}}\\ &= \frac{1}{|\pi\mathbf{C}_{\tilde{\mathbf{x}}}|}\int_{\mathbb{C}^{N+1}}\exp(-\tilde{\mathbf{x}}^H(\gamma\tilde{\mathbf{a}}_m\tilde{\mathbf{a}}_m^H + \mathbf{C}_{\tilde{\mathbf{x}}}^{-1})\tilde{\mathbf{x}})\tilde{x}_n\tilde{x}_{n'}^*\mathbf{d}\tilde{\mathbf{x}}\\ &= \frac{1}{|\pi\mathbf{C}_{\tilde{\mathbf{x}}}|}\int_{\mathbb{C}^{N+1}}\exp(-\tilde{\mathbf{x}}^H\widetilde{\mathbf{K}}^{-1}\tilde{\mathbf{x}})\tilde{x}_n\tilde{x}_{n'}^*\mathbf{d}\tilde{\mathbf{x}},\end{aligned}$$

where $n = 1,\ldots,N+1$, $n' = 1,\ldots,N+1$. We can further rewrite this expression as

$$\frac{1}{|\pi\mathbf{C}_{\tilde{\mathbf{x}}}|}\int_{\mathbb{C}^{N+1}}\exp(-\tilde{\mathbf{x}}^H\widetilde{\mathbf{K}}^{-1}\tilde{\mathbf{x}})\tilde{x}_n\tilde{x}_{n'}^*\mathbf{d}\tilde{\mathbf{x}} = \frac{|\pi\widetilde{\mathbf{K}}|}{|\pi\widetilde{\mathbf{K}}||\pi\mathbf{C}_{\tilde{\mathbf{x}}}|}\int_{\mathbb{C}^{N+1}}\exp(-\tilde{\mathbf{x}}^H\widetilde{\mathbf{K}}^{-1}\tilde{\mathbf{x}})\tilde{x}_n\tilde{x}_{n'}^*\mathbf{d}\tilde{\mathbf{x}}.$$

It is now key to realize that

$$\frac{1}{|\pi\widetilde{\mathbf{K}}|}\int_{\mathbb{C}^{N+1}}\exp(-\tilde{\mathbf{x}}^H\widetilde{\mathbf{K}}^{-1}\tilde{\mathbf{x}})\tilde{x}_n\tilde{x}_{n'}^*\mathbf{d}\tilde{\mathbf{x}} = \mathbb{E}[\tilde{x}_n\tilde{x}_{n'}^*] = [\widetilde{\mathbf{K}}]_{n,n'}$$

and hence we have

$$\begin{aligned}&\mathbb{E}\left[\exp(-\gamma|\tilde{\mathbf{a}}_m^H\tilde{\mathbf{x}}|^2)\tilde{x}_n\tilde{x}_{n'}^*\right]\\ &= \frac{|\widetilde{\mathbf{K}}|}{|\mathbf{C}_{\tilde{\mathbf{x}}}|}[\widetilde{\mathbf{K}}]_{n,n'} = \frac{1}{|\widetilde{\mathbf{K}}^{-1}||\mathbf{C}_{\tilde{\mathbf{x}}}|}[\widetilde{\mathbf{K}}]_{n,n'}\\ &= \frac{1}{|\gamma\tilde{\mathbf{a}}_m\tilde{\mathbf{a}}_m^H + \mathbf{C}_{\tilde{\mathbf{x}}}^{-1}||\mathbf{C}_{\tilde{\mathbf{x}}}|}[\widetilde{\mathbf{K}}]_{n,n'}\\ &= \frac{1}{|\gamma\tilde{\mathbf{a}}_m\tilde{\mathbf{a}}_m^H\mathbf{C}_{\tilde{\mathbf{x}}} + \mathbf{I}_{N+1}|}[\widetilde{\mathbf{K}}]_{n,n'}.\end{aligned}$$

We can now use the matrix-determinant lemma to simplify

$$\begin{aligned}|\gamma\tilde{\mathbf{a}}_m\tilde{\mathbf{a}}_m^H\mathbf{C}_{\tilde{\mathbf{x}}} + \mathbf{I}_{N+1}| &= \gamma\tilde{\mathbf{a}}_m^H\mathbf{C}_{\tilde{\mathbf{x}}}\tilde{\mathbf{a}}_m + 1\\ &= \gamma(\sigma_x^2\|\mathbf{a}_m\|^2 + \sigma_m^2) + 1\end{aligned}$$

and the matrix inversion lemma to simplify

$$\begin{aligned}\widetilde{\mathbf{K}} &= (\gamma\tilde{\mathbf{a}}_m\tilde{\mathbf{a}}_m^H + \mathbf{C}_{\tilde{\mathbf{x}}}^{-1})^{-1}\\ &= \mathbf{C}_{\tilde{\mathbf{x}}} - \frac{\gamma\mathbf{C}_{\tilde{\mathbf{x}}}\tilde{\mathbf{a}}_m\tilde{\mathbf{a}}_m^H\mathbf{C}_{\tilde{\mathbf{x}}}}{\gamma\tilde{\mathbf{a}}_m^H\mathbf{C}_{\tilde{\mathbf{x}}}\tilde{\mathbf{a}}_m + 1}\\ &= \mathbf{C}_{\tilde{\mathbf{x}}} - \frac{\gamma\mathbf{C}_{\tilde{\mathbf{x}}}\tilde{\mathbf{a}}_m\tilde{\mathbf{a}}_m^H\mathbf{C}_{\tilde{\mathbf{x}}}}{\gamma(\sigma_x^2\|\mathbf{a}_m\|^2 + \sigma_m^2) + 1}.\end{aligned}$$

By using these two simplifications, we have

$$\begin{aligned}&\mathbb{E}\left[\exp(-\gamma|\tilde{\mathbf{a}}_m^H\tilde{\mathbf{x}}|^2)\tilde{x}_n\tilde{x}_{n'}^*\right]\\ &= \frac{1}{\gamma(\sigma_x^2\|\mathbf{a}_m\|^2 + \sigma_m^2) + 1}\\ &\quad\times\left[\mathbf{C}_{\tilde{\mathbf{x}}} - \frac{\gamma\mathbf{C}_{\tilde{\mathbf{x}}}\tilde{\mathbf{a}}_m\tilde{\mathbf{a}}_m^H\mathbf{C}_{\tilde{\mathbf{x}}}}{\gamma(\sigma_x^2\|\mathbf{a}_m\|^2 + \sigma_m^2) + 1}\right]_{n,n'}\end{aligned}$$

and since we are only interested in the upper $N \times N$ part of the matrix $\widetilde{\mathbf{K}}$, we have

$$\begin{aligned}&\mathbb{E}\left[\exp(-\gamma|\mathbf{a}_m^H\mathbf{x} + [\mathbf{e}^z]_m|^2)x_n x_{n'}^*\right]\\ &= \frac{1}{\gamma(\sigma_x^2\|\mathbf{a}_m\|^2 + \sigma_m^2) + 1}\\ &\quad\times\left[\sigma_x^2\mathbf{I}_N - \frac{\gamma\sigma_x^4\mathbf{a}_m\mathbf{a}_m^H}{\gamma(\sigma_x^2\|\mathbf{a}_m\|^2 + \sigma_m^2) + 1}\right]_{n,n'}\\ &= \frac{1}{\gamma[\mathbf{C}_\mathbf{z}]_{m,m} + 1}\left[\sigma_x^2\mathbf{I}_N - \frac{\gamma\sigma_x^4\mathbf{a}_m\mathbf{a}_m^H}{\gamma[\mathbf{C}_\mathbf{z}]_{m,m} + 1}\right]_{n,n'}\end{aligned}$$

since for our assumptions

$$\sigma_x^2\|\mathbf{a}_m\|^2 + \sigma_m^2 = [\mathbf{C}_\mathbf{z}]_{m,m}.$$

In compact matrix form, we have

$$\begin{aligned}&\mathbb{E}\left[\exp(-\gamma|\mathbf{a}_m^H\mathbf{x} + [\mathbf{e}^z]_m|^2)\mathbf{x}\mathbf{x}^H\right]\\ &\quad= \frac{1}{\gamma[\mathbf{C}_\mathbf{z}]_{m,m} + 1}\left(\sigma_x^2\mathbf{I}_N - \frac{\gamma\sigma_x^4\mathbf{a}_m\mathbf{a}_m^H}{\gamma[\mathbf{C}_\mathbf{z}]_{m,m} + 1}\right).\end{aligned}$$

As a second step, we use definition (39) and obtain

$$\mathbb{E}[\exp(-\gamma[\mathbf{e}^y]_m)] = [\mathbf{p}_\gamma]_m.$$

By combining both steps, we obtain

$$\mathbf{V}_m = \frac{[\mathbf{p}_\gamma]_m}{\gamma[\mathbf{C}_\mathbf{z}]_{m,m}+1}\left(\sigma_x^2\mathbf{I}_N - \frac{\gamma\sigma_x^4\mathbf{a}_m\mathbf{a}_m^H}{\gamma[\mathbf{C}_\mathbf{z}]_{m,m}+1}\right) - \frac{[\mathbf{p}_\gamma]_m}{\gamma[\mathbf{C}_\mathbf{z}]_{m,m}+1}\sigma_x^2\mathbf{I}_N$$

$$= -\frac{\gamma\sigma_x^4[\mathbf{p}_\gamma]_m}{(\gamma[\mathbf{C}_\mathbf{z}]_{m,m}+1)^2}\mathbf{a}_m\mathbf{a}_m^H,$$

which is what we desperately wanted to show.