A. Appendix: Proof of Lemma 1

Lemma 1 (Fano’s inequality Fano (1949)).
For any Markov chain $V \rightarrow X$ $\rightarrow \hat{V}$, we have

$$h(P(\hat{V} \neq V)) + P(\hat{V} \neq V)(\log(|\mathcal{Y}| - 1)) \geq H(V|\hat{V}),$$

where

$$h(p) = -p \log p - (1 - p) \log(1 - p).$$

$\mathcal{Y}$ is the set of possible value of $V$, and $H(V|\hat{V})$ is the entropy of $V$ conditioned on $\hat{V}$.

Proof of Lemma 1.
This inequality follows by expanding the entropy in two different ways. Let $E$ be the indicator random variable for the event that $\hat{V} \neq V$, that is, $E = 1$ if $\hat{V} \neq V$ and is 0 otherwise. Then we have

$$H(V,E|\hat{V}) = H(V|\hat{V}) + H(E|\hat{V})$$

$$= P(E = 1)H(V|E = 1, \hat{V}) + P(E = 0)H(V|E = 0, \hat{V}) + H(E|\hat{V})$$

$$= P(E = 1)H(V|E = 1, \hat{V}) + H(E|\hat{V}),$$

where the last equation follows because there is no error, and $V$ has no variability given $\hat{V}$ if $E = 0$. Expanding the entropy by the chain rule in a different order, we have

$$H(V,E|\hat{V}) = H(V|\hat{V}) + H(E|V, \hat{V})$$

$$= H(V|\hat{V}),$$

because $E$ is perfectly determined by $V$ and $\hat{V}$. Combining these two equations, we have

$$H(V|\hat{V}) = P(E = 1)H(V|E = 1, \hat{V}) + H(E|\hat{V})$$

$$\leq P(\hat{V} \neq V)(|\mathcal{Y}| - 1) + h(P(\hat{V} \neq V)),$$

where the first inequality follows because $V$ can take on at most $|\mathcal{Y}| - 1$ values when there is an error, and the last inequality follows by $H(E|\hat{V}) \leq H(E) = h(P(\hat{V} \neq V))$. The proof is complete.

B. Appendix: Detailed Derivation of Inference Algorithm

In this section, we show the detailed derivation of an inference algorithm for the proposed WC model.

B.1. Empirical Variational Inference

To estimate latent variables and parameters, we adopt the strategy of empirical variational inference.

We want to maximize a following marginal log likelihood or model evidence.

$$\log p(X|\alpha, \beta, \Lambda) = \log \int p(X,G,\pi|\rho, \tau, \Lambda)dGd\pi.$$

However it is impossible to optimize the RHS’s integral analytically. Therefore, we will instead maximize the following lower bound (ELBO).

$$\log \int p(X,G,\pi|\rho, \tau, \Lambda)dGd\pi$$

$$\geq \int q(G,\pi) \log \left( \frac{p(X,G,\pi|\rho, \tau, \Lambda)}{q(G,\pi)} \right)dGd\pi$$

$$:= ELBO.$$

We make an assumption of the mean field approximation for $q(G,\pi)$, that is,

$$q(G,\pi) = \left[ \prod_{i=1}^{n} q(G_i) \right] \left[ \prod_{j=1}^{n} q(\pi^j) \right].$$
On the basis on the variational method, maximizing ELBO yields the following expression.

\[ q(G_i) = \text{Mutinomial}(G_i|\theta), \]
\[ q(\pi^j) = \text{Mutinomial}(f_K^{-1}(\pi^j)|\phi_j), \]

where \( \theta = (\theta_{j,k}^m)_{j=1}^m \), \( \phi = (\phi_{j,l})_{j=1}^m \) are variational parameters. They satisfy the following equations.

\[ \hat{\theta}_{i,k} = \frac{\exp(\sum_{j=1}^m \frac{1}{Z} \sum_{k=1}^K \sum_{l=1}^L \phi_{j,l} \log g_{X_i,j}^{L,k} + \log \Lambda_{i,k,l} + \log \rho_k)}{\sum_{k'=1}^K \sum_{l'=1}^L \phi_{j,l'} \exp(\sum_{m'=1}^m \frac{1}{Z} \sum_{k'=1}^K \sum_{l'=1}^L \phi_{j,l'} \log g_{X_i,j}^{L,k'} + \log \Lambda_{i,k',l'} + \log \rho_{k'})}, \]

(1)

\[ \hat{\phi}_{j,l} = \frac{\exp(\sum_{m=1}^n \frac{1}{Z} \sum_{k=1}^K \sum_{l=1}^L \phi_{j,l} \log g_{X_i,j}^{L,k})}{\sum_{m'=1}^n \sum_{k'=1}^K \sum_{l'=1}^L \phi_{j,l'} \exp(\sum_{m'=1}^n \frac{1}{Z} \sum_{k'=1}^K \sum_{l'=1}^L \phi_{j,l'} \log g_{X_i,j}^{L,k'})}. \]

(2)

We can calculate ELBO analytically as follows.

\[ \text{ELBO} = \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^K \sum_{l=1}^L \hat{\theta}_{i,k} \hat{\phi}_{j,l} \log (X_{i,j} = k') \log \Lambda_{i,k,l} \]

\[ + \sum_{i=1}^n \sum_{k=1}^K \hat{\theta}_{i,k} (\log \rho_k - \log \hat{\theta}_{i,k}) + \sum_{j=1}^m \sum_{l=1}^L \hat{\phi}_{j,l} (\log \tau_l - \log \hat{\phi}_{j,l}). \]

(3)

We want to separate all groups. The ability of each group \( l \in \{1, 2, ..., L\} \) is measured by \( \Lambda_l \). The bigger the diagonal components of each \( \Lambda_l \), are the higher the accuracy of group \( l \) is. Thus, we first calculate the trace norm of each \( \pi^j \), namely \( ||\pi^j|| \), and then rearrange all workers in descending order according to \( ||\pi^j|| \) to obtain the permutation of workers \( \sigma = \left( \begin{array}{ccc} 1 & 2 & \ldots & m \\ \sigma(1) & \sigma(2) & \ldots & \sigma(m) \end{array} \right) \)

where \( \sigma(j') = j \) means that the trace norm of the worker \( j \) is the \( j' \)-th largest. Second, we separate \( ||\pi^j|| \) into \( L \) groups. In other words, let \( J_l = \left\{ \sigma \left( \frac{m-l+1}{L} \right), \ldots, \sigma \left( \frac{m}{L} \right) \right\} \). Using this set, the initial values of \( \Lambda \) and \( \phi \) are below.

\[ \Lambda_{i,k,l} = L \sum_{j=1}^m \pi_{i,k,l}^{j}, \]

\[ \phi_{j,l} = \delta(j \in J_l). \]

### Algorithm 1

**Empirical Variational Inference for Worker Clustering Model**

**Input:** \( X \)

**Output:** \( \hat{G}, \hat{\pi} \)

1. Initialize \( \hat{\theta}, \hat{\phi}, \rho, \tau, \Lambda \) appropriately (discussed later).
2. Update variational parameters by using (1) and (2)
3. Update hyper parameters by using (4)–(6)
4. Calculate ELBO (3), then compare it with previous ELBO value. If the increment is smaller than the threshold, go to the step [5], otherwise repeat from step [2]
5. Calculate estimated values \( \hat{G}, \hat{\pi} \) by using (7) and (8)

The initialization process of \( \hat{\theta} \) is the same as that of Dawid & Skene (1979) given by

\[ \hat{\theta}_{i,k} = \frac{\sum_{j=1}^m \delta(X_{i,j} = k)}{\sum_{k=1}^K \sum_{j=1}^m \delta(X_{i,j} = k)}. \]

The initialization process of \( \hat{\phi} \) and \( \Lambda \) is a bit more complicated. First, we approximately calculate confusion matrices \( \pi = \{\pi^j\}_{j=1}^m \) of each worker \( j \) as done by Dawid & Skene (1979) as follows.

\[ \pi_{k,l}^j = \frac{\sum_{i=1}^n \hat{\theta}_{i,k} \delta(X_{i,j} = k')}{\sum_{k'=1}^K \sum_{l'=1}^L \hat{\theta}_{i,k} \delta(X_{i,j} = k')}. \]

We want to separate all \( m \) workers into \( L \) groups. The ability of each group \( l \in \{1, 2, ..., L\} \) is measured by \( \Lambda_l \). The bigger the diagonal components of each \( \Lambda_l \), are the higher the accuracy of group \( l \) is. Thus, we first calculate the trace norm of each \( \pi^j \), namely \( ||\pi^j|| \), and then rearrange all workers in descending order according to \( ||\pi^j|| \) to obtain the permutation of workers \( \sigma = \left( \begin{array}{ccc} 1 & 2 & \ldots & m \\ \sigma(1) & \sigma(2) & \ldots & \sigma(m) \end{array} \right) \)

where \( \sigma(j') = j \) means that the trace norm of the worker \( j \) is the \( j' \)-th largest. Second, we separate \( ||\pi^j|| \) into \( L \) groups. In other words, let \( J_l = \left\{ \sigma \left( \frac{m-l+1}{L} \right), \ldots, \sigma \left( \frac{m}{L} \right) \right\} \). Using this set, the initial values of \( \Lambda \) and \( \phi \) are below.

\[ \Lambda_{i,k,l} = L \sum_{j=1}^m \pi_{i,k,l}^{j}, \]

\[ \phi_{j,l} = \delta(j \in J_l). \]

### References
