Supplement of “Network Global Testing by Counting Graphlets”

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Abstract
This is the supplementary material of (Jin et al., 2018). It contains Proposition A.1 and the proofs of Theorems 3.2-3.3, Corollary 3.1, and secondary lemmas.

A. An Alternative Expression of the GC Test Statistics
We rewrite the test statistic $\hat{\chi}_{gc}$ (as well as $\hat{L}_2$, $\hat{L}_3$ and $\hat{C}_1$) explicitly as a function of the adjacency matrix $A$. The following proposition is proved in Section D.4.

Proposition A.1 The following are true:

$$\hat{L}_2 = \frac{1}{6(\frac{\theta}{3})} [1' A^2 1 - \text{tr}(A^2)],$$

$$\hat{L}_3 = \frac{1}{24(\frac{\theta}{4})} [1' A^3 1 - 2(1'A^2 1) + 1' A 1 - \text{tr}(A^3)],$$

and

$$\hat{C}_1 = \frac{1}{24(\frac{\theta}{4})} \left[ \text{tr}(A^4) - 2(1'A^2 1) + 1' A 1 \right].$$

Furthermore,

$$\hat{\chi}_{gc} = \frac{\text{tr}(A^4) - 2(1'A^2 1) + 1' A 1}{n(n-1)(n-2)(n-3)}$$

$$- \frac{1}{(n-3)^4} \left[ 1'A^3 1 - 2(1'A^2 1) + 1' A 1 - \text{tr}(A^3) \right]^4.$$

B. Proof of Theorem 3.2
We prove the case $m = 4$. The case of $m = 3$ is similar and thus omitted. From now on, we omit the superscripts "(4)"

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in all related quantities (e.g., we write $\delta_{gc}^{(4)}$ as $\delta_{gc}$). Write

$$\sqrt{\frac{3(n)}{4}}\frac{(\hat{\chi}_{gc} - \chi_{gc})}{\sqrt{\hat{C}_4}} = \sqrt{\frac{C_4}{C_4}} \cdot (I + II)$$

(1)

where

$$I = \sqrt{\frac{3(n)}{4}}(\hat{C}_4 - C_4),$$

$$II = -\sqrt{\frac{3(n)}{4}} \left[ (\hat{L}_3 - \frac{1}{L_2})^4 - \left( \frac{L_3}{L_2} \right)^4 \right].$$

Using the Slutsky’s theorem, it suffices to show that

$$\hat{C}_4 / C_4 \overset{D}{\rightarrow} 1,$$

$$I \overset{D}{\rightarrow} N(0, 1),$$

and

$$II \overset{D}{\rightarrow} 0.$$ (4)

The following lemma is useful, and its proof can be found in Section D.

Lemma B.1 Under the assumptions of Theorem 3.2,

$$C_4 \approx n^{-4} \|\theta\|^8, \quad L_2 \approx n^{-3} \|\theta\|_1^2 \|\theta\|^2,$$

$$L_3 \approx n^{-4} \|\theta\|_1^4 \|\theta\|^4.$$ Moreover,

$$\text{Var}(\hat{C}_4) \leq Cn^{-8}\|\theta\|^8, \quad \text{Var}(\hat{L}_2) \leq Cn^{-6}\|\theta\|_1^3 \|\theta\|^3,$$

$$\text{Var}(\hat{L}_3) \leq Cn^{-8}\|\theta\|_1^4 \|\theta\|^4.$$ We now show (2)-(4). The proof of (3) is relatively long, so we prove it in the end.

First, we prove (2). Recall that $C_4 = E[\hat{C}_4]$. By Lemma B.1,

$$E((\hat{C}_4 / C_4 - 1)^2) = C_4^{-2} \text{Var}(\hat{C}_4) = O(\|\theta\|^{-8}),$$

where the right hand side $\rightarrow 0$ as $\|\theta\| \rightarrow \infty$. The claim follows by elementary probability theory.

Second, we prove (4). Define $\tilde{L}_2 = (\|\theta\|^2 / n)\hat{L}_2$ and $L^*_2 = (\|\theta\|^2 / n)L_2$. Using Lemma B.1, it follows from direct calculations that

$$L_3 / L^*_2 = O(1).$$ (5)

where the right hand side $\rightarrow 0$ as $\|\theta\| \rightarrow \infty$. The claim follows by elementary probability theory.
With these notations, we have
\[
|II| = \frac{\sqrt{3}^\theta}{\sqrt{C_4}} \cdot n^4 \left( \frac{\hat{L}_3}{L_2} \right)^4 - \left( \frac{L_3}{L_2} \right)^4 \leq C \|\theta\|^4 \left( \frac{\hat{L}_3}{L_2} \right)^4 - \left( \frac{L_3}{L_2} \right)^4,
\]
where we have used $C_4 \approx n^{-4} \|\theta\|^4$ in the second equality; see Lemma B.1. Note that for any $(x, y)$, $|x^4 - y^4| = |(x-y)(x^3 + x^2 y + x y^3 + y^3)| \leq 3|x-y| \cdot (|x| + |y|)^3$. It follows that
\[
|II| \leq C \cdot \|\theta\|^4 |Z| \cdot \left( \frac{L_3}{L_2} + |Z| \right)^3,
\]
where for short we write
\[
Z = \frac{\hat{L}_3}{L_2} - \frac{L_3}{L_2}.
\]
Recall that $L_3/L_2$ is bounded. Therefore, to show (4), it suffices to show
\[
\|\theta\|^4 \left( \frac{\hat{L}_3}{L_2} - \frac{L_3}{L_2} \right) \overset{p}{\to} 0. \tag{6}
\]
Below, we show (6). Write the term on the left by
\[
\frac{\|\theta\|^4}{L_2^4} (\hat{L}_3 - L_3) + \|\theta\|^4 \left( \frac{\hat{L}_3}{L_2^2} - \frac{L_3}{L_2^2} \right) \equiv II_a + II_b.
\]
To show (6), it suffices to show
\[
II_a \overset{p}{\to} 0. \tag{7}
\]
and
\[
II_b \overset{p}{\to} 0. \tag{8}
\]
Consider (7). Note that $L_3 = \mathbb{E}[\hat{L}_3]$. It follows from Lemma B.1 that
\[
\text{Var}(II_a) = \frac{\|\theta\|^8 \text{Var}(\hat{L}_3)}{(L_2^* )^2} \leq C \frac{\|\theta\|^8 \cdot n^{-8} \|\theta\|^4 n^8}{(n^{-4} \|\theta\|_1^4 \|\theta\|^4)^2} \leq C \|\theta\|^8, \tag{9}
\]
where the last term → 0 for $\|\theta\|_3 \to 0$ as $n \to \infty$; this is due to equation (7) of Jin et al., 2018. By elementary probability, (7) follows.

Consider (8). To show the claim, we first show
\[
\frac{\hat{L}_3}{L_2} \overset{p}{\to} 1, \quad \frac{\hat{L}_3}{L_3} \overset{p}{\to} 1; \tag{9}
\]
as the proofs are similar, we only show the first one. By Lemma B.1, \(\text{Var}(\hat{L}_3) = O(n^{-6} \|\theta\|_1^4 \|\theta\|^4)\) and $L_2 \asymp n^{-3} \|\theta\|_2^2$. Using $\mathbb{E}[\hat{L}_3] = L_2$, $\mathbb{E}[L_3/L_2 - 1]^2 = L_2^{-2} \text{Var}(\hat{L}_2) \leq C (\|\theta\|^3/(\|\theta\|^3 \|\theta\|^4))$, which $\leq C/\|\theta\|^2$ since $\|\theta\|^3 \leq \|\theta\|_1^4 \|\theta\|^4$. This shows (9).

Using (9) and recalling $L_3/L_2 \leq C$ (see (5)), to show (8), it is sufficient to show
\[
\|\theta\|^4 \left( \frac{\hat{L}_3}{L_2} - \frac{L_3}{L_2} \right) \overset{p}{\to} 0,
\]
and since $\hat{L}_3/L_2 = \hat{L}_2/L_2$, it is equivalent to show
\[
\|\theta\|^4 \left( \hat{L}_2 - 1 \right) \overset{p}{\to} 0. \tag{10}
\]
Last, we prove (3). We need some notations. Given 4 distinct nodes, there are 3 different possible cycles, denoted as $CC(i_1, i_2, i_3, i_4) = \{(i_1, i_2, i_3, i_4), (i_1, i_2, i_4, i_3), (i_1, i_3, i_2, i_4)\}$; moreover, for $B \subset \{1, 2, ..., n\}$, let $CC(B) = \bigcup_{(i_1, i_2, i_3, i_4) \in B} CC(i_1, i_2, i_3, i_4)$. For $1 \leq m \leq n$, let $I_m$ be the collection of $(i_1, i_2, i_3, i_4)$ such that $1 \leq i_1 < i_2 < i_3 < i_4 = m$. Write $\Omega_{ij} = \Omega_{ij} (1 - \Omega_{ij})$. Let $W = A - \Omega$. Define
\[
S_{n,n} = \sum_{CC(I_m)} W_{i_1 i_2} W_{i_2 i_3} W_{i_3 i_4} W_{i_4 i_1} \sqrt{\sum_{CC(I_m)} \Omega_{i_1 i_2} \Omega_{i_2 i_3} \Omega_{i_3 i_4} \Omega_{i_4 i_1}}.
\]
The following lemma is proved in Section D.

Lemma B.2 Under the conditions of Theorem 3.2,
\[
\frac{\sqrt{3}^n}{\sqrt{C_4}} (\hat{C}_4 - C_4) - S_{n,n} \overset{p}{\to} 0.
\]
By Lemma B.2, to show (3), it suffices to show that
\[
S_{n,n} \overset{d}{\to} N(0, 1). \tag{11}
\]
Below, we prove (11). For $1 \leq m \leq n$, define the $\sigma$-algebra $\mathcal{F}_{n,m} = \sigma(\{ A_{ij} \}_{1 \leq i < j \leq m})$ and
\[
X_{n,m} = S_{n,m} - S_{n,m-1},
\]
where $S_{n,0} = 0$ and
\[
S_{n,m} = \frac{\sum_{CC(I_m)} W_{i_1 i_2} W_{i_2 i_3} W_{i_3 i_4} W_{i_4 i_1}}{\sqrt{\sum_{CC(I_m)} \Omega_{i_1 i_2} \Omega_{i_2 i_3} \Omega_{i_3 i_4} \Omega_{i_4 i_1}}}, \quad 1 \leq m \leq n.
\]
It is easy to see that $\mathbb{E}[S_{n,m} | \mathcal{F}_{n,m-1}] = S_{n,m-1}$. Hence, $\{X_{n,m}\}_{m=1}^n$ is a martingale difference sequence relative to the filtration $\{\mathcal{F}_{n,m}\}_{m=1}^n$, and $S_{n,n} = \sum_{m=1}^n X_{n,m}$. To show (11), we apply the martingale central limit theorem in (Hall & Heyde, 2014) and check:

(a) $\sum_{m=1}^n \mathbb{E}[(X_{n,m}^2 | \mathcal{F}_{n,m-1})] \overset{p}{\to} 1$. 

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\[(b) \quad \sum_{m=1}^{n} \mathbb{E}(X_{n,m}^2 | (X_{n,m} > \epsilon) \mathcal{F}_{n,m-1}) \xrightarrow{p} 0, \text{ for any } \epsilon > 0.\]

Note that once we have checked that both conditions (a) and (b) are satisfied, then by the martingale central limit theorem, \(\mathcal{S}_{n,m} \xrightarrow{d} N(0,1)\). Combining it with Lemma B.2, we have proved (3).

It remains to check (a)-(b). For preparation, we first derive an alternative expression of \(\mathbb{E}(X_{n,m} | \mathcal{F}_{n,m-1})\) as (14) below. By definition,

\[X_{n,m} = \frac{1}{\sqrt{M_n}} \sum_{CC(I_m) \setminus CC(I_{m-1})} W_{i_1i_2} W_{i_2i_3} W_{i_3i_4} W_{i_4i_1},\]

where \(M_n = \sum_{CC(I_m)} \Omega_{i_1i_2}^{*} \Omega_{i_2i_3}^{*} \Omega_{i_3i_4}^{*} \Omega_{i_4i_1}^{*}\), and the summation is over all 4-cycles in \(CC(I_m) \setminus CC(I_{m-1})\). Note that a cycle in \(CC(I_m) \setminus CC(I_{m-1})\) has to include the node \(m\). Hence, we can consider the following way to get all such cycles: First, select 2 indices \((i, j)\) from \([1, 2, \ldots, m-1]\) and use them as the two neighboring nodes of \(m\); second, select an index \(k \in [1, 2, \ldots, m-1] \setminus \{i, j\}\) as the last node in the cycle. This allows us to write

\[X_{n,m} = \frac{1}{\sqrt{M_n}} \sum_{1 \leq i < j \leq m-1} W_{mi} W_{mj} \cdot Y_{(m-1)ij},\]

where

\[Y_{(m-1)ij} = \sum_{1 \leq k \leq m-1, k \notin \{i,j\}} W_{ki} W_{kj}.\]

Conditioning on \(\mathcal{F}_{n,m-1}\), \(\{W_{mi} W_{mj}\}_{1 \leq i < j \leq m-1}\) are mutually uncorrelated and \(Y_{(m-1)ij}\) is a constant. Hence, it follows from (12)-(13) that

\[\mathbb{E}(X_{n,m}^2 | \mathcal{F}_{n,m-1}) = \frac{1}{M_n} \sum_{1 \leq i < j \leq m-1} Y_{(m-1)ij}^{*} \Omega_{mi}^{*} \Omega_{mj}^{*}.\]  \hspace{1cm} (14)

We now check (a). It suffices to show that

\[\mathbb{E}\left[\sum_{m=1}^{n} \mathbb{E}(X_{n,m}^2 | \mathcal{F}_{n,m-1})\right] = 1,\]  \hspace{1cm} (15)

and

\[\text{Var}\left(\sum_{m=1}^{n} \mathbb{E}(X_{n,m}^2 | \mathcal{F}_{n,m-1})\right) \to 0.\]  \hspace{1cm} (16)

Consider (15). In the definition (13), the terms in the sum are (unconditionally) mutually uncorrelated. As a result, \[\mathbb{E}[Y_{(m-1)ij}^2] = \sum_{k \neq m, k \notin \{i,j\}} \Omega_{ki}^{*} \Omega_{kj}^{*}.\]

It follows that

\[\mathbb{E}\left[\sum_{m=1}^{n} \mathbb{E}(X_{n,m}^2 | \mathcal{F}_{n,m-1})\right] = \frac{1}{M_n} \sum_{m=1}^{n} \left(\sum_{1 \leq i < j \leq m-1} \sum_{k \neq \ell} \Omega_{ki}^{*} \Omega_{kj}^{*} \Omega_{mi}^{*} \Omega_{mj}^{*}\right)\]

\[= \frac{1}{M_n} \sum_{m=1}^{n} \sum_{(i,j,k) \in CC(I_m)} \Omega_{mi}^{*} \Omega_{mj}^{*} = 1.\]  \hspace{1cm} (17)

This proves (17).

Consider (16). We first decompose the random variable \[\sum_{m=1}^{n} \mathbb{E}(X_{n,m}^2 | \mathcal{F}_{n,m-1})\] into the sum of two parts, and then calculate its variance. By (13),

\[Y_{(m-1)ij}^2 = \sum_{k} W_{ki}^{2} W_{kj}^{2} + \sum_{k \neq \ell} W_{ki} W_{kj} W_{\ell i} W_{\ell j},\]

where \(k \text{ and } \ell\) range in \([1, 2, \ldots, m-1] \setminus \{i, j\}\). Plugging it into (14), we have a decomposition

\[\sum_{m=1}^{n} \mathbb{E}(X_{n,m}^2 | \mathcal{F}_{n,m-1}) = I_a + I_b,\]  \hspace{1cm} (18)

where

\[I_a = \frac{1}{M_n} \sum_{m=1}^{n} \sum_{1 \leq i < j \leq m-1} \sum_{k \neq \ell} W_{ki}^{2} W_{kj}^{2} \Omega_{mi}^{*} \Omega_{mj}^{*},\]

\[I_b = \frac{1}{M_n} \sum_{m=1}^{n} \sum_{1 \leq i < j \leq m-1} \sum_{k \neq \ell} W_{ki} W_{kj} W_{\ell i} W_{\ell j} \Omega_{mi}^{*} \Omega_{mj}^{*}.\]

Then,

\[\text{Var}\left(\sum_{m=1}^{n} \mathbb{E}(X_{n,m}^2 | \mathcal{F}_{n,m-1})\right) = \text{Var}(I_a) + \text{Var}(I_b) + 2\text{Cov}(I_a, I_b) \leq \left(\sqrt{\text{Var}(I_a)} + \sqrt{\text{Var}(I_b)}\right)^2 \]  \hspace{1cm} (19)

It suffices to show that both \(\text{Var}(I_a) \to 0\) and \(\text{Var}(I_b) \to 0\).

Consider the variance of \(I_a\). In the sum of \(I_a\), all 4-cycles \((k, i, m, j)\) involved are selected in this way: We first select \(m\), then select a pair \((i, j)\) from \([1, 2, \ldots, m-1]\) and connect both \(i\) and \(j\) to \(m\), and finally select \(k\) to close the cycle. In fact, these 4-cycles can be selected in an alternative way: First, select a V-shape \((i, k, j)\) with \(k\) being the middle point. Second, select \(m > \max\{i, j, k\}\) to make the V-shape a cycle. Hence, we can rewrite

\[I_a = \frac{1}{M_n} \sum_{k=1}^{n} \sum_{1 \leq i < j \leq k} W_{ki}^{2} W_{kj}^{2} \sum_{m=1}^{\max\{i,j,k\}} \Omega_{mi}^{*} \Omega_{mj}^{*}.\]
The terms $W^2_{k_1} W^2_{k_2}$ corresponding to different $k$ are independent of each other. We now fix $k$ and calculate the covariance between $W^2_{k_1} W^2_{j_1}$ and $W^2_{k_2} W^2_{j_2}$ for $(i, j) \neq (i', j')$. There are three cases. Case (i): $(i, j) = (i', j')$. In this case, $\text{Var}(W^2_{k_1} W^2_{j_1}) \leq \mathbb{E}[W^2_{k_1} W^2_{j_1}] \leq \mathbb{E}[W^2_{k_i} W^2_{k_j}] \leq \Omega^*_{k_i} \Omega^*_{k_j}$. Case (ii): $i = i'$ but $j \neq j'$. In this case, we have $\text{Cov}(W^2_{k_i} W^2_{j_1}, W^2_{k_i} W^2_{j_2}) = \text{Var}(W^2_{k_i}) \mathbb{E}[W^2_{j_1}] \mathbb{E}[W^2_{j_2}] \leq \Omega^*_{k_i} \Omega^*_{k_j} \Omega^*_{k_j}$. Case (iii): $(i, j) \cap (i', j') = \emptyset$. The two terms are independent, and their covariance is zero. Combining the above gives

$$\text{Var}(I_b) \leq \frac{1}{M_n} \sum_{k=1}^{n} \left( \sum_{1 \leq i < j \leq n} b^2_{kij} \Omega^*_{k_i} \Omega^*_{k_j} \right) \sum_{i,j \neq k} b_{kij} b_{ki'j'} \Omega^*_{k_i} \Omega^*_{k_j} \Omega^*_{k_j}.$$

We now bound the right hand side. By condition (9), $\Omega^*_{ij} \leq C \theta_i \theta_j$. Hence, $b_{kij} \leq C \sum_{m>k} \theta^2 \theta \theta \leq C \|\theta\|^2 \theta \theta$. As a result,

$$\text{Var}(I_b) \leq \frac{C}{M_n^2} \left[ \sum_{k,j} \|\theta\|^4 \theta^2 \theta^2 \theta^2 + \sum_{k,j,j'} \|\theta\|^4 \theta^2 \theta^2 \theta^2 \right] \leq \frac{C}{M_n^2} \left[ \|\theta\|^6 \|\theta\|^6 + \|\theta\|^8 \|\theta\|^8 \right].$$

By (7), $\|\theta\| \to \infty$, so the second term dominates. Moreover, $\Omega^*_{ij} = \Omega_{ij}(1 - \Omega_{ij}) \leq C \theta_{ij}$ (in our setting, all $\Omega_{ij}$’s are bounded away from 1). As a result, we have $M_n \geq c \sum_{k \neq k_j} \Omega_{k_i} \Omega_{k_j} \Omega_{k_j} \Omega_{k_j} \Omega_{k_j} \geq C^{-1} n^4 C_4$. By Lemma B.1, $n^4 C_4 \approx \|\theta\|^8$. Combining the above gives $\text{Var} = O(\|\theta\|^8 / \|\theta\|^8)$, i.e.,

$$\text{Var}(I_b) \leq \frac{C}{M_n} \left( \sum_{i,j} \|\theta\|^4 \right)^2 \leq \frac{C \theta_{\text{max}}}{\sum_{i,j} \|\theta\|^8} = o(1). \quad (20)$$

Consider the variance of $I_b$. Rewrite

$$I_b = \frac{1}{M_n} \sum_{k \neq k_i \neq j} c_{kij} G_{kij},$$

where

$$G_{kij} \equiv W_{k_i} W_{k_j} W_{k_i} W_{k_j}, \quad c_{kij} = \sum_{m>1} \Omega_{m} \Omega_{m}.$$

Since $I_b$ has a mean zero, $\text{Var}(I_b) = \mathbb{E}(I_b^2)$. Additionally, for 2 cycles $(k, l, i, j)$ and $(k', l', i', j')$, only when they are exactly equal, we have $\mathbb{E}[G_{kij} G_{k'ij} ] \neq 0$. As a result,

$$\text{Var}(I_b) = \frac{1}{M_n} \sum_{k \neq k_i \neq j} c^2_{kij} \mathbb{E}[G^2_{kij}] \sum_{k \neq k_i \neq j} c^2_{kij} \Omega^*_{k_i} \Omega^*_{k_j}.$$

Similarly to how we get the bound for $b_{kij}$, we can derive that $\epsilon_{kij} \leq C \|\theta\|^2 \theta \theta$. Moreover, $\Omega^*_{k_i} \Omega^*_{k_j} \Omega^*_{k_j} \leq C \theta^2 \theta^2 \theta^2$. Hence,

$$\text{Var}(I_b) \leq \frac{C}{\|\theta\|^8} \sum_{k \neq k_i \neq j} \|\theta\|^4 \theta^2 \theta^2 \theta^2 \leq \frac{C \theta_{\text{max}}}{\|\theta\|^8} = o(1). \quad (21)$$

As a result,

$$\sqrt{\text{Var}(I_b)} \leq \frac{C}{\|\theta\|^4} \left( \sum_{i,j} \theta_i \right)^2 \leq \frac{C \theta_{\text{max}}}{\|\theta\|^4} = o(1). \quad (22)$$

We now check (b). By the Cauchy-Schwarz inequality and the Chebyshev’s inequality,

$$\sum_{m=1}^{n} \mathbb{E}(X^2_{n,m} | \mathcal{F}_{n,m-1}) \leq \sum_{m=1}^{n} \sqrt{\mathbb{E}(X^4_{n,m} | \mathcal{F}_{n,m-1})} \mathbb{E}(|X_{n,m}| \geq c | \mathcal{F}_{n,m-1}) \leq \|\theta\|^2 \sum_{m=1}^{n} \mathbb{E}(X^4_{n,m} | \mathcal{F}_{n,m-1}).$$

Therefore, it suffices to show that the right hand side converges to zero in probability. Then, it suffices to show that its $L^1$-norm converges to zero. Since the right hand side is a nonnegative random variable, we only need to prove that its expectation converges to zero, i.e.,

$$\mathbb{E} \left( \sum_{m=1}^{n} X^4_{n,m} \right) = o(1). \quad (22)$$

We now prove (22). We use the expression of $X_{n,m}$ in (12). Conditioning on $\mathcal{F}_{n,m-1}$, the $Y_{(m-1)ij}$’s are non-random. It follows that

$$\mathbb{E}(X^4_{n,m} | \mathcal{F}_{n,m-1}) = \frac{1}{M_n^4} \sum_{i,j=1}^{m-1} Y_{(m-1)ij} \mathbb{E}[W^2_{m} W^2_{m}].$$

First, we shall use the independence across entries of $W$ and the fact that $\mathbb{E}[W^4_{ij}] \leq \mathbb{E}[W^2_{ij}] \leq \Omega_{ij} \leq C \theta_{ij}$. Second, in proving (17), we have seen that $\mathbb{E}[Y^2_{(m-1)ij}] = \sum_{k<m,k \neq i,j} \Omega^*_{k_i} \Omega^*_{k_j} \leq C \sum_k \theta^2 \theta \theta \theta$. Therefore, it suffices to show that the right hand side converges to zero in probability. Then, it suffices to show that its $L^1$-norm converges to zero. Since the right hand side is a nonnegative random variable, we only need to prove that its expectation converges to zero, i.e.,

$$\mathbb{E} \left( \sum_{m=1}^{n} X^4_{n,m} \right) = o(1). \quad (22)$$

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$$\mathbb{E}(X^4_{n,m} | \mathcal{F}_{n,m-1}) = \frac{1}{M_n^4} \sum_{i,j=1}^{m-1} Y_{(m-1)ij} \mathbb{E}[W^2_{m} W^2_{m}].$$

First, we shall use the independence across entries of $W$ and the fact that $\mathbb{E}[W^4_{ij}] \leq \mathbb{E}[W^2_{ij}] \leq \Omega_{ij} \leq C \theta_{ij}$. Second, in proving (17), we have seen that $\mathbb{E}[Y^2_{(m-1)ij}] = \sum_{k<m,k \neq i,j} \Omega^*_{k_i} \Omega^*_{k_j} \leq C \sum_k \theta^2 \theta \theta \theta$. Therefore, it suffices to show that the right hand side converges to zero in probability. Then, it suffices to show that its $L^1$-norm converges to zero. Since the right hand side is a nonnegative random variable, we only need to prove that its expectation converges to zero, i.e.,

$$\mathbb{E} \left( \sum_{m=1}^{n} X^4_{n,m} \right) = o(1). \quad (22)$$
Third, from (13), it is easy to see that when \((i, j, i', j')\) are distinct, \(\mathbb{E}[Y_{(m-1)ij}Y_{(m-1)i'j'}] = 0\); moreover, for \(j \neq j'\), \(\mathbb{E}[Y_{(m-1)ij}Y_{(m-1)i'j'}] = \sum_k \mathbb{E}[W_{k1}W_{kj}] = 0\). Last, in proving (20), we have seen that \(M_n \geq c\|\theta\|^8\). Combining the above, we find that
\[
\mathbb{E}[X^4_{n,m}] = \frac{1}{M^2} \sum_{i,j=1}^{m-1} \mathbb{E}[Y^2_{(m-1)ij}W_{m1}W^4_{mj}]
\leq \frac{C}{\|\theta\|^8} \sum_{i,j=1}^{m-1} (|\theta|^2\theta_i\theta_j)(\theta_m\theta_i)(\theta_m\theta_j)
\leq C\|\theta\|^2/\|\theta\|^{10}.
\]
As a result,
\[
\sum_{n=1}^{n} \mathbb{E}[X^4_{n,m}] \leq C\|\theta\|^{-8} = o(1).
\]
This gives (22) and (b) follows.

**C. Proof of Theorem 3.3 and Corollary 3.1**

Consider Theorem 3.3 first. For short, let
\[
Z^{(m)}_n = \sqrt{\frac{B_{n,m}}{2m}} \tilde{C}_{m-1/2} \chi_{gc}^{(m)} , \quad x^*_0 = \mathbb{P}(Z^{(m)} \geq z_\alpha).
\]
It suffices to show that under the null and alternative,
\[
|x^*_0 - \Phi(\delta^{(m)} - z_\alpha)| \leq o(1).
\]  
(23)

Denote \(a_n = (C_m/\tilde{C}_{m})^{1/2}\) for short. It is seen that
\[
a_n \xrightarrow{P} 1,
\]  
(24)

and
\[
\frac{1}{a_n} Z^{(m)}_n = \sqrt{\frac{B_{n,m}}{2m}} C_{m-1/2} \chi_{gc}^{(m)}.
\]  
(25)

Combining Theorem 3.1 and the proof of Theorem 3.2, we have shown that
\[
\sqrt{\frac{B_{n,m}}{2m}} C_{m-1/2} [\chi_{gc}^{(m)} - \chi_{gc,0}] \xrightarrow{P} N(0,1),
\]  
(26)

where by definitions,
\[
\sqrt{\frac{B_{n,m}}{2m}} C_{m-1/2} \chi_{gc,0} = \delta^{(m)}_g.
\]  
(27)

Combining (25)-(27) gives
\[
\frac{1}{a_n} Z^{(m)}_n - \delta^{(m)}_g \xrightarrow{d} N(0,1).
\]  
(28)

Denote the CDF of \(\frac{1}{a_n} Z^{(m)}_n - \delta^{(m)}_g\) by \(F_n\). Recall that \(\Phi\) denotes the CDF of \(N(0,1)\). It follows from (28) that
\[
\sup_x |F_n(x) - \Phi(x)| \to 0.
\]  
(29)

We now rewrite
\[
x^*_0 = \mathbb{P}\left(\frac{1}{a_n} Z^{(m)}_n - \delta^{(m)}_g \geq \frac{1}{a_n} z_\alpha - \delta^{(m)}_g\right),
\]

and introduce a proxy by
\[
x_0 = \mathbb{P}\left(\frac{1}{a_n} Z^{(m)}_n - \delta^{(m)}_g \geq z_\alpha - \delta^{(m)}_g\right).
\]

By triangle inequality,
\[
|x^*_0 - x_0| = |x^*_0 - x_0| + |x_0 - \Phi(\delta^{(m)}_g - z_\alpha)|.
\]

where by (29),
\[
|x_0 - \Phi(\delta^{(m)}_g - z_\alpha)| \to 0.
\]  
(30)

Moreover, for any fixed \(\epsilon > 0\), it is seen that
\[
|x^*_0 - x_0| \leq I + II,
\]

where
\[
I = \mathbb{P}(|a_n - 1| \geq \epsilon),
\]

and
\[
II = \mathbb{P}(\frac{1}{a_n} Z^{(m)}_n - \delta^{(m)}_g \text{ falls between } (1 \pm \epsilon) z_\alpha - \delta^{(m)}_g).
\]

which by (29) does not exceed
\[
\mathbb{P}(N(0,1) \text{ falls between } (1 \pm \epsilon) z_\alpha - \delta^{(m)}_g) + o(1);
\]

note the first term does not exceed \((2/\sqrt{2\pi})z_\alpha\epsilon\). Combining these gives that for any \(\epsilon > 0\),
\[
|x^*_0 - \Phi(\delta^{(m)}_g - z_\alpha)| \leq (2/\sqrt{2\pi})z_\alpha\epsilon + \mathbb{P}(|a_n - 1| \geq \epsilon) + o(1).
\]

Recall that \(a_n \xrightarrow{P} 1\), the claim follows.

Next, consider Corollary 3.1. It is seen that \(\delta^{(m)}_g = 0\) under the null and that under the alternative,
\[
\delta^{(m)}_g \geq \sum_{k=2}^{K} \frac{\lambda_k}{\lambda_k^m} / \left(\sum_{k=1}^{K} \lambda_k^m\right)^{1/2}.
\]

When \(m = 4\), by Lemma 6.1, \(\delta^{(4)}_g \geq c_4\|\theta\|^4\) for some constant \(c_4 > 0\). When \(m = 3\) and \(P\) is positive definite, \(\lambda_k\) are the eigenvalues of \(\Theta \Pi_P \Pi_\theta\), so for \(1 \leq k \leq K\), \(\lambda_k \geq 0\). Using Lemma 6.1, \(\delta^{(3)}_g \geq c_3\|\theta\|^3\) for some constant \(c_3\). Combining these with Theorem 3.3 gives the claim.

**D. Proof of Secondary Lemmas**

**D.1. Proof of Lemma 6.1**

We first consider the claim about \(\lambda_k\’s\). Recall that \(\lambda_k\’s\) are the eigenvalues of the matrix \(G^{1/2}P G^{1/2}\). First, we
Consider the second item. By (17) of (Jin et al., 2018),
\[ L_2 = \frac{1}{B_{n,3}} \left[ \sum_{k=1}^{K} \lambda_k^2 (\eta, \xi_k)^2 + O(\|\theta\|^2 \|\theta\|^4) \right], \]
where by Lemma 6.1 of (Jin et al., 2018),
\[ \sum_{k=1}^{K} \lambda_k^2 (\eta, \xi_k)^2 \leq \|\theta\|^2 \|\theta\|^4. \]
By similar argument, \( \|\theta\|^2 \|\theta\|^2 \|\theta\|^2 \leq o(\|\theta\|^2 \|\theta\|^4) \), so the claim follows by noting \( B_{n,3} \sim n^{-3} \).

Consider the third item. By similar argument, it is seen that
\[ L_3 = \frac{1}{B_{n,4}} \sum_{1 \leq i_1, i_2, i_3, i_4 \leq n} \Omega_{i_1i_2} \Omega_{i_2i_3} \Omega_{i_3i_4}. \]
As before, let \( \mathcal{N}_1 \) denote the set of pure nodes in community 1. It is not hard to see that
\[ L_3 \geq \frac{K}{B_{n,4}} \sum_{k=1}^{K} \sum_{i_1, i_2, i_3, i_4 \in \mathcal{N}_k} \Omega_{i_1i_2} \Omega_{i_2i_3} \Omega_{i_3i_4}. \]
In our model, all diagonal entries of \( P \) are 1, so for any \( i, j \in \mathcal{N}_1, \Omega_{i_j} = \theta_i \theta_j \). Therefore,
\[ L_3 \geq \frac{K}{B_{n,4}} \sum_{k=1}^{K} \sum_{i_1, i_2, i_3, i_4 \in \mathcal{N}_k} \theta_i \theta_i \theta_i^2 \theta_i \theta_i. \]
Now, we can lower bound the right hand side of (34) by
\[ I - II - III - IV, \]
where
\[ I = \sum_{k=1}^{K} \sum_{i_1, i_2, i_3, i_4 \in \mathcal{N}_k} \theta_i \theta_i \theta_i^2 \theta_i \theta_i, \]
\[ II = \sum_{k=1}^{K} \sum_{i_1, i_2, i_3, i_4 \in \mathcal{N}_k} \theta_i \theta_i \theta_i^2 \theta_i \theta_i, \]
\[ III = \sum_{i, j, k, l \in \mathcal{N}_1} \theta_i \theta_i \theta_i \theta_i \theta_i^2 \theta_i. \]
and \[ IV = 4 \sum_{k=1}^{K} \sum_{i_1,i_2,i_3,i_4 \in N_k \atop i_1 = i_2} \theta_{i_1} \theta_{i_2} \theta_{i_3}^2. \]

First, by (8) of (Jin et al., 2018),

\[ I = \sum_{k=1}^{K} \left( \sum_{i \in N_k} \theta_i \right)^2 \left( \sum_{i \in N_k} \theta_i^2 \right)^2 \]

\[ \geq C \| \theta \|^4 \sum_{k=1}^{K} \left( \sum_{i \in N_k} \theta_i \right)^2 \]

\[ \geq C \| \theta \|^4 \| \theta \|^4, \]

where the last inequality is Cauchy-Schwarz inequality.

Second, by (7) of (Jin et al., 2018) that \( \theta_{\text{max}} \leq \| \theta \|_3 = o(1) \), we obtain \( \| \theta \|^2 \leq o(1) \cdot \| \theta \|_1 \) (note \( \| \theta \| \to \infty \)), and

\[ II \leq C \sum_{k=1}^{K} \sum_{i_1,i_2,i_3 \in N_k} \theta_{i_1} \theta_{i_2} \theta_{i_3}^2 \]

\[ \leq C \| \theta \|_1 \| \theta \|^4 \]

\[ = o \left( \| \theta \|^3 \| \theta \|^4 \right). \]

Similarly, we have \( \| \theta \|_3 \leq o(1) \cdot \| \theta \|_2 \) and \( \| \theta \|_1 \leq o(1) \cdot \| \theta \|_3 \), which implies

\[ III = o \left( \| \theta \|^3 \| \theta \|^4 \right), \quad \text{and} \quad IV = o \left( \| \theta \|^3 \| \theta \|^4 \right). \]

Combining these gives \( L_3 \geq \varepsilon \| \theta \|^3 \| \theta \|^4 \), and the claim follows.

We now prove the next three items (on the variances). In the Proof of Lemma B.2, we’ve already shown that the dominating term in \( \widehat{C}_4 - C_4 \) is the term \( G_{i_1,i_2,i_3,i_4}(W) \) is the dominating term of \( \widehat{C}_4 - C_4 \), and that

\[ \text{Var} \left( \frac{1}{B_{n,4}} \sum_{i_1,\ldots,i_4 \text{ distinct}} G_{i_1,i_2,i_3,i_4}(W) \right) \]

\[ \leq C n^{-4} \sum_{i_1,\ldots,i_4 \text{ distinct}} G_{i_1,i_2,i_3,i_4}(\Omega) \]

\[ \leq C n^{-4} \sum_{i_1,\ldots,i_4 \text{ distinct}} \theta_{i_1}^2 \theta_{i_2} \theta_{i_3}^2 \theta_{i_4} = C n^{-4} \| \theta \|^8. \]

Combining it with \( C_4 \asymp n^{-4} \| \theta \|^8 \), we get \( \text{Var} (\widehat{C}_4) = O(n^{-4} C_4) \).

Consider \( \text{Var} (\widehat{L}_2) \). By definitions and that \( B_{n,m} \asymp n^m \), we bound

\[ \text{E} (\widehat{L}_2 - L_2)^2 \leq C n^{-6} \sum_{i_1 < i_2 < i_3} (A_{i_1,i_2,i_3} - \Omega_{i_1,i_2,i_3})^2. \]

(35)

Recall that when \( i \neq j \), \( A_{ij} = \Omega_{ij} + W_{ij} \). Since for any numbers \( x, y, a, b \), \( (a + x)(b + y) - ab = xy + ay + bx \), we can write

\[ \sum_{i_1 < i_2 < i_3} (A_{i_1,i_2,i_3} - \Omega_{i_1,i_2} - \Omega_{i_2,i_3}) = I + II + III, \]

where

\[ I = \sum_{i_1 < i_2 < i_3} W_{i_1,i_2} W_{i_2,i_3}, \]

\[ II = \sum_{i_1 < i_2 < i_3} \Omega_{i_1,i_2} W_{i_2,i_3}, \]

and

\[ III = \sum_{i_1 < i_2 < i_3} \Omega_{i_2,i_3} W_{i_1,i_2}. \]

Inserting this into (35) and using Cauchy-Schwarz inequality,

\[ \text{E} (\widehat{L}_2 - L_2)^2 \leq C n^{-6} (\text{E} [I]^2 + \text{E} [II]^2 + \text{E} [III]^2]). \]

It then suffices to show

\[ \text{E} [I]^2 \lesssim \| \theta \|^3 \| \theta \|^3, \quad \text{E} [II]^2 \lesssim \| \theta \|^3 \| \theta \|^3, \quad \text{E} [III]^2 \lesssim \| \theta \|^3 \| \theta \|^3. \]

(36)

(37)

(38)

We now show (36)-(38) separately.

Consider (36). Note that for two sets of indices \( (i_1, i_2, i_3) \) and \( (j_1, j_2, j_3) \) such that \( i_1 < i_2 < i_3, j_1 < j_2 < j_3 \), by basic statistics, we have that when \( (i_1, i_2, i_3) \neq (j_1, j_2, j_3) \),

\[ \text{E} [W_{i_1,i_2} W_{i_2,i_3} W_{j_1,j_2} W_{j_2,j_3}] = 0. \]

and when \( (i_1, i_2, i_3) = (j_1, j_2, j_3) \),

\[ \text{E} [W_{i_1,i_2} W_{i_2,i_3} W_{j_1,j_2} W_{j_2,j_3}] = \text{E} [W_{i_2,i_3} W_{i_2,i_3}] = \Omega_{i_1,i_2} (1 - \Omega_{i_1,i_2}) \Omega_{i_2,i_3} (1 - \Omega_{i_2,i_3}). \]

Therefore,

\[ \text{E} [I]^2 = \sum_{i_1 < i_2 < i_3} \sum_{j_1 < j_3} \text{E} [W_{i_1,i_2} W_{i_2,i_3} W_{j_1,j_2} W_{j_2,j_3}] \]

\[ \leq \sum_{i_1 < i_2 < i_3} \text{E} [W_{i_1,i_2}^2 W_{i_2,i_3}^2] \]

\[ \leq \sum_{i_1 < i_2 < i_3} \Omega_{i_1,i_2} (1 - \Omega_{i_1,i_2}) \Omega_{i_2,i_3} (1 - \Omega_{i_2,i_3}). \]

Recall that for any \( i < j \),

\[ \Omega_{ij} (1 - \Omega_{ij}) \leq \Omega_{ij} \leq \theta_i \theta_j, \]
it follows that
\[
E[(J)]^2 \leq \sum_{i_1 < i_2 < i_3} \theta_{i_1} \theta_{i_2} \theta_{i_3} \leq \|\theta\|_2^4 \|\theta\|_2^2,
\]
and the claim follows by Cauchy Schwarz inequality that
\[
\|\theta\|_2^4 \|\theta\|_2^2 \leq \|\theta\|_2^4 \|\theta\|_3^2 \leq \|\theta\|_2^4 \|\theta\|_3^3.
\]
Consider (37)-(38). Since the proofs are similar, we only show (37).

Note that for two sets of indices \((i_1, i_2, i_3)\) and \((j_1, j_2, j_3)\) such that \(i_1 < i_2 < i_3, j_1 < j_2 < j_3\), by basic statistics, we have that when \((i_2, i_3) \neq (j_2, j_3),\)
\[
E[W_{i_2 i_3} W_{j_2 j_3}] = 0.
\]
and when \((i_2, i_3) = (j_2, j_3),\)
\[
E[W_{i_2 i_3} W_{j_2 j_3}] = E[W_{i_2 i_3}^2] = \Omega_{i_2 i_3} (1 - \Omega_{i_2 i_3}).
\]
Therefore,
\[
E[(I I)^2] = \sum_{i_1 < i_2 < i_3} \sum_{j_1 < j_2 < j_3} E[\Omega_{i_1 i_2} \Omega_{j_1 j_2} W_{i_2 i_3} W_{j_2 j_3}] = \sum_{i_1 < i_2 < i_3} \sum_{j_1 < j_2 < j_3} E[\Omega_{i_1 i_2} W_{i_2 i_3} (\sum_{j_1 < j_2 < j_3} \Omega_{j_1 j_2})]
\]
\[
= \sum_{i_1 < i_2 < i_3} E[\Omega_{i_1 i_2} \Omega_{i_2 i_3} (1 - \Omega_{i_2 i_3})] (\sum_{j_1 < j_2 < j_3} \Omega_{j_1 j_2})
\]
Again by \(\Omega_{i_1} (1 - \Omega_{i_1}) \leq \Omega_{i_1} \leq i_1 \theta_{i_1}\) for any \(i < j\), we find
\[
E[(I I)^2] \leq \sum_{i_1 < i_2 < i_3} E[\Omega_{i_1 i_2} \Omega_{i_2 i_3} (\sum_{j_1 < j_2 < j_3} \Omega_{j_1 j_2})]
\]
\[
\leq \sum_{i_1 < i_2 < i_3} \theta_{i_1} \theta_{i_2} \theta_{i_3} (\sum_{j_1 < j_2 < j_3} \theta_{j_1} \theta_{j_2} \theta_{j_3})
\]
\[
\leq \|\theta\|_2^4 \|\theta\|_3^3.
\]
last, we prove the claim on \(\text{Var}(\hat{\theta}_3)\). It suffices to control the covariance between \((A_{i_1 i_2} A_{i_3 i_4}, A_{j_1 j_2} A_{j_3 j_4})\) and \((A_{j_1 j_2} A_{j_3 j_4}, A_{j_1 j_2} A_{j_3 j_4})\). To be more specific, define the set
\[
\mathcal{J} = \{ (i_1, i_2), (i_2, i_3), (i_3, i_4), (j_1, j_2), (j_2, j_3), (j_3, j_4) \},
\]
whose elements are pairs of unordered integers, i.e. we treat \((i_1, i_2)\) and \((i_2, i_1)\) as the same element.

Let \(|\mathcal{J}|\) be the number of distinct elements of \(\mathcal{J}\), where \(3 \leq |\mathcal{J}| \leq 6\) under the condition that \(i_1 < i_2 < i_3 < i_4\) and \(j_1 < j_2 < j_3 < j_4\). To control the variance of \(\hat{\theta}_3\), it suffices to bound the following quantity
\[
\sum_{s=3}^{6} \sum_{|\mathcal{J}|=s} \text{Cov}(A_{i_1 i_2} A_{i_3 i_4}, A_{j_1 j_2} A_{j_3 j_4}).
\]
Furthermore, it suffices to show for \(3 \leq s \leq 6,\)
\[
\sum_{|\mathcal{J}|=s} \text{Cov}(A_{i_1 i_2} A_{i_3 i_4}, A_{j_1 j_2} A_{j_3 j_4}) \leq \|\theta\|_1 \|\theta\|_3^5.
\]
When \(|\mathcal{J}| = 6,\) it’s not hard to see \((A_{i_1 i_2} A_{i_3 i_4}, A_{j_1 j_2} A_{j_3 j_4})\) and \((A_{i_1 i_2} A_{i_3 i_4}, A_{j_1 j_2} A_{j_3 j_4})\) are independent because the six elements in \(\mathcal{J}\) are all distinct, which indicates
\[
\sum_{|\mathcal{J}|=6} \text{Cov}(A_{i_1 i_2} A_{i_3 i_4}, A_{j_1 j_2} A_{j_3 j_4}) = 0.
\]
The following basic property is frequently used in the discussion of remaining cases. For non-negative random variables \(X\) and \(Y\), we have
\[
\text{Cov}(X, Y) \leq E[XY].
\]
Consider the case where \(|\mathcal{J}| = 5\). By symmetry, it’s enough to consider three situations where \((i_1, i_2) = (j_1, j_2),\)
\((i_1, i_2) = (j_2, j_3)\) and \((i_2, i_3) = (j_2, j_3)\), separately. If \((i_1, i_2) = (j_1, j_2),\) we have
\[
\sum_{(i_1, i_2) = (j_1, j_2)} \text{Cov}(A_{i_1 i_2} A_{i_3 i_4}, A_{j_1 j_2} A_{j_3 j_4}) \leq \sum_{(i_1, i_2) = (j_1, j_2)} E[A_{i_1 i_2} A_{i_3 i_4} A_{j_1 j_2} A_{j_3 j_4}]
\]
\[
\leq C \sum_{(i_1, i_2) = (j_1, j_2)} \theta_{i_1} \theta_{i_2} \theta_{j_1} \theta_{j_2} \theta_{j_3} \theta_{j_4}
\]
\[
= C \|\theta\|^4 \|\theta\|_3 \|\theta\|_3 \leq C \|\theta\|^4 \|\theta\|_3^3,
\]
where the last inequality is due to \(\|\theta\|^4 \leq \|\theta\|_1 \|\theta\|_3^3\) by Cauchy-Schwarz inequality.
If \((i_1, i_2) = (j_2, j_3),\) we have
\[
\sum_{(i_1, i_2) = (j_2, j_3)} \text{Cov}(A_{i_1 i_2} A_{i_3 i_4}, A_{j_1 j_2} A_{j_3 j_4}) \leq \sum_{(i_1, i_2) = (j_2, j_3)} E[A_{i_1 i_2} A_{i_3 i_4} A_{j_1 j_2} A_{j_3 j_4}]
\]
\[
\leq C \sum_{(i_1, i_2) = (j_2, j_3)} \theta_{i_1} \theta_{i_2} \theta_{j_1} \theta_{j_2} \theta_{j_3} \theta_{j_4}
\]
\[
= C \|\theta\|^4 \|\theta\|_3 \|\theta\|_3 \leq C \|\theta\|^4 \|\theta\|_3^3,
\]
where the last inequality is due to \(\|\theta\|^4 \leq \|\theta\|_1 \|\theta\|_3^3\).
If \((i_2, i_3) = (j_2, j_3)\), we have
\[
\sum_{(i_2, i_3) = (j_2, j_3)} \text{Cov}(A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4}, A_{j_1 j_2} A_{j_2 j_3} A_{j_3 j_4}) \\
\leq \sum_{(i_2, i_3) = (j_2, j_3)} \mathbb{E}\left[ A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4} A_{j_1 j_2} A_{j_2 j_3} A_{j_3 j_4} \right] \\
\leq C \sum_{i_2, i_3 = (j_2, j_3)} \theta_i \theta_j \theta_k \theta_l \theta_m \theta_n = C \|\theta\|_1^4 \|\theta\|_3^6.
\]
Combining above three inequalities, we derive
\[
\sum_{|\mathcal{J}| = 5} \text{Cov}(A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4}, A_{j_1 j_2} A_{j_2 j_3} A_{j_3 j_4}) \leq C \|\theta\|_1^4 \|\theta\|_3^6.
\]
Consider the case where \(|\mathcal{J}| = 4\). By symmetry, \(\mathcal{J}\) either equals to \(\mathcal{J}_1 = \{(i_1, i_2), (i_2, i_3), (i_3, i_4), (j_1, j_2)\}\) or \(\mathcal{J}_2 = \{(i_1, i_2), (i_2, i_3), (i_3, i_4), (j_2, j_3)\}\).
Therefore, we decompose and bound
\[
\sum_{|\mathcal{J}| = 4} \text{Cov}(A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4}, A_{j_2 j_3} A_{j_3 j_4}) \\
\leq \sum_{\mathcal{J}_1} \text{Cov}(A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4}, A_{j_2 j_3} A_{j_3 j_4}) \\
+ \sum_{\mathcal{J}_2} \text{Cov}(A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4}, A_{j_2 j_3} A_{j_3 j_4}) \\
\leq \sum_{\mathcal{J}_1} \mathbb{E}\left[ A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4} A_{j_1 j_2} \right] + \sum_{\mathcal{J}_2} \mathbb{E}\left[ A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4} A_{j_2 j_3} \right]
\]
It then suffices to show
\[
\sum_{\mathcal{J}_1} \mathbb{E}\left[ A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4} A_{j_1 j_2} \right] \leq C \|\theta\|_1^4 \|\theta\|_3^6, \tag{41}
\]
and
\[
\sum_{\mathcal{J}_2} \mathbb{E}\left[ A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4} A_{j_2 j_3} \right] \leq C \|\theta\|_1^4 \|\theta\|_3^6. \tag{42}
\]
For (41), \(j_2\) must equal to one of \(i_1, \cdots, i_4\) since \((j_2, j_3)\) equals to some \((i_2, i_3)\) by definition of \(\mathcal{J}_1\). By symmetry, we only need to consider \(j_2 = i_1\) and \(j_2 = i_2\). Again by \(\Omega_{ij} \leq \theta_i \theta_j\), we obtain
\[
\sum_{\mathcal{J}_1} \mathbb{E}\left[ A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4} A_{j_1 j_2} \right] \\
\leq \sum_{j_2 = i_1} \theta_i \theta_j^2 \theta_k \theta_l \theta_m \theta_n + \sum_{j_2 = i_2} \theta_i \theta_j^2 \theta_k \theta_l \theta_m \theta_n \\
\leq \sum_{j_2 = i_1} \theta^2 \theta_1^2 \theta_i \theta_k \theta_l \theta_m \theta_n + \sum_{j_2 = i_2} \theta^2 \theta_1^2 \theta_i \theta_k \theta_l \theta_m \theta_n \\
= \|\theta\|_1^4 \|\theta\|_6 + C \|\theta\|_1^4 \|\theta\|_2^2 \|\theta\|_3^3 \\
\leq \|\theta\|_1^4 \|\theta\|_3^6.
\]

Here we explain the last inequality. By Cauchy-Schwarz inequality, \(\|\theta\|_1^4 \leq \|\theta\|_1 \|\theta\|_3^3\). Combining with (7) that \(\|\theta\| \to \infty\), \(\|\theta\|_1^4 \|\theta\|_3^6 \leq \|\theta\|_1^4 \|\theta\|_3^6 \leq \|\theta\|_1^4 \|\theta\|_3^6 \). Moreover, \(\|\theta\|_1^4 \|\theta\|_3^6 \leq \|\theta\|_1^4 \|\theta\|_3^6 \leq \|\theta\|_1^4 \|\theta\|_3^6 \). For (42), we similarly found \(j_2, j_3\) must equal to some \(i_1, \cdots, i_4\). By (7), \(\theta_j \leq C\). Thus we only need to discuss the cases where \(j_2 = i_1\) or \(j_2 = i_2\).
\[
\sum_{j_2} \mathbb{E}\left[ A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4} A_{j_2 j_3} \right] \\
\leq \sum_{j_2 = i_1} \theta_i \theta_j^2 \theta_k \theta_l \theta_m \theta_n + \sum_{j_2 = i_2} \theta_i \theta_j^2 \theta_k \theta_l \theta_m \theta_n \\
\leq \sum_{j_2 = i_1} \theta_1^2 \theta_2 \theta_3 \theta_4 \theta_5 \theta_6 + \sum_{j_2 = i_2} \theta_1^2 \theta_2 \theta_3 \theta_4 \theta_5 \theta_6 \\
= C \|\theta\|_1^2 \|\theta\|_6 + C \|\theta\|_1^2 \|\theta\|_2^2 \|\theta\|_3^3 \leq C \|\theta\|_1^4 \|\theta\|_3^6.
\]
where the last inequality has been explained in the proof of (41).
Combining (41) and (42), we bound
\[
\sum_{|\mathcal{J}| = 4} \text{Cov}(A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4}, A_{j_1 j_2} A_{j_2 j_3} A_{j_3 j_4}) \leq C \|\theta\|_1^4 \|\theta\|_3^6.
\]
Finally, consider the case where \(|\mathcal{J}| = 3\). In this case, the covariance is in fact variance. Therefore,
\[
\sum_{|\mathcal{J}| = 3} \text{Cov}(A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4}, A_{j_1 j_2} A_{j_2 j_3} A_{j_3 j_4}) \\
= \sum_{i_1, \cdots, i_4} \text{Var}(A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4}) \\
\leq \sum_{i_1, \cdots, i_4} \mathbb{E}\left[ A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4} \right] \\
\leq \sum_{i_1, \cdots, i_4} \theta_i \theta_j \theta_k \theta_l \theta_m \theta_n \\
= C \|\theta\|_1^2 \|\theta\|_6 \leq \|\theta\|_1^4 \|\theta\|_3^6.
\]
where the second last inequality is by (7) that \(\|\theta\| \to \infty\) and last inequality is Cauchy-Schwarz inequality. This proves (39).

\[\square\]

D.3. Proof of Lemma B.2 
Write for short \(T_n = \frac{\sqrt{\beta_n - \beta_n(C_4 - C_4)}}{\sqrt{G(\Omega)}}\). We introduce some useful notations. For any \(n \times n\) matrix \(M\) and distinct indices \((i_1, i_2, i_3, i_4)\), define
\[
G_{i_1 i_2 i_3 i_4}(M) = M_{i_1 i_2} M_{i_2 i_3} M_{i_3 i_4} M_{i_4 i_1}, \\
G(M) = \sum_{(i_1, i_2, i_3, i_4) \in C_4(n)} G_{i_1 i_2 i_3 i_4}(M).
\]
Additionally, let \(W = A - \Omega\) and let \(\Omega^*\) be the matrix where \(\Omega^*_{ij} = \Omega_{ij} (1 - \Omega_{ij})\) for all \(1 \leq i, j \leq n\). We now rewrite
\[
T_n = \frac{G(A) - G(\Omega)}{\sqrt{G(\Omega)}}, \quad S_{n,n} = \frac{G(W)}{\sqrt{G(\Omega^*)}}. \tag{43}
\]
Therefore,
\[
T_n - S_{n,n} = \frac{G(A) - G(\Omega)}{\sqrt{G(\Omega)}} + S_{n,n} \left[ \frac{\sqrt{G(\Omega^*)}}{\sqrt{G(\Omega)}} - 1 \right]
\equiv J_1 + S_{n,n} \cdot J_2.
\]

In the proof of Theorem 3.2, we have shown \( S_{n,n} \overset{d}{\rightarrow} N(0,1) \). Hence, to show \( (T_n - S_{n,n}) \overset{p}{\rightarrow} 0 \), it suffices to show that
\[
J_1 \overset{p}{\rightarrow} 0
\]  
(44)
and
\[
J_2 \overset{p}{\rightarrow} 0.
\]  
(45)

First, we prove (44). We can decompose \( G_{i_1,i_2,i_3,i_4}(A) - G_{i_1,i_2,i_3,i_4}(W) \) as the sum of three terms
\[
\Delta_{i_1i_2i_3i_4}^{(1)} = W_{i_1i_2} \Omega_{i_1i_3} \Omega_{i_1i_4} \Omega_{i_2i_3} \Omega_{i_2i_4} + \Omega_{i_1i_2} W_{i_1i_3} \Omega_{i_1i_4} \Omega_{i_2i_3} \Omega_{i_2i_4} + \Omega_{i_1i_2} \Omega_{i_1i_3} W_{i_2i_3} \Omega_{i_1i_4} \Omega_{i_2i_4} + \Omega_{i_1i_2} \Omega_{i_1i_3} \Omega_{i_1i_4} W_{i_2i_3} \Omega_{i_2i_4},
\]
\[
\Delta_{i_1i_2i_3i_4}^{(2)} = W_{i_1i_2} \Omega_{i_2i_3} \Omega_{i_1i_4} \Omega_{i_2i_4} + W_{i_1i_2} \Omega_{i_1i_3} \Omega_{i_2i_4} \Omega_{i_1i_4} + W_{i_1i_2} \Omega_{i_1i_3} \Omega_{i_2i_4} \Omega_{i_1i_3} + W_{i_1i_2} \Omega_{i_1i_3} \Omega_{i_2i_4} \Omega_{i_2i_3},
\]
\[
\Delta_{i_1i_2i_3i_4}^{(3)} = \Omega_{i_1i_2} W_{i_1i_3} W_{i_1i_4} W_{i_2i_3} W_{i_2i_4} + \Omega_{i_1i_2} W_{i_1i_3} W_{i_1i_4} W_{i_2i_3} W_{i_2i_4} + \Omega_{i_1i_2} W_{i_1i_3} W_{i_2i_4} W_{i_2i_3} W_{i_2i_4} + \Omega_{i_1i_2} W_{i_1i_3} W_{i_1i_4} W_{i_2i_3} W_{i_2i_4}.
\]

It is easy to see that
\[
E \left[ \sum_{CC(I_n)} \Delta_{i_1i_2i_3i_4}^{(1)} \right] = 0.
\]  
(46)

We then study the variance of this term. Note that the four terms in \( \Delta_{i_1i_2i_3i_4}^{(1)} \) are independent of each other. Let \((j,s,m,\ell)\) be any cycle on the four nodes \(\{i_1, i_2, i_3, i_4\}\). Then, the variance of \(W_{js} \Omega_{js} \Omega_{sm} \Omega_{m\ell} \Omega_{\ell j}\) is bounded by \(\Omega_{js} \Omega_{sm} \Omega_{m\ell} \Omega_{\ell j} = O(\theta_3^4 \theta_2^4 \theta_2^4 \theta_2^4)\). Hence,
\[
\sum_{CC(I_n)} \text{Var}(\Delta_{i_1i_2i_3i_4}^{(1)}) \leq C \sum_{j,s,m,\ell} \theta_3^4 \theta_2^4 \theta_2^4 \theta_2^4 \leq C \|\theta\|^4 \|\theta\|^4 \leq C \|\theta\|^8 \|\theta\|^6 = o(\|\theta\|^8),
\]
where the last inequality is from the condition (7) and the fact that \(\|\theta\|^4 = \sum_{j,s} \theta_3^4 \leq \theta_2^4 \max_j (\sum_{j} \theta_2^4) = O(\|\theta\|^2) = o(\|\theta\|^4)\). We then look at the covariance between \(\Delta_{i_1i_2i_3i_4}^{(1)}\) and \(\Delta_{i_1i_2i_3i_4}^{(1)}\). Let \((j,s,m,\ell)\) be any cycle on the four nodes \(\{i_1, i_2, i_3, i_4\}\) and \((j',s',m',\ell')\) be any cycle on the four nodes \(\{i_1', i_2', i_3', i_4'\}\). As long as \((j,s) \neq (j',s')\), the two terms \(W_{js} \Omega_{js} \Omega_{sm} \Omega_{m\ell} \Omega_{\ell j}\) and \(W_{js} \Omega_{js} \Omega_{sm} \Omega_{m\ell} \Omega_{\ell j}\) are independent, hence, their covariance is zero. If \((j,s) = (j',s')\), their covariance is bounded by \(\Omega_{js} \Omega^{(1)}_{i_1i_2i_3i_4} \Omega_{s'm} \Omega_{m'\ell} \Omega_{\ell'j} = O(\theta_3^4 \theta_2^4 \theta_2^4 \theta_2^4 \theta_2^4)\). As a result,
\[
\sum_{j,s,m,\ell} \text{Cov}(\Delta_{i_1i_2i_3i_4}^{(1)}, \Delta_{i_1i_2i_3i_4}^{(1)}) \leq C \sum_{j,s,m,\ell} \theta_3^4 \theta_2^4 \theta_2^4 \theta_2^4 \theta_2^4 \leq C \|\theta\|^6 \|\theta\|^4 \leq o(\|\theta\|^8).
\]
Note that \(G(\Omega) \sim n^4 C_4 \sim \|\theta\|^8\) by Lemma B.1. Additionally, from the condition (7), \(\|\theta\|_3 = o(1)\). Hence, the above imply
\[
\text{Var} \left( \sum_{CC(I_n)} \Delta_{i_1i_2i_3i_4}^{(1)} \right) \approx G(\Omega).
\]  
(47)

Combining (46)-(47) gives
\[
\frac{1}{\sqrt{G(\Omega)}} \sum_{CC(I_n)} \Delta_{i_1i_2i_3i_4}^{(1)} \overset{p}{\rightarrow} 0.
\]  
(48)

We can consider other terms similarly. By direct calculations,
\[
\text{Var} \left( \sum_{CC(I_n)} \Delta_{i_1i_2i_3i_4}^{(2)} \right) \leq \sum_{j,s,m,\ell} \Omega_{js} \Omega_{sm} \Omega_{m\ell} \Omega_{\ell j},
\]
\[
\leq C \sum_{j,s,m,\ell} \theta_3^3 \theta_2^3 \theta_2^3 \theta_2^4 + C \sum_{j,s,m,\ell} \theta_3^2 \theta_2^3 \theta_2^4 \theta_2^4 \theta_2^4 \leq C \|\theta\|^2 \|\theta\|^4 + C \|\theta\|^6 \|\theta\|^6 = o(\|\theta\|^8),
\]
and
\[
\text{Var} \left( \sum_{CC(I_n)} \Delta_{i_1i_2i_3i_4}^{(3)} \right) \leq \sum_{j,s,m,\ell} \Omega_{js} \Omega_{sm} \Omega_{m\ell} \Omega_{\ell j},
\]
\[
\leq C \sum_{j,s,m,\ell} \theta_3^4 \theta_2^2 \theta_2^2 \theta_2^2 \leq C \|\theta\|^6 \|\theta\|^4 = o(\|\theta\|^8).
\]

Hence, for the terms related to \(\Delta_{i_1i_2i_3i_4}^{(2)}\) and \(\Delta_{i_1i_2i_3i_4}^{(3)}\), we also have a similar convergence as that of (48). These together imply \(J_1 \overset{p}{\rightarrow} 0\). Hence, (44) is true.

Next, we prove (45). It is seen that
\[
0 \leq G(\Omega) - G(\Omega^*) \leq C \sum_{j,s,m,\ell} \Omega_{js} \Omega_{sm} \Omega_{m\ell} \Omega_{mj},
\]
\[
\leq C \sum_{j,s,m,\ell} \theta_3^3 \theta_2^3 \theta_2^3 \theta_2^4 \leq C \|\theta\|^6 \|\theta\|^4 = o(\|\theta\|^8).
\]
As a result, \(|G(\Omega^*) / G(\Omega) - 1| = o(1)\). This proves (45).
D.4. Proof of Proposition A.1

The last item follows once the first three items are proved, so we only consider the first three items.

Consider the first item. Write

\[ 1'A^21 = \sum_{1 \leq i_1,i_2,i_3 \leq n} A_{i_1i_2}A_{i_2i_3}, \]

Recall that all diagonal entries of \( A \) are 0, we can exclude the case \( i_1 = i_2 \) or \( i_2 = i_3 \) from the summation. Therefore, we only need to sum over either the cases where \( i_1, i_2, i_3 \) are distinct and the cases \( i_1 = i_3 \) but \( i_1 \neq i_2 \). It follows

\[ 1'A^21 = \left( \sum_{\substack{i_1,i_2,i_3 \text{ are distinct} \\ i_1 \neq i_3,i_1 \neq i_2}} + \sum_{\substack{i_1,i_2,i_3 \text{ are distinct} \\ i_1 \neq i_3,i_1 \neq i_2}} \right) A_{i_1i_2}A_{i_2i_3} = I + II. \]

Now, first, by definition,

\[ I = B_{n,3} \hat{L}_2, \quad \text{where } B_{n,3} = 6\binom{n}{3}, \]

and second (recall all diagonal entries of \( A \) are 0),

\[ II = \sum_{i_1,i_2} A^2_{i_1i_2} = \text{tr}(A^2). \]

Combining these gives

\[ \hat{L}_2 = \frac{1}{6\binom{n}{3}}(1'A^21 - \text{tr}(A^2)), \]

and the claim follows.

Consider the second item. Using similar arguments, we decompose

\[ 1'A^31 = \sum_{1 \leq i_1,i_2,i_3,i_4 \leq n} A_{i_1i_2}A_{i_2i_3}A_{i_3i_4} = I + II + III + IV, \]

where

\[ I = \sum_{\substack{i_1,i_2,i_3,i_4 \text{ are distinct}}} A_{i_1i_2}A_{i_2i_3}A_{i_3i_4} = B_{n,4} \hat{L}_3, \]

with \( B_{n,4} = 24\binom{n}{4} \),

\[ II = \left( \sum_{\substack{i_1,i_2,i_3,i_4 \text{ are distinct} \\ i_1 = i_3}} + \sum_{\substack{i_1,i_2,i_3,i_4 \text{ are distinct} \\ i_1 \neq i_3}} \right) A_{i_1i_2}A_{i_2i_3}A_{i_3i_4} = 2\cdot(1'A^21), \]

\[ III = \sum_{i_1,i_2,i_3,i_4} A_{i_1i_2}A_{i_2i_3}A_{i_3i_4} = 1'A^31, \]

and

\[ IV = -\sum_{\substack{i_1,i_2,i_3,i_4 \text{ are distinct} \\ i_1 \neq i_3,i_2 \neq i_4}} A_{i_1i_2}A_{i_2i_3}A_{i_3i_4} = -1'A1. \]

Combining these gives

\[ \hat{L}_3 = \frac{1}{24\binom{n}{4}} [1'A^31 - 2 \cdot 1'A^21 + 1'A1 - \text{tr}(A^3)], \]

and the claim follows.

Consider the third item. Note first

\[ \text{tr}(A^4) = \sum_{1 \leq i_1,i_2,i_3,i_4 \leq n} A_{i_1i_2}A_{i_2i_3}A_{i_3i_4}A_{i_4i_1} = I + II + III, \]

Similarly, we have

\[ \text{tr}(A^4) = \sum_{i_1,i_2,i_3,i_4} A_{i_1i_2}A_{i_2i_3}A_{i_3i_4}A_{i_4i_1} = I + II + III, \]

where

\[ I = \sum_{\substack{i_1,i_2,i_3,i_4 \text{ are distinct}}} A_{i_1i_2}A_{i_2i_3}A_{i_3i_4}A_{i_4i_1} = 24\binom{n}{4} \hat{C}_4, \]

\[ II = \left( \sum_{\substack{i_1,i_2,i_3,i_4 \text{ are distinct} \\ i_1 = i_3}} + \sum_{\substack{i_1,i_2,i_3,i_4 \text{ are distinct} \\ i_1 \neq i_3}} \right) A_{i_1i_2}A_{i_2i_3}A_{i_3i_4}A_{i_4i_1} = 2\cdot(1'A^21), \]

and

\[ III = -\sum_{\substack{i_1,i_2,i_3,i_4 \text{ are distinct} \\ i_1 \neq i_3,i_2 \neq i_4}} A_{i_1i_2}A_{i_2i_3}A_{i_3i_4}A_{i_4i_1} = -1'A1. \]

Combining these gives

\[ \hat{C}_4 = \frac{1}{24\binom{n}{4}} \left( \text{tr}(A^4) - 2 \cdot 1'A^21 + 1'A1 \right), \]

and the claim follows. \( \square \)

References
