D²: Decentralized Training over Decentralized Data

Hanlin Tang ¹ Xiangru Lian ¹ Ming Yan ² ³ Ce Zhang ⁴ Ji Liu ⁵

Abstract

While training a machine learning model using multiple workers, each of which collects data from its own data source, it would be useful when the data collected from different workers are unique and different. Ironically, recent analysis of decentralized parallel stochastic gradient descent (D-PSGD) relies on the assumption that the data hosted on different workers are not too different. In this paper, we ask the question: Can we design a decentralized parallel stochastic gradient descent algorithm that is less sensitive to the data variance across workers? In this paper, we present D², a novel decentralized parallel stochastic gradient descent algorithm designed for large data variance among workers (imprecisely, “decentralized” data). The core of D² is a variance reduction extension of D-PSGD. It improves the convergence rate from $O\left(\frac{\sigma}{\sqrt{nT}} + \frac{(n\varsigma^2)^{\frac{1}{3}}}{T^{\frac{2}{3}}/\sqrt{n}}\right)$ to $O\left(\frac{\sigma}{\sqrt{nT}}\right)$ where $\varsigma^2$ denotes the variance among data on different workers. As a result, D² is robust to data variance among workers. We empirically evaluated D² on image classification tasks, where each worker has access to only the data of a limited set of labels, and find that D² significantly outperforms D-PSGD.

1. Introduction

Training machine learning models in a decentralized way has attracted intensive interests recently (Lian et al., 2017a; Yuan et al., 2016; Colin et al., 2016). In the decentralized setting, there is a set of workers, each of which collects data from different data sources. Instead of sending all data to a centralized place, these workers only communicate with their neighbors. The goal is to get a model that is the same as if all data are collected in a centralized place. Decentralized learning algorithms are important in scenarios where the centralized communication is expensive or impossible, or the underlying communication network has high latency.

For decentralized learning to provide benefits, each user should provide data that is somehow unique, i.e., the variance of data collected from different workers are large. However, many recent theoretical results (Lian et al., 2017a;b; Nedic & Ozdaglar, 2009; Yuan et al., 2016) assume a bounded data variance across workers — when data hosted on different workers are very different, these approaches converge slowly, both empirically and theoretically. In this paper, we aim at bringing this discrepancy between the current theoretical understanding and the requirements from some practical scenarios.

In this paper, we present D², a novel decentralized learning algorithm designed to be robust under high data variance. D² is built upon decentralized parallel stochastic gradient descent (D-PSGD), but benefits from an additional variance reduction component. In D², each worker stores the stochastic gradient and its local model in the previous iterate and linearly combines them with the current stochastic gradient and local model. It results in an improved convergence rate over D-PSGD by eliminating the data variation among workers. In particular, the convergence rate is improved from $O\left(\frac{\sigma}{\sqrt{nT}} + \frac{(n\varsigma^2)^{\frac{1}{3}}}{T^{\frac{2}{3}}/\sqrt{n}}\right)$ to $O\left(\frac{\sigma}{\sqrt{nT}}\right)$ where $\varsigma^2$ is the data variation among all workers, $\sigma^2$ is the data variance within each worker, $n$ is the number of workers, and $T$ is the number of iterations. We empirically show $D²$ can significantly outperform D-PSGD by training an image classification model where each worker has access to only the data of a limited set of labels.

Throughout this paper, we consider the following decentralized optimization:

$$\min_{x \in \mathbb{R}^N} f(x) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{\xi \sim D_i} F_i(x; \xi),$$  \hspace{1cm} (1)
where $n$ is the number of workers and $D_i$ is the local data distribution for worker $i$. All workers are connected through a connected graph. Each worker can only exchange information with its neighbors.

### Definitions and notation

Throughout this paper, we use the following notation and definitions:

- $\| \cdot \|_F$ denotes the Frobenius norm of matrices.
- $\| \cdot \|$ denotes the $\ell_2$ norm for vectors and the spectral norm for matrices.
- $\nabla f(\cdot)$ denotes the gradient of a function $f$.
- $f^*$ denotes the optimal solution of (1).
- $\lambda_i(\cdot)$ denotes the $i$th largest eigenvalue of a matrix.
- $x^{(i)}$ denotes the local model of worker $i$.
- $\nabla F_i(x^{(i)}; \xi^{(i)})$ denotes a local stochastic gradient of worker $i$.
- $1 = [1, 1, \cdots, 1] \in \mathbb{R}^n$ denotes the all-one vector.
- In order to organize the algorithm more clearly, here we define the concatenation of all local variables, stochastic gradients, and their averages respectively:

$$X := [x^{(1)}, \ldots, x^{(n)}] \in \mathbb{R}^{N \times n},$$

$$\bar{X} := \frac{1}{n} \sum_{i=1}^{n} x^{(i)},$$

$$G(X; \xi) := [\nabla F_1(x^{(1)}; \xi^{(1)}), \ldots, \nabla F_n(x^{(n)}; \xi^{(n)})] \in \mathbb{R}^{N \times n},$$

$$\bar{G}(X; \xi) := G(X; \xi) \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(x^{(i)}; \xi^{(i)}),$$

$$\nabla f(X) := \sum_{i=1}^{n} \frac{1}{n} \nabla f_i(X),$$

$$\nabla f(X) := \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(x^{(i)}),$$

where $\xi$ is the collection of randomly sampled data from all workers.

### Organization

This paper is organized as follows: Section 2 reviews related work about the proposed approach; Section 3 introduces the state-of-the-art decentralized stochastic gradient descent method and its convergence rate; Section 4 introduces the proposed algorithm and its intuition why it improves the state-of-the-art approach; Section 5 provides the theoretical guarantee; and Section 6 validates the proposed approaches via empirical study; and Section 7 concludes this paper.

### 2. Related work

In this section, we review the stochastic gradient descent algorithm and its decentralized variants, decentralized algorithms, and previous variance reduction technologies.

#### Stochastic gradient descent (SGD)

The SGD approaches (Ghadimi & Lan, 2013; Moulies & Bach, 2011; Nemirovski et al., 2009) is quite powerful for solving large-scale machine learning problems. It achieves a convergence rate of $O\left(\frac{1}{\sqrt{T}}\right)$. As an implementation of SGD, the Centralized Parallel Stochastic Gradient Descent (C-PSGD), has been widely used in parallel computation. In C-PSGD, a central worker, whose job is to perform the variable updates, is connected to many leaf workers that are used to compute stochastic gradients in parallel. C-PSGD has been applied to many deep learning frameworks, such as CNTK (Seide & Agarwal, 2016), MXNet (Chen et al., 2015), and TensorFlow (Abadi et al., 2016). The convergence rate of C-PSGD is $O\left(\frac{1}{\sqrt{nT}}\right)$, which shows that it can achieve linear speedup with regards to the number of leaf workers.

#### Decentralized algorithms

Centralized algorithms require a central server to communicate with all other workers (Suresh et al., 2017). In contrast, decentralized algorithms work on any connected network and only rely on the information exchange between neighbor workers (Kashyap et al., 2007; Lavaei & Murray, 2012; Nedic et al., 2009).

Decentralized algorithms are especially useful under a network with limited bandwidth or high latency. It is more favorable when data privacy is sensitive. These advantages have led to successful applications. The decentralized approach for multi-task reinforcement learning was studied in Omidsfai et al. (2017); Mhamdi et al. (2017). In Colin et al. (2016), a dual based decentralized algorithm was proposed to solve the pairwise function optimization. Shi et al. (2014) and Mokhtari & Ribeiro (2015) analyzed the decentralized version of the ADMM optimization algorithm. An information theoretic approach was used to analyze decentralization in Dobbe et al. (2017). The decentralized version of (sub-)gradient descent was studied in Nedic & Ozdaglar (2009); Yuan et al. (2016). Its $O(1/\sqrt{T})$ convergence requires a diminishing stepsize or a constant stepsize that depends on the total number of iterations. This phenomenon happens because of the variance between the data in different workers, which we call “outer variance” to differentiate it from the variance in SGD. Recently, there are several deterministic decentralized optimization algorithms that allows a constant stepsize. For example, EXTRA (Shi et al., 2015a) is the first modification of decentralized gradient descent that converges under a constant stepsize. Later this algorithm is extended for problems with the sum of smooth and nonsmooth functions at each node (Shi et al., 2015b).
The algorithm DIGing is proposed in Nedić et al. (2017), where two exchanges are needed in each iteration. However, their stepsizes depend on both the Lipschitz constant of the differentiable function and the network structure. NIDS is the first algorithm that has a constant network independent stepsize (Li et al., 2017). This algorithm was simultaneously proposed by Yuan et al. (2017) for the smooth case only using a different approach.

Decentralized parallel stochastic gradient descent (D-PSGD) The D-PSGD algorithm (Nedic & Ozdaglar, 2009; Ram et al., 2010a;b) requires each worker to compute a stochastic gradient and exchange its local model with neighbors. In Duchi et al. (2012), a dual averaging based method is proposed for solving the constrained decentralized SGD optimization. In Yuan et al. (2016), the convergence rate for D-PSGD was analyzed when the gradient is assumed to be bounded. In Lan et al. (2017), a decentralized primal-dual type method was proposed with a computational complexity of $O(n/\varepsilon^2)$ for general convex objectives. Lian et al. (2017a) proved that D-PSGD can admits linear speedup with respect to the number of workers with a similar convergence rate as C-PSGD.

Variance reduction technology There have been many methods developed for reducing the variance in SGD, including SVRG (Johnson & Zhang, 2013), SAGA (Defazio et al., 2014), SAG (Schmidt et al., 2017), MISD (Mairal, 2015), and mS2GD (Konečný et al., 2016). However, most of these technologies are designed for centralized approaches. The DSA algorithm (Mokhtari & Ribeiro, 2016) applied the variance reduction similar to SAGA on strongly convex decentralized optimization problems and proved a linear convergence rate. However, the speedup property is unclear and a table of all stochastic gradients need to be stored.

3. Preliminary: decentralized stochastic gradient descent

The decentralized stochastic gradient descent (Lian et al., 2017a; Zhang et al., 2017; Shahrampour & Jadbabaie, 2017) allows each worker (say worker $i$) maintaining its own local variable $x_i$. During each iteration (say, iteration $t$), each worker performs the following steps:

1. Query its neighbors’ local variables.
2. Take weighted average with its local variable and its neighbors’ local variables:

$$x_{t+1} = \sum_{j=1}^{n} W_{ij} x_{ij},$$

where $W_{ij}$ is the $(i, j)$ element of the matrix $W$. $W_{ij} = 0$ means worker $i$ and worker $j$ are not connected.

3. Perform one stochastic gradient descent step

$$x_{t+1} = x_{t+1} - \gamma \nabla f(x_{t+1}; x_{t})$$

where $x_{t}$ represents the data sampled in worker $i$ at the iteration $t$ following the distribution $D_i$.

From a global point of view, the update rule of D-PSGD can be viewed as

$$X_{t+1} = X_t W - \gamma G(X_t; x_{t}).$$

It admits the following rate shown in Theorem 1.

**Theorem 1** (Convergence rate of D-PSGD (Lian et al., 2017a)). Under certain assumptions, the output of D-PSGD admits the following inequality

$$\frac{1 - \gamma L^2}{2T} \sum_{t=0}^{T-1} \mathbb{E} \|\nabla f(X_t)\|^2 + \frac{D_1}{T} \sum_{t=0}^{T-1} \mathbb{E} \|\nabla f(X_t)\|^2$$

$$\leq \frac{f(0) - f^*}{\gamma T} + \frac{\gamma L^2 n \sigma^2}{2} + \frac{\gamma^2 L^2 n \sigma^2}{2} + \frac{9\gamma^2 L^2 n \sigma^2}{2} + \frac{1}{(1 - \sqrt{\lambda})^2 D_2},$$

where $\sigma$ reflects the property of the network, $D_1$ and $D_2$ are defined to be

$$D_1 := \left(1 - \frac{9\gamma^2 L^2 n}{(1 - \sqrt{\lambda})^2 D_2}\right),$$

$$D_2 := \left(1 - \frac{18\gamma^2}{(1 - \sqrt{\lambda})^2 n L^2}\right),$$

and $\sigma$ and $\zeta$ measure the variation within each worker and among all workers respectively

$$\mathbb{E}_{\zeta \sim \mathcal{D}} \|\nabla F_i(x; \zeta) - \nabla f_i(x)\|^2 \leq \sigma^2, \quad \forall i, \forall x, \quad (2)$$

$$\frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(x) - \nabla f(x)\|^2 \leq \zeta^2, \quad \forall i, \forall x. \quad (3)$$

Choosing the optimal stepsize $\gamma = \frac{1}{L + \sigma \sqrt{\frac{n}{T} + \frac{1}{2} \zeta^2}}$ we have the following convergence rate:

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \|\nabla f(X_t)\|^2 \leq O \left(\frac{\sigma}{\sqrt{n} T} + \frac{n \zeta^2}{T^2} + \frac{1}{T}\right).$$

The proposed $D^2$ algorithm can improve the convergence rate by removing the dependence to the global bound of outer variance $\zeta$.  


The complete algorithm is summarized in Algorithm 1.

Algorithm 1 The $D^2$ algorithm

1: Input: Initial point $x_0^{(i)} = 0$, step length $\gamma > 0$, confusion matrix $W$, and the total number of iterations $T$.
2: for $t = 0,1,\ldots,T$ do
3: Randomly sample $\xi_t^{(i)}$ from the local data of the $i$th worker.
4: Compute a local stochastic gradient based on $\xi_t^{(i)}$ and current variable $x_t^{(i)}$: $\nabla F_i(x_t^{(i)}; \xi_t^{(i)})$.
5: if $t=0$ then
6: $x_t^{(i)} = x_0^{(i)} - \gamma \nabla F_i(x_0^{(i)}; \xi_t^{(i)})$.
7: else
8: $x_t^{(i)} = 2x_t^{(i)} - x_{t-1}^{(i)} - \gamma \nabla F_i(x_t^{(i)}; \xi_t^{(i)}) + \gamma \nabla F_i(x_{t-1}^{(i)}; \xi_{t-1}^{(i)})$.
9: end if
10: Each worker sends $x_t^{(i)}$ to its neighbors and takes the weighted average
11: $x_{t+\frac{1}{2}}^{(i)} = \frac{1}{\frac{n}{2}} \sum_{j=1}^{n} W_{ij}x_t^{(j)}$, where $x_t^{(j)}$ is from the worker $j$.
12: Output: $\frac{1}{n} \sum_{i=1}^{n} x_T^{(i)}$.

4. The $D^2$ algorithm

In $D^2$ algorithm, each worker (say, worker $i$) repeats the following updating rule (say, at iteration $t$):

1. Compute a local stochastic gradient $\nabla F_i(x_t^{(i)}; \xi_t^{(i)})$ by sampling $\xi_t^{(i)}$ from distribution $D^{(i)}$.
2. Update the local model $x_{t+\frac{1}{2}}^{(i)} := 2x_t^{(i)} - x_{t-1}^{(i)} - \gamma \nabla F_i(x_t^{(i)}; \xi_t^{(i)}) + \gamma \nabla F_i(x_{t-1}^{(i)}; \xi_{t-1}^{(i)})$ using the local models and stochastic gradients in both the $t$th iteration and the $(t-1)$th iteration.
3. When the synchronization barrier is met, exchange $x_{t+\frac{1}{2}}^{(i)}$ with neighbors:

$$x_{t+1}^{(i)} = \sum_{j=1}^{n} W_{ij}x_{t+\frac{1}{2}}^{(j)}.$$ From a global point of view, the update rule of $D^2$ can be viewed as:

$$x_{t+1} = (2x_t - x_{t-1} - \gamma G(x_t; \xi_t) + \gamma G(x_{t-1}; \xi_{t-1})) W.$$ The complete algorithm is summarized in Algorithm 1.

$D^2$ essentially runs the stochastic gradient descent step. To understand the intuition of $D^2$, let us consider the mean value $X_t$, which is updated just like the standard stochastic gradient descent:

$$x_{t+1} = (2x_t - x_{t-1} - \gamma G(x_t; \xi_t) + \gamma G(x_{t-1}; \xi_{t-1})) W \frac{1}{n}.$$ 

or equivalently

$$X_{t+1} = X_t - X_{t-1} - \gamma G(X_t; \xi_t) + \gamma G(X_{t-1}; \xi_{t-1}),$$

Why $D^2$ improves the D-PSGD? We may notice that D-PSGD also essentially updates in the form of stochastic gradient descent in (4). Then why $D^2$ can improve D-PSGD? Assume that $x_t$ has achieved the optimum $X^* := x^* \mathbf{1}^T$ with all local models equal to the optimum $x^*$ to (1). Then for D-PSGD, the next update will be

$$x_{t+1} = X^* - \gamma G(X^*; \xi_t).$$

It shows that the convergence when we approach a solution is affected by $E[\|G(X^*; \xi_t)\|^2]$, which is bounded by

$$O(\sigma^2 + \zeta^2),$$

as we can see from the following:

$$E[\|G(X^*; \xi_t)\|^2]$$

$$= E \sum_{i=1}^{n} \|\nabla f_i(x^*; \xi_t^{(i)}) - \nabla f_i(x^*)\|^2$$

$$\leq 2E \sum_{i=1}^{n} \|\nabla f_i(x^*; \xi_t^{(i)}) - \nabla f_i(x^*)\|^2$$

$$+ 2\|\nabla f_i(x^*) - \nabla f(x^*)\|^2$$

$$\leq 2\sigma^2 + 2\zeta^2.$$ Next we apply a similar analysis for $D^2$ by assuming that both $x_{t-1}$ and $x_t$ have reached the optimal solution $X^*$. The next update for $D^2$ will be:

$$x_{t+1} = (X^* - \gamma G(X^*; \xi_t) - \gamma G(X^*; \xi_{t-1})) W.$$ It shows that for $D^2$, the convergence when we approach a solution relies on the magnitude of $E[\|G(X^*; \xi_t) - G(X^*; \xi_{t-1})\|^2]$, which is bounded by:

$$O(\sigma^2),$$
which can be seen from:
\[
\mathbb{E}[||G(X^*; \xi_l) - G(X^*; \xi_{l-1})||_F^2] = \mathbb{E} \sum_{i=1}^{n} ||\nabla F_i(x^*; \xi_i^{(l)}) - \nabla f_i(x^*)||^2
- \mathbb{E} \sum_{i=1}^{n} ||\nabla F_i(x^*; \xi_i^{(l-1)}) - \nabla f_i(x^*)||^2 \leq 2\sigma^2.
\]

5. Theoretical guarantee

This section provides the theoretical guarantee for the proposed \(D^2\) algorithm. We first give the assumptions required below.

**Assumption 1.** Throughout this paper, we make the following commonly used assumptions:

1. **Lipschitzian gradient:** All function \(f_i(\cdot)\)'s are with \(L\)-Lipschitzian gradients.
2. **Bounded variance:** Assume bounded variance of stochastic gradient within each worker
   \[
   \mathbb{E}_{\xi \sim \mathcal{D}_i} \left\| \nabla F_i(x; \xi) - \nabla f_i(x) \right\|^2 \leq \sigma^2, \quad \forall i, \forall x.
   \]
3. **Symmetric confusion matrix:** The confusion matrix \(W\) is symmetric and satisfies \(W1 = 1\).
4. **Spectral gap:** Let the eigenvalues of \(W \in \mathbb{R}^{n \times n}\) be \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\). Denote by for short
   \[
   \lambda := \max_{i \in \{2, \ldots, n\}} \lambda_i = \lambda_2.
   \]
   We assume \(\lambda < 1\) and \(\lambda_n > -\frac{1}{3}\).
5. **Initialization:** W.l.o.g., assume all local variables are initialized by zero, that is, \(X_0 = 0\).

Existing decentralized consensus algorithms (Shi et al., 2015b; Li et al., 2017) use a modification of the doubly stochastic matrix such that \(\lambda > 0\), i.e., choose \(W = (W + I)/2\) where \(W\) is a doubly stochastic matrix. Recently, Li & Yan (2017) show that \(\lambda_n > -1/3\) is optimal in the convergence of EXTRA. However, the optimal \(\lambda_n\) for NIDS (Li et al., 2017) is unknown. In this paper, we proved that \(-\frac{1}{3}\) is the infimum of \(\lambda_n\), and when it reduces to deterministic case, this condition is weaker than that in (Li et al., 2017). This is important, because we actually can use a \(W\) that performs better.

Given Assumption 1, we have following convergence guarantee for \(D^2\):

**Theorem 2** (Convergence of Algorithm 1). Choose the step length \(\gamma\) in Algorithm 1 to be a constant satisfying \(1 - 24C_2\gamma^2L^2 > 0\). Under Assumption 1, we have the following convergence rate for Algorithm 1:

\[
\begin{align*}
A_1\|\nabla f(0)\|^2 &+ \sum_{i=1}^{T-1} \left( \mathbb{E}\|\nabla f(X_t)\|^2 + A_2\mathbb{E}\|\nabla f(X_t)\|^2 \right) \\
& \leq \frac{2(f(0) - f^*)}{\gamma} + \frac{LT\gamma}{n} + \frac{6L^2C_2\gamma^2\zeta_0}{C_3} + \frac{12L^2C_2\gamma^2\sigma^2T}{C_3} + \frac{6L^2C_2\gamma^4L^2\sigma^2T}{nC_3} + \frac{6L^2C_1\gamma^2\sigma^2}{C_3},
\end{align*}
\]

where

\[
\begin{align*}
\zeta_0 &:= \frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(0) - \nabla f_i(0)\|^2, \\
v &:= \lambda_n - \sqrt{\lambda_n^2 - \lambda_n}, \\
C_1 &:= \max \left\{ \frac{1}{1 - |v|^2}, \frac{1}{(1 - \lambda)^2} \right\} \geq 1, \\
C_2 &:= \max \left\{ \frac{\lambda_n^2}{1 - |v|^2}, \frac{\lambda_n^2}{(1 - \sqrt{\lambda})^2(1 - \lambda)} \right\}, \\
C_3 &:= 1 - 24C_2\gamma^2L^2, \\
A_1 &:= 1 - \frac{6L^2C_1\gamma^2}{C_3}, \\
A_2 &:= 1 - \frac{6L^2C_2\gamma^4L^2}{C_3}.
\end{align*}
\]

By appropriately specifying the step length \(\gamma\), we reach the following corollary:

**Corollary 3.** Choose the step length \(\gamma\) in Algorithm 1 to be \(\gamma = \frac{1}{8C_2L^2 + 6\sqrt{C_1L} + \sigma \sqrt{T}}\), where \(C_1\) and \(C_2\) are defined in Theorem 2. Under Assumption 1, the following convergence rate holds

\[
\begin{align*}
\frac{1}{T} \sum_{t=0}^{T} \mathbb{E}\|\nabla f(X_t)\|^2 &\leq \frac{\sigma^2}{\sqrt{nT}} + \frac{1}{T} + \frac{\zeta_0^2}{T + \sigma^2T^2} + \frac{\zeta_0^2}{1 + \sigma^2T},
\end{align*}
\]

where \(\zeta_0\) is defined in Theorem 2 and we treat \(f(0) - f^*\), \(L\), \(\lambda_n\), and \(\lambda\) as constants.

Note that we can obtain even better constants by choosing different parameters and applying tighter inequalities, however, the main result of this corollary is to show the order of the convergence. We highlight a few key observations from our theoretical results in the following.

**Tightness of the convergence rate** Setting \(\sigma = 0\) and \(\zeta_0 = 0\), which reduces the VR-SGD to a normal GD
6. Experiments

We evaluate the effectiveness of D² by comparing it with both centralized and decentralized SGD algorithms.

6.1. Experiment Settings

We conduct experiments in two settings.

1. TRANSFERLEARNING: We test the case that each worker has access to a local pre-trained neural network as feature extractor, and we want to train a logistic regression model among all these workers. In our experiment, we select the first 16 classes of ImageNet and use InceptionV4 as the feature extractor to extract 2048 features for each image. We conduct data augmentation and generate a blurback version for each image. In total this dataset contains 16 × 1300 × 2 images.

2. LE.NET: We test the case that all workers collaboratively train a neural network model. We train a LeNet on the CIFAR10 dataset. In total this dataset contains 50,000 images of size 32 × 32.

One caveat of training more recent neural networks is that modern architectures often have a batch normalization layer, which inherently assumes that the data distribution is uniform across different batches, which is not the case that we are interested in. In principle, we could also flow the batch information through the network in a decentralized way; however, we leave this as future work.

By default, each worker only has exclusive access to a subset of classes. For TRANSFERLEARNING, we use 16 workers and each worker has access to one class; for LE.NET, we use 5 workers and each worker has access to two classes. For comparison, we also consider a case when the datasets are first shuffled and then uniformly partitioned among all the workers, we call this the unshuffled case, and the default one the shuffled case. We use a ring topology for both experiments.

Parameter Tuning. For TRANSFERLEARNING, we use constant learning rates and tune it from {0.01, 0.025, 0.05, 0.075, 0.1}. For LE.NET, we use constant learning rate 0.05 which is tuned from {0.5, 0.1, 0.05, 0.01} for centralized algorithms and batch size 128 on each worker.

Metrics. In this paper, we mainly focus on the convergence rate of different algorithms instead of the wall clock speed. This is because the implementation of D² is a minor change over the standard D-PSGD algorithm, and thus they has almost the same speed to finish one epoch of training, and both are no slower than the centralized algorithm. When the network has high latency, if a decentralized algorithm (D² or D-PSGD) converges with a similar speed as the centralized algorithm, it can be up to one order of magnitude faster (Lian et al., 2017a). However, the convergence rate depending on the “outer variance” is different for both algorithms.

6.2. Unshuffled Case

variation across workers is maximized. Figure 1 shows the result. In the unshuffled case, we see that D-PSGD converges slower than the centralized case. This is consistent with the original D-PSGD paper (Lian et al., 2017a). On the other hand, D² converges much faster than D-PSGD, and achieves almost the same loss as the centralized algorithm. When the learning rate is tuned from {0.01, 0.025, 0.05, 0.075, 0.1}, we use constant learning rate 0.05 which is tuned from {0.5, 0.1, 0.05, 0.01} for centralized algorithms and batch size 128 on each worker.

6.3. Shuffled Case

As a sanity check, Figure 2 shows the result of three different algorithms on the shuffled data. In this case, the data variation among workers is small (in expectation, they are drawn from the same distribution). We see that, all strategies have similar convergence rate. This validate that D² is more effective for larger data variation between different workers.

---

1We can tune the learning rate 50x smaller for D-PSGD to converge in this case, but doing so will make D-PSGD stuck at the starting point for quite a long time.
Decentralized Training over Decentralized Data

7. Conclusion
In this paper, we propose a decentralized algorithm, namely, \( D^2 \) algorithm. \( D^2 \) algorithm integrates the D-PSGD algorithm with the variance reduction technology, by which we improve the convergence rate of D-PSGD. The variance reduction technology used in this paper is different from the commonly used ones such as SVRG and SAGA, that are designed for centralized approaches. Experiments validate the advantage of \( D^2 \) over D-PSGD — \( D^2 \) converges with a rate that is similar to centralized SGD while D-PSGD does not converge to a solution with a similar quality when the data variance is large. While being robust to large data variance among workers, the same performance benefit of D-PSGD over the centralized strategy still holds for \( D^2 \).

Acknowledgements
This project is supported in part by NSF CCF1718513, NEC fellowship, IBM faculty award, NSF DMS-1621798, Swiss NSF NRP 75 407540_167266, IBM Zurich, Mercedes-Benz Research & Development North America, Oracle...
Decentralized Training over Decentralized Data

Labs, Swisscom, Zurich Insurance, and Chinese Scholarship Council.

References


Lan, G., Lee, S., and Zhou, Y. Communication-efficient algorithms for decentralized and stochastic optimization. 01 2017.


Decentralized Training over Decentralized Data


Supplemental Materials

This supplement material includes the proofs for Theorem 2.

Because the confusion matrix $W$ is symmetric, it can be decomposed as $W = P\Lambda P^\top$, where $P = (v_1, v_2, \ldots, v_n)$ is an orthogonal matrix, i.e., $P^\top P = PP^\top = I$, and $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$ is a diagonal matrix with diagonal entries being the eigenvalues of $W$ in the nonincreasing order. Then applying the decomposition to the iteration (from $W_t$ and $W_{t-1}$ to $W_{t+1}$)

$$X_{t+1} = 2X_t W - X_{t-1} W - \gamma G(X_t; \xi_t) W + \gamma G(X_{t-1}; \xi_{t-1}) W$$

gives

$$X_{t+1} = 2X_t P\Lambda P^\top - X_{t-1} P\Lambda P^\top - \gamma G(X_t; \xi_t) P\Lambda P^\top + \gamma G(X_{t-1}; \xi_{t-1}) P\Lambda P^\top.$$  

Denote $Y_t = X_t P$, $H(X_t; \xi_t) = G(X_t; \xi_t) P$, and use $y^{(i)}_t$ and $h^{(i)}_t$ to indicate the $i$-th column of $Y_t$ and $H(X_t; \xi_t)$, respectively. Then

$$Y_{t+1} = 2Y_t \Lambda - Y_{t-1} \Lambda - \gamma H(X_t; \xi_t) \Lambda + \gamma H(X_{t-1}; \xi_{t-1}) \Lambda,$$

or in the columns of $Y_t$ and $H(X_t; \xi_t)$,

$$y^{(i)}_{t+1} = \lambda_i(2y^{(i)}_t - y^{(i)}_{t-1} - \gamma h^{(i)}_t + \gamma h^{(i)}_{t-1}).$$

From the properties of $W$ in Assumption 1 and the decomposition, we have $\lambda_1 = 1$ and $v_1 = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^\top$. Therefore $y^{(1)}_t = \sqrt{n}$. For all other eigenvalues $-\frac{1}{2} < \lambda_i < 1$, the equation (7) shows that all $y^{(i)}_t$ would “decay to zero”, which explains how the confusion matrix works.

**Lemma 4.** Given two non-negative sequences \(\{a_t\}_{t=1}^\infty\) and \(\{b_t\}_{t=1}^\infty\) that satisfying

$$a_t = \sum_{s=1}^{t} \rho^{t-s} b_s,$$

with $\rho \in [0, 1)$, we have

$$S_k := \sum_{t=1}^{k} a_t \leq \frac{b_k}{1 - \rho},$$

$$D_k := \sum_{t=1}^{k} a_t^2 \leq \frac{1}{(1 - \rho)^2} \sum_{s=1}^{k} b_s^2.$$  

**Proof.**

$$S_k := \sum_{t=1}^{k} a_t = \sum_{t=1}^{k} \sum_{s=1}^{t} \rho^{t-s} b_s = \sum_{s=1}^{k} \sum_{t=s}^{k} \rho^{t-s} b_s = \sum_{s=1}^{k} \rho^0 b_s \leq \sum_{s=1}^{k} b_s \leq \frac{b_k}{1 - \rho}. (9)$$

$$D_k := \sum_{t=1}^{k} a_t^2 = \sum_{t=1}^{k} \sum_{s=1}^{t} \rho^{t-s} b_s^2 \leq \sum_{t=1}^{k} \sum_{s=1}^{t} \rho^{2t-s-r} b_s^2 b_r \leq \frac{1}{1 - \rho} \sum_{t=1}^{k} \sum_{s=1}^{t} \rho^{t-s} b_s^2 \leq \frac{1}{(1 - \rho)^2} \sum_{s=1}^{k} b_s^2 (10)$$

where the last inequality holds because of (9).
Lemma 5. For any matrix $X_t \in \mathbb{R}^{N \times n}$, we have

$$
\sum_{i=2}^{n} \left\| X_t v^{(i)} \right\|^2 \leq \sum_{i=1}^{n} \left\| X_t v^{(i)} \right\|^2 = \| X_t \|_F^2.
$$

$$
\sum_{i=1}^{n} \left\| X_t P^T e^{(i)} \right\|^2 = \| X_t P^T \|_F^2 = \| X_t \|_F^2.
$$

where $e^{(i)} \in \mathbb{R}^{n \times 1}$ with the $i$-th component being 1 and all others being 0.

Proof. From the definition of the Frobenius norm for a matrix, we have

$$
\sum_{i=1}^{n} \left\| X_t v^{(i)} \right\|^2 = \| X_t P \|_F^2 = Tr \left( X_t P P^T X^T_t \right) = Tr \left( X_t X^T_t \right) = \| X_t \|_F^2.
$$

Since $\| X_t v^{(1)} \|^2 \geq 0$, so

$$
\sum_{i=2}^{n} \left\| X_t v^{(i)} \right\|^2 \leq \sum_{i=1}^{n} \left\| X_t v^{(i)} \right\|^2 = \sum_{i=1}^{n} \| X_t \|_F^2.
$$

In the same way, we have

$$
\sum_{i=1}^{n} \left\| X_t P^T e^{(i)} \right\|^2 = \| X_t P^T \|_F^2 = \| X_t \|_F^2.
$$

The result is proved.

Lemma 6. Given $\rho \in (-\frac{1}{3}, 0) \cup (0, 1)$, for any two sequence $\{a_t\}_{t=0}^{\infty}$ and $\{b_t\}_{t=0}^{\infty}$ that satisfy

$$
a_0 = b_0 = 0,\quad a_1 = b_1,\quad a_{t+1} = \rho (2a_t - a_{t-1}) + b_t - b_{t-1}, \quad \forall t \geq 1,
$$

we have

$$
a_{t+1} = a_1 \left( \frac{a^{t+1} - v^{t+1}}{u - v} \right) + \sum_{s=1}^{t} \beta_s \frac{a^{t-s+1} - v^{t-s+1}}{u - v}, \quad \forall t \geq 0,
$$

where

$$
\beta_s = b_s - b_{s-1}, \quad u = \rho + \sqrt{\rho^2 - \rho}, \quad v = \rho - \sqrt{\rho^2 - \rho}.
$$

More specifically, if $0 < \rho < 1$, we have

$$
a_{t+1} \sin \theta = a_1 \rho^{t/2} \sin \left( (t + 1) \theta \right) + \sum_{s=1}^{t} \beta_s \rho^{(t-s)/2} \sin \left( (t + 1 - s) \theta \right), \quad \forall t \geq 0
$$

where

$$
\beta_s = b_s - b_{s-1}, \quad \theta = \arccos(\sqrt{\rho}).
$$

Proof. When $t = 0$, the results is easy to verify. Next we consider the case $t \geq 1$. Since

$$
a_{t+1} = 2 \rho a_t - \rho a_{t-1} + \beta_t,
$$
We can find
\[ u = \rho + \sqrt{\rho^2 - \rho}, \quad v = \rho - \sqrt{\rho^2 - \rho}, \]
such that
\[ a_{t+1} - ua_t = (a_t - ua_{t-1})v + \beta_t. \]  \hspace{1cm} (11)

Note that \( u \) and \( v \) are complex numbers when \( 0 < \rho < 1 \). That is
\[ u = \sqrt{\rho}e^{i\theta}, \quad v = \sqrt{\rho}e^{-i\theta}, \]
with \( \theta = \arccos(\sqrt{\rho}) \).

Recursively applying (11) gives
\[ a_{t+1} - ua_t = (a_t - ua_{t-1})v + \beta_t = (a_{t-1} - ua_{t-2})v^2 + \beta_{t-1}v + \beta_t \]
\[ = (a_1 - ua_0)v^t + \sum_{s=1}^{t} \beta_sv^{t-s} \]
\[ = a_1v^t + \sum_{s=1}^{t} \beta_sv^{t-s}. \] (due to \( a_0 = 0 \))

Diving both sides by \( u^{t+1} \), we obtain
\[ \frac{a_{t+1}}{u^{t+1}} = \frac{a_t}{u^t} + u^{-(t+1)} \left( a_1v^t + \sum_{s=1}^{t} \beta_sv^{t-s} \right) \]
\[ = \frac{a_{t-1}}{u^{t-1}} + u^{-t} \left( a_1v^{t-1} + \sum_{s=1}^{t-1} \beta_sv^{t-1-s} \right) + u^{-(t+1)} \left( a_1v^t + \sum_{s=1}^{t} \beta_sv^{t-s} \right) \]
\[ = \frac{a_1}{u} + \sum_{k=1}^{t} u^{-k} \left( a_1v^k + \sum_{s=1}^{k} \beta_sv^{k-s} \right) \]

Then we multiply both sides by \( u^{t+1} \) and have
\[ a_{t+1} = a_1u^t \sum_{k=0}^{t} u^{t-k} \left( a_1v^k + \sum_{s=1}^{k} \beta_sv^{k-s} \right) \]
\[ = a_1u^t \left( 1 + \sum_{k=1}^{t} \left( \frac{v}{u} \right)^k \right) + u^t \sum_{s=1}^{k} \beta_sv^{s} \left( \frac{v}{u} \right)^k \]
\[ = a_1u^t \sum_{k=0}^{t} \left( \frac{v}{u} \right)^k + u^t \sum_{s=1}^{k} \beta_sv^{s} \left( \frac{v}{u} \right)^k \] (due to \( \sum_{k=1}^{t} a_kv_k = \sum_{s=1}^{t} \sum_{k=s}^{t} a_kv_k \))
\[ = a_1u^t \left( 1 - \frac{v^{t+1}}{1 - \frac{v}{u}} \right) + u^t \sum_{s=1}^{k} \beta_sv^{s} \left( \frac{v}{u} \right)^s \frac{1 - \left( \frac{v}{u} \right)^{t+1}}{1 - \frac{v}{u}} \]
\[ = a_1 \left( \frac{u^{t+1} - v^{t+1}}{u - v} \right) + \sum_{s=1}^{t} \beta_s u^{t-s+1} - u^{t-s+1} \]
\[ \frac{u^{t-s+1}}{u - v}. \]

When \( \rho \in (0, 1) \), since \( u = \sqrt{\rho}e^{i\theta} \) and \( v = \sqrt{\rho}e^{-i\theta} \), we have
\[ a_{t+1} = a_1\rho^{t/2} \frac{\sin[(t+1)\theta]}{\sin \theta} + \sum_{s=1}^{t} \beta_s \rho^{(t-s)/2} \frac{\sin[(t-s+1)\theta]}{\sin \theta}. \]
Lemma 7. Under Assumption 1, we have

\[
(1 - 24C_2\gamma^2 L^2) \sum_{i=1}^{n} \sum_{t=0}^{T} \left\| \mathbf{X}_t - x_t^{(i)} \right\|^2 \\
\leq 2C_1 \| X_1 \|^2 + 12C_2\gamma^2 \sigma^2 T + 6C_2\gamma^4 L^2 \sigma^2 T + 6C_2\gamma^4 L^2 n \sum_{t=1}^{T-1} \left\| \nabla f(X_t) \right\|^2 ,
\]

where \( \gamma, L, \sigma, \theta, C_1 \) and \( C_2 \) are defined in Theorem 2.

Proof. To estimate the difference of the local models and the global mean model, we have

\[
\sum_{i=1}^{n} \left\| \mathbf{X}_t - x_t^{(i)} \right\|^2 = \sum_{i=1}^{n} \left\| X_t e^{(i)} - X_t \frac{1_n}{\sqrt{n}} \right\|^2 = \left\| X_t - X_t \frac{1_n}{\sqrt{n}} \right\|_F^2 = \left\| X_t PP^T - X_t v_t v_t^T \right\|_F^2.
\]

where \( y_t^{(i)} \) is the \( i \)-th column of \( X_t P \). Note that we have, from (7),

\[
y_t^{(i)} = \lambda_i (2y_t^{(i)} - y_{t-1}^{(i)} - \gamma h_t^{(i)} + h_{t-1}^{(i)}) = \lambda_i (2y_t^{(i)} - y_{t-1}^{(i)}) + \lambda_i \beta_t^{(i)},
\]

where \( \beta_t^{(i)} = -\gamma h_t^{(i)} + h_{t-1}^{(i)} \). For all \( y^{(i)} \) that corresponding to \(-\frac{1}{3} < \lambda_i < 0\), Lemma 6 shows

\[
y_{t+1}^{(i)} = y_t^{(i)} \left( \frac{u_t^{t+1} - v_t^{t+1}}{u_t - v_t} \right) + \lambda_i \sum_{s=1}^{t} \beta_s^{(i)} \frac{u_t^{t-s+1} - v_t^{t-s+1}}{u_t - v_t},
\]

where \( u_t = \lambda_t + \sqrt{\lambda_t^2 - \lambda_i} \) and \( v_t = \lambda_t - \sqrt{\lambda_t^2 - \lambda_i} \). Therefore, we have

\[
\left\| y_{t+1}^{(i)} \right\|^2 \leq 2 \left\| y_t^{(i)} \right\|^2 \left( \frac{u_t^{t+1} - v_t^{t+1}}{u_t - v_t} \right)^2 + 2\lambda_i^2 \left( \sum_{s=1}^{t} \beta_s^{(i)} \left\| \frac{u_t^{t-s+1} - v_t^{t-s+1}}{u_t - v_t} \right\| \right)^2.
\]

For \( \frac{u_t^{n+1} - v_t^{n+1}}{u_t - v_t} \), we have

\[
\left| \frac{u_t^{n+1} - v_t^{n+1}}{u_t - v_t} \right| \leq \left| \frac{u_t^{n} - v_t^{n}}{u_t - v_t} \right| \leq \left| v_t \right|^n \quad \text{(due to } |u_t| < |v_t|\).
\]

Using (13), we obtain

\[
\left\| y_{t+1}^{(i)} \right\|^2 \leq 2 \left\| y_t^{(i)} \right\| \left\| v_t \right\|^{2t} + 2\lambda_i^2 \left( \sum_{s=1}^{t} \beta_s^{(i)} \left\| v_t \right\|^{t-s} \right)^2.
\]

Summing from \( t = 0 \) to \( t = T - 1 \) gives

\[
\sum_{t=0}^{T-1} \left\| y_{t+1}^{(i)} \right\|^2 = \sum_{t=1}^{T} \left\| y_t^{(i)} \right\|^2 \leq 2 \left\| y_0^{(i)} \right\| \sum_{t=0}^{T-1} \left\| v_t \right\|^{2t} + 2\lambda_i^2 \sum_{t=1}^{T-1} \left( \sum_{s=1}^{t} \beta_s^{(i)} \left\| v_t \right\|^{t-s} \right)^2.
\]
Denote $a_t = \sum_{s=1}^{t} \beta_s^{(i)} |v_i|^{t-s}$, which has the same structure as the sequence in Lemma 4. Therefore, when $\lambda_i < 0$, we have

$$\sum_{t=1}^{T} \left\| y_t^{(i)} \right\|^2 \leq \frac{2 \left\| y_1^{(i)} \right\|^2}{1 - |v|} + \frac{2 \lambda_i^2}{(1 - |v|)^2} \sum_{t=1}^{T-1} \left\| \beta_t^{(i)} \right\|^2 \leq \frac{2 \left\| y_1^{(i)} \right\|^2}{1 - |v|} + \frac{2 \lambda_i^2}{(1 - |v|)^2} \sum_{t=1}^{T-1} \left\| \beta_t^{(i)} \right\|^2, \tag{14}$$

where $v = \lambda_n - \sqrt{\lambda_n^2 - \lambda_0}$.

For all $y_t^{(i)}$ that satisfies $0 \leq \lambda_i < 1$, from (7) and Lemma 6, we have

$$y_{t+1}^{(i)} \sin \theta_i = y_1^{(i)} \lambda_i^{t/2} \sin [(t + 1) \theta_i] + \lambda_i \sum_{s=1}^{t} \beta_s^{(i)} \lambda_i^{(t-s)/2} \sin [(t + 1 - s) \theta_i],$$

where $\beta_s^{(i)} = -\gamma h_s^{(i)} + \gamma h_{s-1}^{(i)}$ and $\theta_i = \arccos(\sqrt{\lambda_i})$.

Then

$$\left\| y_{t+1}^{(i)} \right\|^2 \sin^2 \theta_i \leq 2 \left\| y_1^{(i)} \right\|^2 \lambda_i^{t} \sin^2 [(t + 1) \theta_i] + 2 \lambda_i^2 \left( \sum_{s=1}^{t} \left\| \beta_s^{(i)} \right\| \lambda_i^{(t-s)/2} \right)^2 \leq 2 \left\| y_1^{(i)} \right\|^2 \lambda_i^{t} + 2 \lambda_i^2 \left( \sum_{s=1}^{t} \left\| \beta_s^{(i)} \right\| \lambda_i^{(t-s)/2} \right)^2,$$

Summing from $t = 0$ to $T - 1$ gives

$$\sum_{t=0}^{T-1} \left\| y_{t+1}^{(i)} \right\|^2 \sin^2 \theta_i = \sum_{t=1}^{T} \left\| y_t^{(i)} \right\|^2 \sin^2 \theta_i \leq 2 \left\| y_1^{(i)} \right\|^2 \sum_{t=0}^{T-1} \lambda_i^{t} + 2 \lambda_i^2 \sum_{t=1}^{T-1} \left( \sum_{s=1}^{t} \left\| \beta_s^{(i)} \right\| \lambda_i^{(t-s)/2} \right)^2.$$

From Lemma 4, $\sum_{s=1}^{t} \left\| \beta_s^{(i)} \right\| \lambda_i^{(t-s)/2}$ has the same structure as the sequence in Lemma 4, so we have

$$\sum_{t=1}^{T} \left\| y_t^{(i)} \right\|^2 \sin^2 \theta_i \leq \frac{2 \left\| y_1^{(i)} \right\|^2}{1 - \lambda_i} + \frac{2 \lambda_i^2}{(1 - \sqrt{\lambda_i})^2} \sum_{t=1}^{T-1} \left\| \beta_t^{(i)} \right\|^2.$$

Then $\sin^2 \theta_i = 1 - \lambda_i$ gives

$$\sum_{t=1}^{T} \left\| y_t^{(i)} \right\|^2 \leq \frac{2 \left\| y_1^{(i)} \right\|^2}{(1 - \lambda_i)^2} + \frac{2 \lambda_i^2}{(1 - \sqrt{\lambda_i})^2(1 - \lambda_i)} \sum_{t=1}^{T-1} \left\| \beta_t^{(i)} \right\|^2 \leq \frac{2 \left\| y_1^{(i)} \right\|^2}{(1 - \lambda)^2} + \frac{2 \lambda_i^2}{(1 - \sqrt{\lambda_i})^2(1 - \lambda)} \sum_{t=1}^{T-1} \left\| \beta_t^{(i)} \right\|^2. \tag{15}$$

Denote $C_1 = \max \left\{ \lambda_i^2 / (1 - |v|)^2, \lambda_i^2 / (1 - \lambda_i)^2 \right\}$ and $C_2 = \max \left\{ \lambda_i^2 / (1 - |v|)^2, \lambda_i^2 / (1 - \lambda_i)^2 \right\}$. From (14) and (15), we have

$$\sum_{t=1}^{T} \left\| y_t^{(i)} \right\|^2 \leq 2C_1 \left\| y_1^{(i)} \right\|^2 + 2C_2 \sum_{t=1}^{T-1} \left\| \beta_t^{(i)} \right\|^2. \tag{16}$$
We next bound $\beta_i^{(i)}$

\[
\mathbb{E} \sum_{i=2}^{n} \beta_i^{(i)} \leq \sum_{i=2}^{n} \gamma^2 \mathbb{E} \left\| h_i^{(i)} \right\|^2 \\
= \gamma^2 \mathbb{E} \left\| G \left( X_i; \xi \right) P e^{(i)} - G \left( X_{i-1}; \xi_{i-1} \right) P e^{(i)} \right\|^2 \\
\leq \gamma^2 \mathbb{E} \left\| G \left( X_i; \xi \right) P e^{(i)} - G \left( X_{i-1}; \xi_{i-1} \right) P e^{(i)} \right\|^2 \\
= \gamma^2 \mathbb{E} \left\| G \left( X_i; \xi \right) - G \left( X_{i-1}; \xi_{i-1} \right) \right\|^2_F \\
= \gamma^2 \mathbb{E} \left\| G \left( X_i; \xi \right) - G \left( X_{i-1}; \xi_{i-1} \right) \right\|^2_F \quad \text{(due to Lemma 5)} \\
= \gamma^2 \mathbb{E} \left\| \nabla F_i \left( x_i^{(i)}; \xi_i^{(i)} \right) - \nabla F_i \left( x_{i-1}^{(i)}; \xi_{i-1}^{(i)} \right) \right\|^2 \\
= \gamma^2 \mathbb{E} \left\| \left( \nabla F_i \left( x_i^{(i)}; \xi_i^{(i)} \right) - \nabla f_i \left( x_i^{(i)} \right) \right) - \left( F_i \left( x_i^{(i)}; \xi_i^{(i)} - \nabla f_i \left( x_i^{(i)} \right) + \left( \nabla f_i \left( x_i^{(i)} \right) - \nabla f_i \left( x_{i-1}^{(i)} \right) \right) \right) \right\|^2 \\
= 3\gamma^2 \mathbb{E} \left\| \nabla F_i \left( x_i^{(i)}; \xi_i^{(i)} \right) - \nabla f_i \left( x_i^{(i)} \right) \right\|^2 + 3\gamma^2 \mathbb{E} \left\| F_i \left( x_i^{(i)}; \xi_i^{(i)} \right) - \nabla f_i \left( x_{i-1}^{(i)} \right) \right\|^2 \\
\leq 6\gamma^2 n \sigma^2 + 3\gamma^2 \mathbb{E} \left\| \nabla f_i \left( x_i^{(i)} \right) - \nabla f_i \left( x_{i-1}^{(i)} \right) \right\|^2 \\
\leq 6\gamma^2 n \sigma^2 + 3\gamma^2 \mathbb{E} \left\| y_i^{(i)} - y_{i-1}^{(i)} \right\|^2 \\
= 6\gamma^2 n \sigma^2 + 3\gamma^2 L^2 \mathbb{E} \left\| Y_i P^T e^{(i)} - Y_{i-1} P^T e^{(i)} \right\|^2 \\
= 6\gamma^2 n \sigma^2 + 3\gamma^2 L^2 \mathbb{E} \left\| Y_i P^T - Y_{i-1} P^T \right\|^2_F \\
= 6\gamma^2 n \sigma^2 + 3\gamma^2 L^2 \mathbb{E} \left\| Y_i - Y_{i-1} \right\|^2_F \quad \text{(due to Lemma 5)} \\
= 6\gamma^2 n \sigma^2 + 3\gamma^2 L^2 \mathbb{E} \left\| y_i^{(i)} - y_{i-1}^{(i)} \right\|^2. \quad \text{(17)}
\]

Combining (16) and (17), we have

\[
\sum_{i=2}^{n} \sum_{t=1}^{T} \left\| y_i^{(i)} \right\|^2 \leq 2C_1 \left\| Y_i \right\|^2_F + 2C_2 \sum_{i=2}^{n} \sum_{t=1}^{T-1} \left\| \beta_t^{(i)} \right\|^2 \\
\leq 2C_1 \left\| Y_i \right\|^2_F + 2C_2 \sum_{i=2}^{n} \sum_{t=1}^{T-1} \left( 6\gamma^2 n \sigma^2 + 3\gamma^2 L^2 \mathbb{E} \left\| y_i^{(i)} - y_{i-1}^{(i)} \right\|^2 \right) \\
\leq 2C_1 \left\| Y_i \right\|^2_F + 12C_2 \gamma^2 n \sigma^2 T + 6C_2 \gamma^2 L^2 \mathbb{E} \left\| y_i^{(i)} - y_{i-1}^{(i)} \right\|^2. \quad \text{(18)}
\]
The next step is to bound $E\|y_t^{(1)} - y_{t-1}^{(1)}\|^2$. Because

$$y_t^{(1)} = X_t P e^{(1)} = X_t v_1 = X_t \frac{1}{\sqrt{n}} 1_n = \overline{X}_t \sqrt{n},$$

what we need to bound is $E\|\overline{X}_{t+1} - \overline{X}_t\|^2$. From (4), we have $\overline{X}_{t+1} = \overline{X}_t - \gamma G_t$. Therefore

$$E\|\overline{X}_{t+1} - \overline{X}_t\|^2 = \gamma^2 E\|G_t - \nabla f(X_t)\|^2 + \gamma^2 \|\nabla f(X_t)\|^2 \leq \frac{\gamma^2 \sigma^2}{n} + \gamma^2 \|\nabla f(X_t)\|^2,$$

and we have the follow bound for $E\|y_t^{(1)} - y_{t-1}^{(1)}\|^2$:

$$E\|y_t^{(1)} - y_{t-1}^{(1)}\|^2 \leq \gamma^2 \sigma^2 + n \gamma^2 \|\nabla f(X_t)\|^2. \quad (19)$$

Combing (18) and (19) we get

$$\sum_{i=2}^{n} \sum_{t=1}^{T} \|y_t^{(i)}\|^2 \leq 2C_1 \|Y_1\|_F^2 + 12C_2 \gamma^2 n \sigma^2 T + 6C_2 \gamma^4 L^2 \sigma^2 T + 6C_2 \gamma^4 L^2 n \sum_{t=1}^{T-1} \|\nabla f(X_t)\|^2$$

$$+ 6C_2 \gamma^2 L^2 \sum_{i=2}^{n} \sum_{t=1}^{T-1} E \|y_t^{(i)} - y_{t-1}^{(i)}\|^2$$

$$\leq 2C_1 \|Y_1\|_F^2 + 12C_2 \gamma^2 n \sigma^2 T + 6C_2 \gamma^4 L^2 \sigma^2 T + 6C_2 \gamma^4 L^2 n \sum_{t=1}^{T-1} \|\nabla f(X_t)\|^2$$

$$+ 6C_2 \gamma^2 L^2 \sum_{i=2}^{n} \sum_{t=1}^{T-1} 2E \left( \|y_t^{(i)}\|^2 + \|y_t^{(i)}\|^2 \right)$$

$$\leq 2C_1 \|Y_1\|_F^2 + 12C_2 \gamma^2 n \sigma^2 T + 6C_2 \gamma^4 L^2 \sigma^2 T + 6C_2 \gamma^4 L^2 n \sum_{t=1}^{T-1} \|\nabla f(X_t)\|^2$$

$$+ 6C_2 \gamma^2 L^2 \sum_{i=2}^{n} \sum_{t=1}^{T-1} 2E \left( \|y_t^{(i)}\|^2 + \|y_t^{(i)}\|^2 \right) \quad \text{(due to } y_0^{(i)} = 0)$$

$$+ 24C_2 \gamma^2 L^2 \sum_{i=2}^{n} \sum_{t=1}^{T-1} E \|y_t^{(i)}\|^2,$$

$$(1 - 24C_2 \gamma^2 L^2) \sum_{i=2}^{n} \sum_{t=1}^{T} \|y_t^{(i)}\|^2 \leq 2C_1 \|Y_1\|_F^2 + 12C_2 \gamma^2 n \sigma^2 T + 6C_2 \gamma^4 L^2 \sigma^2 T + 6C_2 \gamma^4 L^2 n \sum_{t=1}^{T-1} \|\nabla f(X_t)\|^2.$$

Together with (12) and $X_0 = 0$, we have

$$(1 - 24C_2 \gamma^2 L^2) \sum_{i=1}^{T} \|X_t - x_t^{(i)}\|^2 \leq 2C_1 \|X_1\|_F^2 + 12C_2 \gamma^2 n \sigma^2 T + 6C_2 \gamma^4 L^2 \sigma^2 T + 6C_2 \gamma^4 L^2 n \sum_{t=1}^{T-1} \|\nabla f(X_t)\|^2$$

$$(\text{due to } \|X_1\|_F = \|Y_1\|_F) \leq 2C_1 \|X_1\|_F^2 + 12C_2 \gamma^2 n \sigma^2 T + 6C_2 \gamma^4 L^2 \sigma^2 T + 6C_2 \gamma^4 L^2 n \sum_{t=1}^{T-1} \|\nabla f(X_t)\|^2.$$

Actually, when $\lambda_n \leq -\frac{1}{3}$, we have $|v_n| \geq 1$, then $\|y_t^{(n)}\|^2 \propto t$ and

$$\frac{1}{T} \sum_{i=1}^{T} \sum_{t=1}^{T} \|X_t - x_t^{(i)}\|^2 \leq T.$$

The algorithm would fail to converge in this situation, and this is why $-1/3$ is the infimum of $\lambda_n$. \qed
**Lemma 8.** Following the Assumption 1, we have

\[
E(f(\mathbf{X}_{t+1}) \leq E(f(\mathbf{X}_t) - \frac{\gamma t}{2} E\|\nabla f(\mathbf{X}_t)\|^2 - \left(\frac{\gamma t}{2} - \frac{L\gamma t^2}{2}\right) E\|\nabla f(\mathbf{X}_t)\|^2 + \frac{\gamma t}{2} E\|\nabla f(\mathbf{X}_t) - \nabla f(\mathbf{X}_t)\|^2 + \frac{L\gamma t^2}{2n} \sigma^2.
\]

**Proof.** From (4), we have

\[
\mathbf{X}_{t+1} = \mathbf{X}_t - \gamma_t \mathcal{G}(\mathbf{X}_t; \xi_t).
\]

From item 1 of Assumption 1, we know that \( f \) has a \( L \)-Lipschitz continuous gradient. So, we have

\[
E(f(\mathbf{X}_{t+1}) \leq E(f(\mathbf{X}_t) + E(\langle \nabla f(\mathbf{X}_t), -\gamma_t \mathcal{G}(\mathbf{X}_t; \xi_t) \rangle) + \frac{L}{2} E\|\nabla f(\mathbf{X}_t)\|^2 + \frac{L\gamma t^2}{2} E\|\nabla f(\mathbf{X}_t)\|^2 - \left(\frac{\gamma t}{2} - \frac{L\gamma t^2}{2}\right) E\|\nabla f(\mathbf{X}_t)\|^2 + \frac{\gamma t}{2} E\|\nabla f(\mathbf{X}_t) - \nabla f(\mathbf{X}_t)\|^2 + \frac{L\gamma t^2}{2n} \sigma^2.
\]

which completes the proof. \( \square \)

**Proof to Theorem 2**

**Proof.** We first estimate the upper bound for \( E\|\nabla f(\mathbf{X}_t) - \nabla f(\mathbf{X}_t)\|^2 \):

\[
E\|\nabla f(\mathbf{X}_t) - \nabla f(\mathbf{X}_t)\|^2 = \frac{1}{n^2} E\left\| \sum_{i=1}^{n} \left( \nabla f_i(\mathbf{X}_t) - \nabla f_i(x_t^{(i)}) \right) \right\|^2 \\
\leq \frac{1}{n} \sum_{i=1}^{n} E\left\| \nabla f_i(\mathbf{X}_t) - \nabla f_i(x_t^{(i)}) \right\|^2 \\
\leq \frac{L^2}{n} E\sum_{i=1}^{n} \| \mathbf{X}_t - x_t^{(i)} \|^2.
\]

(21)
Combining (20) in Lemma 8 and (21) yields

$$
\frac{\gamma_t}{2} \mathbb{E} \| \nabla f(\overline{X}_t) \|^2 + \left( \frac{\gamma_t}{2} - \frac{L \gamma_t^2}{2} \right) \mathbb{E} \| \nabla f(X_t) \|^2
$$

$$
\leq \mathbb{E} f(\overline{X}_t) - \mathbb{E} f(\overline{X}_{t+1}) + \frac{\gamma_t}{2} \mathbb{E} \| \nabla f(\overline{X}_t) - \nabla f(X_t) \|^2 + \frac{L \gamma_t^2}{2n} \sigma^2
$$

$$
\leq \mathbb{E} f(\overline{X}_t) - \mathbb{E} f(\overline{X}_{t+1}) + \frac{L^2 \gamma_t}{2n} \sum_{i=1}^{n} \| \overline{X}_t - x_t^{(i)} \|^2 + \frac{L \gamma_t}{2n} \sigma^2.
$$

Setting $\gamma_t = \gamma$, we obtain

$$
\mathbb{E} \| \nabla f(\overline{X}_t) \|^2 + (1 - L \gamma) \mathbb{E} \| \nabla f(X_t) \|^2 \leq \frac{2}{\gamma} (\mathbb{E} f(\overline{X}_t) - f^* - (\mathbb{E} f(\overline{X}_{t+1}) - f^*)) + \frac{L^2}{n} \sum_{i=1}^{n} \| \overline{X}_t - x_t^{(i)} \|^2 + \frac{L \gamma}{n} \sigma^2.
$$

(22)

From Lemma 7, we have

$$
(1 - 24C_2^2 \gamma^2 L^2) \sum_{i=1}^{n} \sum_{t=0}^{T} \left\| \overline{X}_t - x_t^{(i)} \right\|^2 \leq 2C_1 \| X_1 \|^2_F + 12C_2 \gamma^2 n \sigma^2 T + 6C_2 \gamma^4 L^2 \sigma^2 T
$$

$$
+ 6C_2 \gamma^4 L^2 n \sum_{t=1}^{T-1} \left\| \nabla f(X_t) \right\|^2.
$$

(23)

If $\gamma$ is not too large that satisfies $1 - 24C_2^2 \gamma^2 L^2 > 0$, then denote $C_3 = 1 - 24C_2 \gamma^2 L^2$, we would have

$$
\sum_{i=1}^{n} \sum_{t=0}^{T} \left\| \overline{X}_t - x_t^{(i)} \right\|^2 \leq \frac{2C_1}{C_3} \| X_1 \|^2_F + \frac{12C_2 \gamma^2 n \sigma^2 T}{C_3} + \frac{6C_2 \gamma^4 L^2 \sigma^2 T}{C_3} + \frac{6C_2 \gamma^4 L^2 n}{nC_3} \sum_{t=1}^{T-1} \left\| \nabla f(X_t) \right\|^2.
$$

(23)

Summarizing both sides of (22) and applying (23) yields

$$
\sum_{t=0}^{T-1} \left( \mathbb{E} \| \nabla f(\overline{X}_t) \|^2 + (1 - L \gamma - \frac{6L^2C_2^2 \gamma^2 L^2}{C_3}) \mathbb{E} \| \nabla f(X_t) \|^2 \right)
$$

$$
\leq \frac{2}{\gamma} (\mathbb{E} f(\overline{X}_0) - f^*) + \frac{L T \gamma}{n} \sum_{i=1}^{n} \sum_{t=0}^{T} \left\| \overline{X}_t - x_t^{(i)} \right\|^2 + \frac{L T \gamma}{n} \sigma^2
$$

$$
\leq \frac{2}{\gamma} \left( \frac{f(0) - f^*}{n} + \frac{2L^2C_1}{nC_3} \| X_1 \|^2_F + \frac{12L^2C_2^2 \gamma^2 n \sigma^2 T}{nC_3} + \frac{6L^2C_2 \gamma^4 L^2 \sigma^2 T}{nC_3} \right)
$$

$$
+ \frac{6L^2C_2 \gamma^4 L^2}{C_3} \sum_{t=1}^{T-1} \left\| \nabla f(X_t) \right\|^2.
$$

It implies

$$
\sum_{t=0}^{T-1} \left( \mathbb{E} \| \nabla f(\overline{X}_t) \|^2 + \left( 1 - L \gamma - \frac{6L^2C_2^2 \gamma^2 L^2}{C_3} \right) \mathbb{E} \| \nabla f(X_t) \|^2 \right)
$$

$$
\leq \frac{2}{\gamma} \left( \frac{f(0) - f^*}{n} + \frac{2L^2C_1}{nC_3} \| X_1 \|^2_F + \frac{12L^2C_2^2 \gamma^2 n \sigma^2 T}{nC_3} + \frac{6L^2C_2 \gamma^4 L^2 \sigma^2 T}{nC_3} \right)
$$

$$
+ \frac{6L^2C_2 \gamma^4 L^2}{C_3} \sum_{t=1}^{T-1} \left\| \nabla f(X_t) \right\|^2.
$$

(24)

However, $\| G(0; \xi_0) \|^2_F$ can be expanded as:

$$
\left\| G(0; \xi_0) \right\|^2_F = \sum_{i=1}^{n} \left\| (\nabla F_i(0, \xi_1) - \nabla F_i(0)) + (\nabla f_i(0) - \nabla f(0)) + \nabla f(0) \right\|^2
$$

$$
\leq 3n \sigma^2 + 3n \xi_0^2 + 3n \| \nabla f(0) \|^2.
$$

(25)
where \( \zeta_0 = \frac{1}{n} \sum_{i=1}^{n} \| \nabla f_i(0) - \nabla f(0) \|^2 \) indicates the difference between different workers’ dataset at the start point.

Combining (24) and (25), then we have

\[
\sum_{i=0}^{T-1} \left( \mathbb{E} \| \nabla f(\mathbf{X}_i) \|^2 + \left( 1 - L\gamma - \frac{6L^2C_2\gamma^4L^2}{C_3} \right) \mathbb{E} \| \nabla \mathbf{f}(\mathbf{X}_i) \|^2 \right) \\
\leq \frac{2(f(0) - f^*)}{\gamma} + \frac{LT\gamma\sigma^2}{n} + \frac{12L^2C_2\gamma^2\sigma^2T}{C_3} + \frac{6L^2C_2\gamma^4L^2\sigma^2T}{nC_3} \\
+ \frac{6L^2C_1\gamma^2\sigma^2}{C_3} + \frac{6L^2C_1\gamma^2\zeta_0^2}{C_3} + \frac{6L^2C_1\gamma^2\zeta_0}{C_3} \| \nabla f(0) \|^2.
\]

Then we have

\[
\left( 1 - \frac{6L^2C_1\gamma^2}{C_3} \right) \| \nabla f(0) \|^2 + \sum_{i=1}^{T-1} \left( \mathbb{E} \| \nabla f(\mathbf{X}_i) \|^2 + \left( 1 - L\gamma - \frac{6L^2C_2\gamma^4L^2}{C_3} \right) \mathbb{E} \| \nabla \mathbf{f}(\mathbf{X}_i) \|^2 \right) \\
\leq \frac{2(f(0) - f^*)}{\gamma} + \frac{LT\gamma\sigma^2}{n} + \frac{12L^2C_2\gamma^2n\sigma^2T}{nC_3} + \frac{6L^2C_2\gamma^4L^2\sigma^2T}{nC_3} + \frac{6L^2C_1\gamma^2\sigma^2}{C_3} + \frac{6L^2C_1\gamma^2\zeta_0^2}{C_3}.
\]

Denote

\[
A_1 = 1 - \frac{6L^2C_1\gamma^2}{C_3} \\
A_2 = 1 - L\gamma - \frac{6L^2C_2\gamma^4L^2}{C_3},
\]

it becomes

\[
A_1 \| \nabla f(0) \|^2 + \sum_{i=1}^{T-1} \left( \mathbb{E} \| \nabla f(\mathbf{X}_i) \|^2 + A_2 \mathbb{E} \| \nabla \mathbf{f}(\mathbf{X}_i) \|^2 \right) \\
\leq \frac{2(f(0) - f^*)}{\gamma} + \frac{LT\gamma\sigma^2}{n} + \frac{12L^2C_2\gamma^2n\sigma^2T}{nC_3} + \frac{6L^2C_2\gamma^4L^2\sigma^2T}{nC_3} + \frac{6L^2C_1\gamma^2\sigma^2}{C_3} + \frac{6L^2C_1\gamma^2\zeta_0^2}{C_3}.
\]

It completes the proof. \( \square \)

**Proof to Corollary 3**

**Proof.** From the value of \( \gamma \), we obtain

\[
C_2\gamma^2L^2 \leq \frac{1}{64}, \quad C_1\gamma^2L^2 \leq \frac{1}{36}.
\]

Therefore

\[
C_3 = 1 - 24C_2\gamma^2L^2 \geq \frac{1}{2} \\
A_1 = 1 - \frac{6L^2C_1\gamma^2}{C_3} \geq \frac{1}{2} \\
A_2 = 1 - L\gamma - \frac{6L^2C_2\gamma^4L^2}{C_3} > 0, \\
\gamma^2 \leq \frac{n}{nL^2 + \sigma^2T}, \\
\gamma^4 \leq \frac{n^2}{n^2L^4 + \sigma^4T^2}.
\]
Then we can remove the $\|\nabla f(X_t)\|^2$ and $\|\nabla f(0)\|^2$ on the left hand side of (5) in Theorem 2, and (5) becomes

$$\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\|\nabla f(X_t)\|^2 \leq \frac{4(f(0) - f^*)L(8\sqrt{C_2} + 6\sqrt{C_1})}{T} + \frac{4(f(0) - f^* )\sigma}{\sqrt{Tn}}$$

$$+ \frac{2L\sigma}{\sqrt{Tn}} + \frac{48nL^2C_2\sigma^2}{nL^2 + \sigma^2T} + \frac{24L^4n\sigma^2C_2}{n^2L^4 + \sigma^4T^2} + \frac{24nL^2C_1\sigma^2}{T(nL^2 + \sigma^2T)} + \frac{24L^2C_1\zeta_0^2}{T(nL^2 + \sigma^2T)},$$

which completes the proof.