Appendix: Proofs

Lemma 1 *Let $f(x)$ be a continuously strictly log-concave differentiable probability density function with support $(-\infty,+\infty)$. $F(x)=\int_{-\infty}^{x} f(t)dt$ is strictly log-concave.*

Proof: The proof is slightly modified from (Bagnoli & Bergstrom, 2005). We will prove $\frac{\partial^2 \ln F(x)}{\partial x^2}=\frac{d}{dx}(\frac{f(x)}{F(x)})=\frac{f'(x)F(x)-f(x)^2}{F(x)^2}<0$. Since $F(x)>0$, we only need to prove $f'(x)F(x)-f(x)^2<0$.

Because $f(x)$ is strictly log-concave, we have that $\frac{d\ln f(x)}{dx}=\frac{f'(x)}{f(x)}$ is decreasing for any $x\in\mathbb{R}$. So we have $\frac{f'(x)}{f(x)}F(x)=\frac{f'(x)}{f(x)}\int_{-\infty}^{x} f(t)dt<\int_{-\infty}^{x}\frac{f'(t)}{f(t)}f(t)dt=f(x)-\lim_{x\to-\infty} f(x)=f(x)$.
This proves the lemma. ■

Lemma 2 *For any alternatives $a_i, a_{i'}$ with distributions $\pi_i,\pi_{i'}>0$ defined on $(-\infty,+\infty)$, we define $L=\theta_i-\theta_{i'}$ and let $p_{ii'}(\vec{\theta})$ denote the probability of $a_i\succ a_{i'}$ given π_i and $\pi_{i'}$. For any $\epsilon>0$, there exists L s.t. $|\frac{\partial p_{ii'}(\vec{\theta})}{\partial\theta_i}|,|\frac{\partial p_{i'i}(\vec{\theta})}{\partial\theta_i}|\le\epsilon$.*

Proof: Because $p_{ii'}(\vec{\theta})+p_{i'i}(\vec{\theta})=1$, for any $1\le l\le m$, we have

$$\frac{\partial p_{ii'}(\vec{\theta})}{\partial\theta_l}+\frac{\partial p_{i'i}(\vec{\theta})}{\partial\theta_l}=0 \tag{8}$$

So we have $|\frac{\partial p_{ii'}(\vec{\theta})}{\partial\theta_i}|=|\frac{\partial p_{i'i}(\vec{\theta})}{\partial\theta_i}|$. We only need to prove $|\frac{\partial p_{ii'}}{\partial\theta_i}|\le\epsilon$.

Let $\theta_{i'}=0$ and $\theta_i=L$. This is without loss of generality because $p_{ii'}(\vec{\theta})$ remains the same under parameter shifts. Let u_i and $u_{i'}$ denote the sampled utilities. We have

$$p_{ii'}(\vec{\theta})=p_{ii'}(L)=\Pr(u_i>u_{i'}|\vec{\theta})=\int_{-\infty}^{\infty}\pi_{i'}(x')\int_{x'}^{\infty}\pi_i(x-L)dxdx'=\int_{-\infty}^{\infty}\pi_{i'}(x')\int_{x'-L}^{\infty}\pi_i(x)dxdx'$$

When L increases, $\int_{x'-L}^{\infty}\pi_i(x)dx$ increases given any x'. So we have $\frac{\partial p_{ii'}(\vec{\theta})}{\partial\theta_i}=\frac{dp_{ii'}(L)}{dL}\frac{\partial L}{\partial\theta_i}=\frac{dp_{ii'}(L)}{dL}>0$. On the other hand, because $0\le p_{ii'}(L)\le 1$ we have $\int_{-\infty}^{+\infty}\frac{dp_{ii'}(L)}{dL}dL=p_{ii'}(L)|_{+\infty}-p_{ii'}(L)|_{-\infty}\le 1$.

Therefore, for any ϵ, any interval I whose length is $1/\epsilon$, we claim there exists an L s.t. $\frac{\partial p_{ii'}}{\partial\theta_i}\le\epsilon$. The reason is as follows. Suppose for all $L\in I$, $\frac{\partial p_{ii'}}{\partial\theta_i}>\epsilon$ holds. Then we have $\int_{-\infty}^{+\infty}\frac{dp_{ii'}(L)}{dL}dL>\int_I\frac{dp_{ii'}(L)}{dL}dL>\int_I\epsilon dL=\epsilon\times\frac{1}{\epsilon}=1$, which is a contradiction. ■

Lemma 3 *For any alternatives $a_i, a_{i'}$ with distributions $\pi_i,\pi_{i'}>0$ defined on $(-\infty,+\infty)$. Define $L=\theta_i-\theta_{i'}$. For any $\epsilon>0$, there exists L s.t.*

$$|\frac{\bar{\kappa}_{ii'}w_{ii'}}{p_{ii'}(\vec{\theta})}\frac{\partial p_{ii'}(\vec{\theta})}{\partial\theta_i}+\frac{\bar{\kappa}_{i'i}w_{i'i}}{p_{i'i}(\vec{\theta})}\frac{\partial p_{i'i}(\vec{\theta})}{\partial\theta_i}|\le\epsilon$$

Proof: Let $\max\{\mathcal{G}\}$ denote the maximum weight on the edges of $\mathcal{G}$. Since $\frac{\bar{\kappa}_{ii'}}{p_{ii'}}$ is upper bounded by $\max\{\mathcal{G}\}$ and $w_{ii'}$ is finite, we let $M=\max\{|\frac{\bar{\kappa}_{ii'}w_{ii'}}{p_{ii'}(\vec{\theta})}|,|\frac{\bar{\kappa}_{i'i}w_{i'i}}{p_{i'i}(\vec{\theta})}|\}$ and $\epsilon'=\frac{\epsilon}{2M}$. By Lemma 2 there exists L s.t. $|\frac{\partial p_{ii'}(\vec{\theta})}{\partial\theta_i}|,|\frac{\partial p_{i'i}(\vec{\theta})}{\partial\theta_i}|\le\epsilon'$. Then we have $|\frac{\bar{\kappa}_{ii'}w_{ii'}}{p_{ii'}(\vec{\theta})}\frac{\partial p_{ii'}(\vec{\theta})}{\partial\theta_i}+\frac{\bar{\kappa}_{i'i}w_{i'i}}{p_{i'i}(\vec{\theta})}\frac{\partial p_{i'i}(\vec{\theta})}{\partial\theta_i}|\le|\frac{\bar{\kappa}_{ii'}w_{ii'}}{p_{ii'}(\vec{\theta})}\frac{\partial p_{ii'}(\vec{\theta})}{\partial\theta_i}|+|\frac{\bar{\kappa}_{i'i}w_{i'i}}{p_{i'i}(\vec{\theta})}\frac{\partial p_{i'i}(\vec{\theta})}{\partial\theta_i}|\le\epsilon'\times 2M=\epsilon$ ■

Lemma 4 *For any pair of alternatives a_i and $a_{i'}$ with equal weights $w_{ii'}=w_{i'i}$, if $\theta_i=\theta_{i'}$, then we have*

$$\frac{\bar{\kappa}_{ii'}w_{ii'}}{p_{ii'}(\vec{\theta})}\frac{\partial p_{ii'}(\vec{\theta})}{\partial\theta_i}+\frac{\bar{\kappa}_{i'i}w_{i'i}}{p_{i'i}(\vec{\theta})}\frac{\partial p_{i'i}(\vec{\theta})}{\partial\theta_i}=0$$

Proof: Since $\theta_i=\theta_{i'}$, we have $p_{ii'}(\vec{\theta})=p_{i'i}(\vec{\theta})$ and $\bar{\kappa}_{ii'}=\bar{\kappa}_{i'i}$, the lemma follows from (8). ■

Lemma 5 *Let $\mathcal{G}^*$ be the graph obtained by labeling the vertices of $\mathcal{G}$ reversely, $\mathcal{M}^*$ be the model obtained by flipping all of the utility distributions of $\mathcal{M}$ around their means, and $\mathcal{W}^*$ be the weight vector where $w^*_{ii'} = w_{i'i}$. For any RUM $\mathcal{M}$, if RBCML$(\mathcal{G}, \mathcal{W})$ is consistent for $\mathcal{M}$, then RBCML$(\mathcal{G}^*, \mathcal{W}^*)$ is consistent for $\mathcal{M}^*$.*

Proof: By Theorem 5, we only need to prove the solution to RBCML$(\mathcal{G}, \mathcal{W})$, which is the ground truth, is the only solution to RBCML$(\mathcal{G}^*, \mathcal{W}^*)$. Due to strict concavity, RBCML$(\mathcal{G}^*, \mathcal{W}^*)$ does not have multiple solutions. So we only need to prove the solution to RBCML$(\mathcal{G}, \mathcal{W})$ is the solution to RBCML$(\mathcal{G}^*, \mathcal{W}^*)$.

For any $i \in \{1, \dots, m\}$ and any $\vec{\theta}$, (7) holds. Since $\mathcal{M}^*$ is flipped $\mathcal{M}$, for any ranking R, we have $\Pr_{\mathcal{M}^*}(R|\vec{\theta}) = \Pr_{\mathcal{M}}(rev(R)|\vec{\theta})$, where $rev(R)$ is the reverse of R. Therefore, for any pair of alternatives a and a', $a \succ a' \in \mathcal{G}^*(R)$ if and only if $a' \succ a \in \mathcal{G}(rev(R))$.

Then for any $i \in \{1, \dots, m\}$, we have

$$\nabla_i \text{ELL}_{\mathcal{M}^*}(\vec{\theta}) = \sum_{i' \neq i} \left(\frac{\bar{\kappa}_{ii'} w^*_{ii'}}{p^*_{ii'}(\vec{\theta})} \frac{\partial p^*_{ii'}(\vec{\theta})}{\partial \theta_i} + \frac{\bar{\kappa}_{i'i} w^*_{i'i}}{p^*_{i'i}(\vec{\theta})} \frac{\partial p^*_{i'i}(\vec{\theta})}{\partial \theta_i}\right) = \sum_{i' \neq i} \left(\frac{\bar{\kappa}_{i'i} w_{i'i}}{p_{i'i}(\vec{\theta})} \frac{\partial p_{i'i}(\vec{\theta})}{\partial (\theta_i)} + \frac{\bar{\kappa}_{ii'} w_{ii'}}{p_{ii'}(\vec{\theta})} \frac{\partial p_{ii'}(\vec{\theta})}{\partial (\theta_i)}\right) = 0.$$

This finishes the proof of the lemma. ■

Lemma 6 *Let $\mathcal{G}_{[k_1,k_2]}$ denote the subgraph $\mathcal{G}$ restricted to nodes between k_1 and k_2 (inclusive). For any RUM $\mathcal{M}$, if RBCML$(\mathcal{G}, \mathcal{W}_u)$ is consistent, then for any $1 \leq k_1 < k_2 \leq m$, RBCML$(\mathcal{G}_{[k_1,k_2]}, \mathcal{W}_u)$ is either empty or consistent for $k_2 - k_1 + 1$ alternatives.*

Proof: We prove that if RBCML$(\mathcal{G}_{[k_1,k_2]}, \mathcal{W}_{\text{u}})$ is not consistent then RBCML$(\mathcal{G}, \mathcal{W}_{\text{u}})$ is not consistent. Suppose RBCML$(\mathcal{G}_{[k_1,k_2]}, \mathcal{W}_{\text{u}})$ is not consistent. For convenience we keep the index of $\mathcal{G}$ in $\mathcal{G}_{[k_1,k_2]}$ and let $\mathcal{M}'$ denote the model with the $k_2 - k_1 + 1$ alternatives. Then there exists θ_i where $k_1 \leq i \leq k_2$ s.t.

$$|\nabla_i \text{ELL}_{\mathcal{M}'}(\vec{\theta})| = \left|\sum_{k_1 \leq i' \leq k_2, i' \neq i} \left(\frac{\bar{\kappa}_{ii'} w_{ii'}}{p_{ii'}(\vec{\theta})} \frac{\partial p_{ii'}(\vec{\theta})}{\partial \theta_i} + \frac{\bar{\kappa}_{i'i} w_{i'i}}{p_{i'i}(\vec{\theta})} \frac{\partial p_{i'i}(\vec{\theta})}{\partial \theta_i}\right)\right| = C > 0$$

We now construct other elements in $\vec{\theta}$ to show that RBCML$(\mathcal{G}, \mathcal{W}_{\text{u}})$ is not consistent. We let $\theta_1 = \ldots = \theta_{k_1 - 1} = L$ and $\theta_{k_2} + 1 = \ldots = \theta_r = -L$. Then when $L \to \infty$, with probability that goes to 1, $a_1, \dots, a_{k_1-1}$ are ranked in the top $k_1 - 1$ positions and $a_{k_2+1}, \dots, a_m$ are ranked in the bottom $m - k_2$ positions.

By Lemma 3 for any $k_1 \leq i \leq k_2$ and $i' < k_1$ (or $i' > k_2$) there exists L s.t. $|\frac{\bar{\kappa}_{ii'} w_{ii'}}{p_{ii'}(\vec{\theta})} \frac{\partial p_{ii'}(\vec{\theta})}{\partial \theta_i} + \frac{\bar{\kappa}_{i'i} w_{i'i}}{p_{i'i}(\vec{\theta})} \frac{\partial p_{i'i}(\vec{\theta})}{\partial \theta_i})| \leq \frac{C}{m}$. Then we have $|\nabla_i \text{ELL}_{\mathcal{M}}(\vec{\theta})| \geq |\nabla_i \text{ELL}_{\mathcal{M}'}(\vec{\theta})| - (m - (k_2 - k_1 + 1))\frac{C}{m} = \frac{(k_2 - k_1 + 1)C}{m} > 0$. So we have $\nabla_i \text{ELL}_{\mathcal{M}}(\vec{\theta}) \neq 0$. RBCML$(\mathcal{G}, \mathcal{W}_{\text{u}})$ is thus not consistent. ■

Lemma 7 *For any $m \geq 3$, RBCML$(\mathcal{G}, \mathcal{W}_u)$ for the Plackett-Luce model is not consistent if $\mathcal{G} = \{g_{1m} = C\}$, where $C > 0$ is a constant.*

Proof: It suffices to prove RBCML$(\mathcal{G}, \mathcal{W}_{\text{u}})$ for the Plackett-Luce model is not consistent if $\mathcal{G} = \{g_{1m} = 1\}$. We prove this lemma by constructing a counter-example. Let $\theta_1 = x$ and $\theta_2 = \ldots = \theta_m = 0$. For any ranking R_1 with alternative a_1 at top, the probability is $\Pr(R_1|\vec{\theta}) = \frac{1}{(m-1)!} \frac{e^x}{e^x + (m-1)}$. For any ranking R_2 with a_1 at bottom, the probability is $\Pr(R_2|\vec{\theta}) = \frac{1}{\prod_{k=1}^{m-1}(e^x + k)}$. For any a_i where $2 \leq i \leq m$, we have $\bar{\kappa}_{1i} = (m-1)! \Pr(R_1|\vec{\theta})$ and $\bar{\kappa}_{i1} = (m-1)! \Pr(R_2|\vec{\theta})$. Therefore, we have $\nabla_i \text{ELL}_{\text{PL}}(\vec{\theta}) = \sum_{i' \neq i}(\bar{\kappa}_{ii'} - (\bar{\kappa}_{ii'} + \bar{\kappa}_{i'i})\frac{1}{e^x + 1}) = (m-1)(\frac{e^{2x}}{e^x + m - 1} - \frac{(m-1)!}{\prod_{k=1}^{m-1}(e^x + k)})$. Let $x = \ln 2$, then we have $\nabla_i \text{ELL}_{\text{PL}}(\vec{\theta}) = \frac{4m-2}{m(m+1)} \neq 0$. This proves the lemma. ■

Lemma 8 *For any $m \geq 3$, RBCML$(\mathcal{G}, \mathcal{W}_u)$ for any RUM location family with the same symmetric pdf is not consistent if $\mathcal{G} = \{g_{1m} = C\}$ where $C > 0$ is a constant.*

Proof: Let π denote the PDF of the utility distribution for all alternatives with mean 0. That is, for any $i \leq m$ and any $x \in \mathbb{R}$, we have $\pi_i(x) = \pi(x - \theta_i)$. Let $B > 0$ be an arbitrary number so that $1 - \epsilon > \int_{-B}^{B} \pi(x)dx > \epsilon$. Let L be a large number that will be specified later.

We first prove the lemma for $m = 3$. Let $\theta_1 = L$ and $\theta_2 = \theta_3 = 0$. Since $\theta_2 = \theta_3$, we have $\frac{\bar{\kappa}_{12}}{p_{12}(\vec{\theta})}\frac{\partial p_{12}(\vec{\theta})}{\partial \theta_1} + \frac{\bar{\kappa}_{21}}{p_{21}(\vec{\theta})}\frac{\partial p_{21}(\vec{\theta})}{\partial \theta_1} = \frac{\bar{\kappa}_{13}}{p_{13}(\vec{\theta})}\frac{\partial p_{13}(\vec{\theta})}{\partial \theta_1} + \frac{\bar{\kappa}_{31}}{p_{31}(\vec{\theta})}\frac{\partial p_{31}(\vec{\theta})}{\partial \theta_1}$. Due to (8), it suffices to prove $\frac{\bar{\kappa}_{12}}{p_{12}(\vec{\theta})} \neq \frac{\bar{\kappa}_{21}}{p_{21}(\vec{\theta})}$, which is equivalent to $\frac{\Pr(a_1 \text{ top and } a_2 \text{ bottom})}{\Pr(a_1 \succ a_2)} \neq \frac{\Pr(a_2 \text{ top and } a_1 \text{ bottom})}{\Pr(a_2 \succ a_1)}$. That is $\frac{p_{132}}{p_{312}+p_{132}+p_{123}} \neq \frac{p_{231}}{p_{321}+p_{231}+p_{213}}$, where p_{123} is the short form of $\Pr(a_1 \succ a_2 \succ a_3)$. Because $p_{123} = p_{132}$ and $p_{231} = p_{321}$, we only need to prove $\frac{p_{132}}{p_{312}} \neq \frac{p_{231}}{p_{213}}$. This is obvious because $p_{312} = p_{213}$ but $p_{132} \neq p_{231}$.

We now prove the lemma for any $m \geq 4$. Let $\theta_1 = \theta_2 = L$ and $\theta_3 = \ldots = \theta_m = 0$. By Lemma 4 we have $\frac{\bar{\kappa}_{12}}{p_{12}(\vec{\theta})}\frac{\partial p_{12}(\vec{\theta})}{\partial \theta_1} + \frac{\bar{\kappa}_{21}}{p_{21}(\vec{\theta})}\frac{\partial p_{21}(\vec{\theta})}{\partial \theta_1} = 0$. For all $3 \leq i \leq m$, we have $\frac{\bar{\kappa}_{1i}}{p_{1i}(\vec{\theta})}\frac{\partial p_{1i}(\vec{\theta})}{\partial \theta_1} + \frac{\bar{\kappa}_{i1}}{p_{i1}(\vec{\theta})}\frac{\partial p_{i1}(\vec{\theta})}{\partial \theta_1} = \frac{\bar{\kappa}_{1m}}{p_{1m}(\vec{\theta})}\frac{\partial p_{1m}(\vec{\theta})}{\partial \theta_1} + \frac{\bar{\kappa}_{m1}}{p_{m1}(\vec{\theta})}\frac{\partial p_{m1}(\vec{\theta})}{\partial \theta_1}$. So we have $\nabla_i \text{ELL}_{\mathcal{M}}(\vec{\theta}) = (m-2)(\frac{\bar{\kappa}_{1m}}{p_{1m}(\vec{\theta})}\frac{\partial p_{1m}(\vec{\theta})}{\partial \theta_1} + \frac{\bar{\kappa}_{m1}}{p_{m1}(\vec{\theta})}\frac{\partial p_{m1}(\vec{\theta})}{\partial \theta_1})$. It suffices to prove $\frac{\bar{\kappa}_{1m}}{p_{1m}(\vec{\theta})} \neq \frac{\bar{\kappa}_{m1}}{p_{m1}(\vec{\theta})}$, which is

$$\frac{\Pr(a_1 \text{ top and } a_m \text{ bottom})}{\Pr(a_1 \succ a_m)} \neq \frac{\Pr(a_m \text{ top and } a_1 \text{ bottom})}{\Pr(a_m \succ a_1)} \tag{9}$$

Because L is large, $\Pr(a_1 \text{ top or } a_2 \text{ top}) \approx 1$. Because π_i's have the same shape, we have that

$$\Pr(a_1 \text{ top and } a_m \text{ bottom}) \approx \Pr(a_1 \succ a_2 \text{ and } a_m \text{ is ranked lower than } a_3, \ldots, a_{m-1})$$

Therefore, the LHS of (9) is $\frac{1}{2(m-2)}$ as $L \to \infty$. We will show that the RHS of (9) is converges to 0 as $L \to \infty$. We define a partition of $\{(u_1, u_m) : u_1 < u_m\} = S_1 \cup S_2$ as follows.

- $S_1 = \{(u_1, u_m) : u_1 < B \text{ and } u_m > L - B\}$,
- $S_2 = \text{others}$.

We further define the following two functions π and π^* for $u_1 < u_m$.

$$\pi(u_1, u_m) = \pi_1(u_1) \times \pi_m(u_m)$$

$$\pi^*(u_1, u_m) = \pi_1(u_m) \times \pi_m(u_m) \times \prod_{i=2}^{m-1} \int_{u_1}^{u_m} \pi_i(u_i) du_i$$

It follows that

$$\frac{\Pr(a_m \text{ top and } a_1 \text{ bottom})}{\Pr(a_m \succ a_1)} = \frac{\int_{S_1} \pi^*(u_1, u_m) + \int_{S_2} \pi^*(u_1, u_m)}{\int_{S_1} \pi(u_1, u_m) + \int_{S_2} \pi(u_1, u_m)}$$

Claim 1 $\lim_{L \to \infty} \dfrac{\int_{S_1} \pi(u_1, u_m)}{\int_{S_2} \pi(u_1, u_m)} = 0.$

Proof: Let $S = \{(u_1, u_m) : u_1 < B < u_m < L - B\}$. We have $\frac{\int_{S_1} \pi(u_1, u_m)}{\int_{S} \pi(u_1, u_m)} = \frac{\int_{L-B}^{\infty} \pi_m(u_m) du_m}{\int_{B}^{L-B} \pi_m(u_m) du_m}$, which converges to 0. The claim follows after observing that $S \subseteq S_2$. ■

Claim 2 $\lim_{\epsilon \to 0} \dfrac{\int_{S_2} \pi^*(u_1, u_m)}{\int_{S_2} \pi(u_1, u_m)} = 0.$

Proof: For any $(u_1, u_m) \in S_2$, either $u_1 > B$ or $u_m < L - B$. If $u_1 > B$, then

$$\prod_{i=2}^{m-1} \int_{u_1}^{u_m} \pi_i(u_i) du_i \leq \int_{u_1}^{u_m} \pi_{m-1}(u_{m-1}) du_{m-1} \leq \int_{B}^{\infty} \pi_{m-1}(u_{m-1}) du_{m-1} \leq \epsilon$$

If $u_m < L - B$, then we have $\prod_{i=2}^{m-1}\int_{u_1}^{u_m}\pi_i(u_i)du_i \le \int_{u_1}^{u_m}\pi_2(u_2)du_2 \le \int_{-\infty}^{L-B}\pi_2(u_2)du_2 \le \epsilon$ Therefore, for any $(u_1, u_m) \in S_2$, $\frac{\pi^*(u_1,u_m)}{\pi(u_1,u_m)} \le \epsilon$. This proves the claim. ■

We are now ready to prove the lemma.

$$\frac{\Pr(a_m \text{ top and } a_1 \text{ bottom})}{\Pr(a_m \succ a_1)} = \frac{\int_{S_1}\pi^*(u_1,u_m) + \int_{S_2}\pi^*(u_1,u_m)}{\int_{S_1}\pi(u_1,u_m) + \int_{S_2}\pi(u_1,u_m)}$$
$$\le \frac{\int_{S_1}\pi(u_1,u_m) + \int_{S_2}\pi^*(u_1,u_m)}{\int_{S_1}\pi(u_1,u_m) + \int_{S_2}\pi(u_1,u_m)} = \frac{\frac{\int_{S_1}\pi(u_1,u_m)}{\int_{S_2}\pi(u_1,u_m)} + \frac{\int_{S_2}\pi^*(u_1,u_m)}{\int_{S_2}\pi(u_1,u_m)}}{\frac{\int_{S_1}\pi(u_1,u_m)}{\int_{S_2}\pi(u_1,u_m)} + 1}$$

Therefore, by combining Claim 1 and Claim 2, we have

$$\lim_{L\to\infty,\epsilon\to 0}\frac{\Pr(a_m \text{ top and } a_1 \text{ bottom})}{\Pr(a_m \succ a_1)} = 0$$

Therefore, there exist L and ϵ so that RBCML($\mathcal{G}, \mathcal{W}_{\text{u}}$) is inconsistent. ■

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be a pair of weighted breakings. Define $\mathcal{G}_1 + \mathcal{G}_2$ to be a breaking with weights being the sum of weights of corresponding edges in $\mathcal{G}_1$ and $\mathcal{G}_2$. Note that no edge between two vertices is equivalent to an edge with zero weight between the two vertices. If weights of all edges of $\mathcal{G}_1$ are no less than those in $\mathcal{G}_2$ (denoted as $\mathcal{G}_1 \ge \mathcal{G}_2$), we define $\mathcal{G}_1 - \mathcal{G}_2$ to be a breaking whose weight on each edge is the difference of the corresponding edge in $\mathcal{G}_1$ and $\mathcal{G}_2$ s.t. weights on all edges are nonnegative.

Lemma 9 *$\mathcal{G}_1$ and $\mathcal{G}_2$ are weighted breakings.*

- *If RBCML($\mathcal{G}_1, \mathcal{W}_u$) and RBCML($\mathcal{G}_2, \mathcal{W}_u$) are both consistent, then RBCML($\mathcal{G}_1 + \mathcal{G}_2, \mathcal{W}_u$) is also consistent. Further, if $\mathcal{G}_1 \ge \mathcal{G}_2$, then RBCML($\mathcal{G}_1 - \mathcal{G}_2, \mathcal{W}_u$) is consistent.*
- *If RBCML($\mathcal{G}_1, \mathcal{W}_u$) is consistent but RBCML($\mathcal{G}_2, \mathcal{W}_u$) is not consistent, then RBCML($\mathcal{G}_1 + \mathcal{G}_2, \mathcal{W}_u$) is not consistent. Further, if $\mathcal{G}_1 \ge \mathcal{G}_2$, then RBCML($\mathcal{G}_1 - \mathcal{G}_2, \mathcal{W}_u$) is not consistent.*

Proof: For any breaking $\mathcal{G}$, let $\text{ELL}_{\mathcal{M}}^{\mathcal{G}}(\vec{\theta})$ denote the expected log-marginal likelihood function under RBCML($\mathcal{G}, \mathcal{W}_{\text{u}}$).

Case 1. Because RBCML($\mathcal{G}_1, \mathcal{W}_{\text{u}}$) and RBCML($\mathcal{G}_2, \mathcal{W}_{\text{u}}$) are both consistent, for any $1 \le i \le m$, we have

$$\nabla_i \text{ELL}_{\mathcal{M}}^{\mathcal{G}_1}(\vec{\theta}) = \sum_{i'\neq i}\Big(\frac{\bar{\kappa}_{ii'}^{\mathcal{G}_1} w_{ii'}}{p_{ii'}(\vec{\theta})}\frac{\partial p_{ii'}(\vec{\theta})}{\partial\theta_i} + \frac{\bar{\kappa}_{i'i}^{\mathcal{G}_1} w_{i'i}}{p_{i'i}(\vec{\theta})}\frac{\partial p_{i'i}(\vec{\theta})}{\partial\theta_i}\Big) = 0$$
$$\nabla_i \text{ELL}_{\mathcal{M}}^{\mathcal{G}_2}(\vec{\theta}) = \sum_{i'\neq i}\Big(\frac{\bar{\kappa}_{ii'}^{\mathcal{G}_2} w_{ii'}}{p_{ii'}(\vec{\theta})}\frac{\partial p_{ii'}(\vec{\theta})}{\partial\theta_i} + \frac{\bar{\kappa}_{i'i}^{\mathcal{G}_2} w_{i'i}}{p_{i'i}(\vec{\theta})}\frac{\partial p_{i'i}(\vec{\theta})}{\partial\theta_i}\Big) = 0$$

It follows that

$$\nabla_i \text{ELL}_{\mathcal{M}}^{\mathcal{G}_1+\mathcal{G}_2}(\vec{\theta}) = \nabla_i \text{ELL}_{\mathcal{M}}^{\mathcal{G}_1}(\vec{\theta}) + \nabla_i \text{ELL}_{\mathcal{M}}^{\mathcal{G}_2}(\vec{\theta}) = 0$$
$$\nabla_i \text{ELL}_{\mathcal{M}}^{\mathcal{G}_1-\mathcal{G}_2}(\vec{\theta}) = \nabla_i \text{ELL}_{\mathcal{M}}^{\mathcal{G}_1}(\vec{\theta}) - \nabla_i \text{ELL}_{\mathcal{M}}^{\mathcal{G}_2}(\vec{\theta}) = 0$$

Case 2. Because RBCML($\mathcal{G}_1, \mathcal{W}_{\text{u}}$) is consistent and RBCML($\mathcal{G}_2, \mathcal{W}_{\text{u}}$) is not consistent, there exists $1 \le i \le m$ s.t.

$$\nabla_i \text{ELL}_{\mathcal{M}}^{\mathcal{G}_1}(\vec{\theta}) = \sum_{i'\neq i}\Big(\frac{\bar{\kappa}_{ii'}^{\mathcal{G}_1} w_{ii'}}{p_{ii'}(\vec{\theta})}\frac{\partial p_{ii'}(\vec{\theta})}{\partial\theta_i} + \frac{\bar{\kappa}_{i'i}^{\mathcal{G}_1} w_{i'i}}{p_{i'i}(\vec{\theta})}\frac{\partial p_{i'i}(\vec{\theta})}{\partial\theta_i}\Big) = 0$$
$$\nabla_i \text{ELL}_{\mathcal{M}}^{\mathcal{G}_2}(\vec{\theta}) = \sum_{i'\neq i}\Big(\frac{\bar{\kappa}_{ii'}^{\mathcal{G}_2} w_{ii'}}{p_{ii'}(\vec{\theta})}\frac{\partial p_{ii'}(\vec{\theta})}{\partial\theta_i} + \frac{\bar{\kappa}_{i'i}^{\mathcal{G}_2} w_{i'i}}{p_{i'i}(\vec{\theta})}\frac{\partial p_{i'i}(\vec{\theta})}{\partial\theta_i}\Big) \neq 0$$

It follows that

$$\nabla_i \text{ELL}_{\mathcal{M}}^{\mathcal{G}_1+\mathcal{G}_2}(\vec{\theta}) = \nabla_i \text{ELL}_{\mathcal{M}}^{\mathcal{G}_1}(\vec{\theta}) + \nabla_i \text{ELL}_{\mathcal{M}}^{\mathcal{G}_2}(\vec{\theta}) \neq 0$$
$$\nabla_i \text{ELL}_{\mathcal{M}}^{\mathcal{G}_1-\mathcal{G}_2}(\vec{\theta}) = \nabla_i \text{ELL}_{\mathcal{M}}^{\mathcal{G}_1}(\vec{\theta}) - \nabla_i \text{ELL}_{\mathcal{M}}^{\mathcal{G}_2}(\vec{\theta}) \neq 0$$

which implies inconsistency. ■

Lemma 10 *Let $m = 3$ and let RUM(π_1, π_2, π_3) be an RUM with symmetric distributions, where for at least one π_i we have $(\ln \pi_i)' = \frac{\pi_i'(x)}{\pi_i(x)}$ is monotonically decreasing and $\lim_{x\to-\infty} \frac{\pi_i'(x)}{\pi_i(x)} \to \infty$, then RBCML$(\mathcal{G}_{\{2\times\{1,2\},\{1,3\}\}}, \mathcal{W}_u)$ is not consistent for RUM(π_1, π_2, π_3).*

Proof: Let $\mathcal{G}_{210}$ denote $\mathcal{G}_{\{2\times\{1,2\},\{1,3\}\}}$. W.l.o.g. suppose $\lim_{x\to-\infty}(\pi_1'(x)) \to \infty$. Let $\theta_1 > 0$ and $\theta_2 = \theta_3 = 0$. We will prove that when θ_1 is sufficiently large, Equation (7) does not hold. Let

$$\Pr(a_1 \succ a_2 \succ a_3) = \Pr(a_1 \succ a_3 \succ a_2) = p_1$$
$$\Pr(a_2 \succ a_1 \succ a_3) = \Pr(a_3 \succ a_1 \succ a_2) = p_2$$
$$\Pr(a_2 \succ a_3 \succ a_1) = \Pr(a_3 \succ a_2 \succ a_1) = p_3$$

We have $p_1 + p_2 + p_3 = \frac{1}{2}$ and $\Pr(a_1 \succ a_2) = 2p_1 + p_2$, $\Pr(a_2 \succ a_1) = p_2 + 2p_3$. Given $\mathcal{G}_{210}$, $\bar{\kappa}_{12} = 3p_1$ and $\bar{\kappa}_{21} = 2p_2 + p_3$. Therefore, Equation (7) becomes

$$\nabla_1 \text{ELL}_{\mathcal{M}}(\vec{\theta}) = \sum_{i=2,3} \left(\frac{\bar{\kappa}_{1i}}{p_{1i}(\vec{\theta})} \frac{\partial p_{1i}(\vec{\theta})}{\partial \theta_1} + \frac{\bar{\kappa}_{i1}}{p_{i1}(\vec{\theta})} \frac{\partial p_{i1}(\vec{\theta})}{\partial \theta_1}\right) = 2\left(\frac{\bar{\kappa}_{12}}{p_{12}(\vec{\theta})} \frac{\partial p_{12}(\vec{\theta})}{\partial \theta_1} + \frac{\bar{\kappa}_{21}}{p_{21}(\vec{\theta})} \frac{\partial p_{21}(\vec{\theta})}{\partial \theta_1}\right)$$
$$= 2\frac{\partial p_{12}(\vec{\theta})}{\partial \theta_1}\left(\frac{3p_1}{2p_1 + p_2} - \frac{2p_2 + p_3}{p_2 + 2p_3}\right) = 0$$

Therefore, the following equation holds for all cases with $\theta_2 = \theta_3 = 0$ and $\theta_1 > 0$.

$$\frac{3p_1}{2p_1 + p_2} = \frac{2p_2 + p_3}{p_2 + 2p_3} \tag{10}$$

As $\theta_1 \to \infty$, $p_1 \to 0.5$ and p_2, p_3 goes to 0. Equation (10) becomes $\frac{2p_2+p_3}{p_2+2p_3} = \frac{3}{2}$. It follows that $\lim_{\theta_1\to\infty} \frac{p_2}{p_3} = 4$. We next prove that $\lim_{\theta_1\to\infty} \frac{p_2}{p_3} = \infty$, which will lead to a contradiction. For $i = 2, 3$, we let CDF$_i$ denote the CDF of π_i. By symmetry, it suffices to prove that $\lim_{\theta_1\to\infty} \frac{\int_{-\infty}^{\infty} \pi_1(U_1-\theta_1)\text{CDF}_2(U_1)(1-\text{CDF}_3(U_1))dU_1}{\int_{-\infty}^{\infty} \pi_1(U_1-\theta_1)(1-\text{CDF}_2(U_1))(1-\text{CDF}_3(U_1))dU_1} = \infty$.

The idea is to choose B and θ_1 so that $U_1 < B$ in the integration of both numerator and denominator can be ignored, and the ratio for the remainders of numeration and denominator can be arbitrarily large. More precisely, for any $K > 0$, let $B > 0$ denote any number such that $\dfrac{\text{CDF}_2(B+1)}{1 - \text{CDF}_2(B+1)} > K + 1$. Let θ_1 be any number such that

$$(\ln \pi_1)'(B + 1 - \theta_1) > \ln\left(K \frac{\int_{-\infty}^{B}(1 - \text{CDF}_2(U_1))(1 - \text{CDF}_3(U_1))dU_1}{\int_{B+1}^{3B+1}(1 - \text{CDF}_2(U_1))(1 - \text{CDF}_3(U_1))dU_1}\right)$$

Such a θ exists because $\lim_{x\to-\infty}\frac{\pi_i'(x)}{\pi_i(x)}\to\infty$. Because $\pi_1(x)$ is monotonically increasing for all $x<0$, we have

$$\begin{aligned}
&\int_{B}^{\infty}\pi_1(U_1-\theta_1)(1-\text{CDF}_2(U_1))(1-\text{CDF}_3(U_1))dU_1\\
>&\int_{B+1}^{3B+1}\pi_1(U_1-\theta_1)(1-\text{CDF}_2(U_1))(1-\text{CDF}_3(U_1))dU_1\\
>&\pi_1(B+1-\theta_1)\times\int_{B+1}^{3B+1}(1-\text{CDF}_2(U_1))(1-\text{CDF}_3(U_1))dU_1\\
>&e^{(\ln\pi_1)'(B+1-\theta_1)}\pi_1(B-\theta_1)\times\int_{B+1}^{3B+1}(1-\text{CDF}_2(U_1))(1-\text{CDF}_3(U_1))dU_1\\
>&K\pi_1(B-\theta_1)\int_{-\infty}^{B}(1-\text{CDF}_2(U_1))(1-\text{CDF}_3(U_1))dU_1\\
>&K\int_{-\infty}^{B}\pi_1(U_1-\theta_1)(1-\text{CDF}_2(U_1))(1-\text{CDF}_3(U_1))dU_1
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\frac{\int_{-\infty}^{\infty}\pi_1(U_1-\theta_1)\text{CDF}_2(U_1)(1-\text{CDF}_3(U_1))dU_1}{\int_{-\infty}^{\infty}\pi_1(U_1-\theta_1)(1-\text{CDF}_2(U_1))(1-\text{CDF}_3(U_1))dU_1}\\
>&\frac{\int_{B+1}^{\infty}\pi_1(U_1-\theta_1)\text{CDF}_2(U_1)(1-\text{CDF}_3(U_1))dU_1}{(1+\frac{1}{K})\int_{B+1}^{\infty}\pi_1(U_1-\theta_1)(1-\text{CDF}_2(U_1))(1-\text{CDF}_3(U_1))dU_1}\\
>&\frac{\text{CDF}_2(B+1)(1-\text{CDF}_3(B+1))}{(1+\frac{1}{K})(1-\text{CDF}_2(B+1))(1-\text{CDF}_3(B+1))}>K
\end{aligned}$$

Therefore, it is impossible that Equation (10) holds for all θ_1, which proves the lemma. ■

Lemma 11 *1. For any location family RUM$(\pi_1,\ldots,\pi_m)$,*

(a) RBCML$(\mathcal{G},\mathcal{W})$ is consistent if and only if RBCML$(k_1\mathcal{G},k_2\mathcal{W})$ is consistent for all $k_1,k_2>0$.

(b) If for any pair of alternatives $a_i,a_{i'}$ we have

$$\frac{\bar{\kappa}_{ii'}}{\bar{\kappa}_{i'i}}=\frac{\Pr_{\vec{\theta}}(a_i\succ a_{i'})}{\Pr_{\vec{\theta}}(a_{i'}\succ a_i)}\tag{11}$$

then RBCML$(\mathcal{G},\mathcal{W})$ is consistent if and only if $\mathcal{W}$ is connected and symmetric.

(c) If $\mathcal{G}$ has positive weight on an adjacent edge $l\to l+1$, then RBCML$(\mathcal{G},\mathcal{W})$ is consistent only if $\mathcal{W}$ is connected and symmetric.

2. For any RUM(π),

(a) RBCML$(\mathcal{G},\mathcal{W})$ is consistent only if for any alternative a_i we have

$$\sum_{i'\neq i}w_{ii'}=\sum_{i'\neq i}w_{i'i}\tag{12}$$

(b) Suppose the breaking graph contains an edge $\{l,l'\}$ that is different from $\{1,m\}$, then RBCML$(\mathcal{G},\mathcal{W})$ is consistent only if the $\mathcal{W}$ is connected and symmetric.

(c) RBCML$(\mathcal{G},\mathcal{W})$ is consistent only if RBCML$(\mathcal{G},\mathcal{W}_u)$ is consistent.

3. For any location family RUM$(\pi_1,\ldots,\pi_m)$ where each π_i is symmetric around 0, if RBCML$(\mathcal{G},\mathcal{W})$ is consistent, then RBCML$(\mathcal{G},\mathcal{W}')$ with symmetric weights $w'_{ii'}=w_{ii'}+w_{i'i}$ is also consistent.

Proof:

1(a). Let $\text{CLL}_{\mathcal{M}}(\vec{\theta}, P)$ be the composite log-likelihood of $\text{RBCML}(\mathcal{G}, \mathcal{W})$. Then the composite log-likelihood for $\text{RBCML}(k_1\mathcal{G}, k_2\mathcal{W})$ is $k_1k_2\text{CLL}_{\mathcal{M}}(\vec{\theta}, P)$. So if $\vec{\theta}^*$ maximizes $\text{CLL}_{\mathcal{M}}(\vec{\theta}, P)$, it also maximizes $k_1k_2\text{CLL}_{\mathcal{M}}(\vec{\theta}, P)$, or vice versa. That is to say, $\text{RBCML}(\mathcal{G}, \mathcal{W})$ and $\text{RBCML}(k_1\mathcal{G}, k_2\mathcal{W})$ are equivalent estimators.

1(b). The "if" direction: by combining (8) and (11), the ground truth is the solution to (7). Due to the strict concavity of $\text{CLL}_{\mathcal{M}}(\vec{\theta}, P)$, the ground truth is the only solution. Consistency follows by Theorem 5.

The "only if" direction: we first prove connectivity, then prove symmetry.

If $\mathcal{W}$ is not connected, then by Theorems 3 and 4, the solution to (7) is unbounded or non-unique. And by Theorem 5, $\text{RBCML}(\mathcal{G}, \mathcal{W})$ is not consistent.

Now we prove symmetry of $\mathcal{W}$ by contradiction. For the purpose of contradiction suppose $w_{12} \neq w_{21}$ (w.l.o.g.). We will construct a counterexample where $\text{RBCML}(\mathcal{G}, \mathcal{W})$ is not consistent. Let $\theta_1 = \theta_2 = 0$ and $\theta_3 = \ldots = \theta_m = L$. By Lemma 3, we have for any $\epsilon > 0$, there exists L s.t. $\nabla_1\text{ELL}_{\mathcal{M}}(\vec{\theta}) = \frac{\bar{\kappa}_{12}w_{12}}{p_{12}(\vec{\theta})}\frac{\partial p_{12}(\vec{\theta})}{\partial\theta_1} + \frac{\bar{\kappa}_{21}w_{21}}{p_{21}(\vec{\theta})}\frac{\partial p_{21}(\vec{\theta})}{\partial\theta_1} + \epsilon = \frac{\bar{\kappa}_{21}(w_{21}-w_{12})}{p_{21}(\vec{\theta})}\frac{\partial p_{21}(\vec{\theta})}{\partial\theta_1} + \epsilon$, where the last equality is obtained due to Lemma 4. Since $w_{12} \neq w_{21}$, we have $\frac{\kappa_{21}(w_{21}-w_{12})}{p_{21}(\vec{\theta})}\frac{\partial p_{21}(\vec{\theta})}{\partial\theta_1} \neq 0$. Let $\epsilon < |\frac{\kappa_{21}(w_{21}-w_{12})}{p_{21}(\vec{\theta})}\frac{\partial p_{21}(\vec{\theta})}{\partial\theta_1}|$, then we have $\nabla_1\text{ELL}_{\mathcal{M}}(\vec{\theta}) \neq 0$. This means the ground truth does not maximize $\text{ELL}_{\mathcal{M}}(\vec{\theta})$. By Theorem 5, the estimator is not consistent.

1(c). The proof for connectivity of $\mathcal{W}$ is the same as in the proof of 1(b). We only prove necessity of symmetry. For the purpose of contradiction suppose $w_{12} \neq w_{21}$. Let $\theta_1 = \theta_2 = 0$, $\theta_3 = \ldots = \theta_{l+1} = -L$, and $\theta_{l+2} = \ldots = \theta_m = L$. By Lemma 3, for any $\epsilon > 0$, we have $\nabla_1\text{ELL}_{\mathcal{M}}(\vec{\theta}) = \frac{\bar{\kappa}_{12}w_{12}}{p_{12}(\vec{\theta})}\frac{\partial p_{12}(\vec{\theta})}{\partial\theta_1} + \frac{\bar{\kappa}_{21}w_{21}}{p_{21}(\vec{\theta})}\frac{\partial p_{21}(\vec{\theta})}{\partial\theta_1} + \epsilon = \frac{\bar{\kappa}_{21}(w_{21}-w_{12})}{p_{21}(\vec{\theta})}\frac{\partial p_{21}(\vec{\theta})}{\partial\theta_1} + \epsilon$, where the last equality is obtained by Lemma 4. Since $w_{12} \neq w_{21}$, we have $\frac{\kappa_{21}(w_{21}-w_{12})}{p_{21}(\vec{\theta})}\frac{\partial p_{21}(\vec{\theta})}{\partial\theta_1} \neq 0$. Let $\epsilon < |\frac{\kappa_{21}(w_{21}-w_{12})}{p_{21}(\vec{\theta})}\frac{\partial p_{21}(\vec{\theta})}{\partial\theta_1}|$, then we have $\nabla_1\text{ELL}_{\mathcal{M}}(\vec{\theta}) \neq 0$. This means the ground truth does not maximize $\text{ELL}_{\mathcal{M}}(\vec{\theta})$. By Theorem 5, the estimator is not consistent.

2(a). Let $\theta_1 = \ldots = \theta_m = 0$. Thus for any pair of alternatives $a_i, a_{i'}$, we have $\bar{\kappa}_{ii'} = \bar{\kappa}_{i'i}$ and $\Pr_{\vec{\theta}}(a_i \succ a_{i'}) = \Pr_{\vec{\theta}}(a_{i'} \succ a_i)$. (12) follows by applying (8) to $\text{ELL}_{\mathcal{M}}(\vec{\theta}) = 0$.

2(b). The proof for connectivity of $\mathcal{W}$ is the same as in the proof of 1(b). For necessity of $\mathcal{W}$, it suffices to prove $w_{12} = w_{21}$. Let $\Delta l = l' - l$ (w.l.o.g. suppose $l < l'$). Let $\theta_1 = \ldots = \theta_{\Delta l+1} = 0$, and $\theta_{\Delta l+2} = \ldots = \theta_{l+\Delta l} = L$, $\theta_{l'+1} = \ldots = \theta_m = -L$. When $L \to +\infty$, with probability approaching 1, θ_1 through $\theta_{\Delta l+1}$ are ranked at positions from l to l'. For any $1 \leq i, i' \leq \Delta l + 1$ and $i' \neq i$, we have $\bar{\kappa}_{ii'} = \bar{\kappa}_{i'i}$ and $\Pr_{\vec{\theta}}(a_i \succ a_{i'}) = \Pr_{\vec{\theta}}(a_{i'} \succ a_i)$. So we have

$$\sum_{i=2}^{\Delta l+1} w_{1i} = \sum_{i=2}^{\Delta l+1} w_{i1} \tag{13}$$

If we swap the values of $\theta_{\Delta l+2}$ and $\theta_{i'}$ where $2 \leq i' \leq \Delta l + 1$, we have

$$\sum_{i=2}^{\Delta l+2} w_{1i} - w_{1i'} = \sum_{i=2}^{\Delta l+2} w_{i1} - w_{i'1} \tag{14}$$

Note that (14) contains Δl equations. Summing up all equations in (13) and (14), we have

$$\sum_{i=2}^{\Delta l+2} w_{1i} = \sum_{i=2}^{\Delta l+2} w_{i1} \tag{15}$$

Let $i' = 2$ in (14), we get

$$\sum_{i=3}^{\Delta l+2} w_{1i} = \sum_{i=3}^{\Delta l+2} w_{i1} \tag{16}$$

(15)-(16), we have $w_{12} = w_{21}$.

2(c). For any $\vec{\theta}$, $\nabla\text{ELL}_{\mathcal{M}}(\vec{\theta}) = \vec{0}$ holds. By relabeling the alternatives (by permuting the elements in $\vec{\theta}$), we can obtain $m!$ similar equations. Equivalently, each $w_{ii'}$ in $\mathcal{W}$ will be the weight of $a_1 \succ a_2$ (or any other pairwise comparison) for $(m-2)!$ times. By summing up all corresponding equations, we obtain another set of equations, which is the gradient of the composite likelihood with uniform $\mathcal{W}' = (m-2)! \sum_{i \neq i'} w_{ii'}$. So RBCML$(\mathcal{G}, \mathcal{W}')$ is also consistent.

3. For any $\vec{\theta}$, we re-write (7)

$$\nabla_i \text{ELL}_{\mathcal{M}}(\vec{\theta}) = \sum_{i' \neq i} (\frac{\bar{\kappa}_{ii'} w_{ii'}}{p_{ii'}(\vec{\theta})} \frac{\partial p_{ii'}(\vec{\theta})}{\partial \theta_i} + \frac{\bar{\kappa}_{i'i} w_{i'i}}{p_{i'i}(\vec{\theta})} \frac{\partial p_{i'i}(\vec{\theta})}{\partial \theta_i}) = 0 \tag{17}$$

Consider the RUM with $\vec{\theta}' = -\vec{\theta}$, we have $p_{ii'}(\vec{\theta}') = p_{i'i}(\vec{\theta})$. So we have

$$\begin{aligned}\nabla_i \text{ELL}_{\mathcal{M}}(\vec{\theta}') &= \sum_{i' \neq i} (\frac{\bar{\kappa}_{ii'} w_{ii'}}{p_{ii'}(\vec{\theta}')} \frac{\partial p_{ii'}(\vec{\theta}')}{\partial \theta'_i} + \frac{\bar{\kappa}_{i'i} w_{i'i}}{p_{i'i}(\vec{\theta}')} \frac{\partial p_{i'i}(\vec{\theta}')}{\partial \theta'_i}) \\ &= \sum_{i' \neq i} (-\frac{\bar{\kappa}_{ii'} w_{i'i}}{p_{ii'}(\vec{\theta})} \frac{\partial p_{ii'}(\vec{\theta})}{\partial \theta_i} - \frac{\bar{\kappa}_{i'i} w_{ii'}}{p_{i'i}(\vec{\theta})} \frac{\partial p_{i'i}(\vec{\theta})}{\partial \theta_i}) = 0 \end{aligned} \tag{18}$$

(17)-(18), we have $\sum_{i' \neq i} (\frac{\kappa_{ii'}(w_{ii'} + w_{i'i})}{p_{ii'}(\vec{\theta})} \frac{\partial p_{ii'}(\vec{\theta})}{\partial \theta_i} + \frac{\kappa_{i'i}(w_{i'i} + w_{i'i})}{p_{i'i}(\vec{\theta})} \frac{\partial p_{i'i}(\vec{\theta})}{\partial \theta_i}) = 0$, which means RBCML$(\mathcal{G}, \mathcal{W}')$ is consistent by Theorem 5. ■

Theorem 1 Let $f(x)$ and $g(x)$ be two continuous and strictly log-concave functions on $\mathbb{R}$. Then $f * g$ is also strictly log-concave on $\mathbb{R}$.

Proof: The proof is done by examining the equality condition for the Prékopa-Leindler inequality. Let $h = f * g$, namely, for any $y \in \mathbb{R}$, $h(y) = \int_{\mathbb{R}} f(y-x)g(x)dx$. Because f and g are continuous, so does h. To prove the strict log-concavity of h, it suffices to prove that for any different $y_1, y_2 \in \mathbb{R}$, $h(\frac{y_1+y_2}{2}) > \sqrt{h(y_1)h(y_2)}$.

Suppose for the sake of contradiction that this is not true. Since log-concavity preserves under convolution (Saumard & Wellner, 2014), h is log-concave. So, there exist $y_1 < y_2$ such that $h(\frac{y_1+y_2}{2}) = \sqrt{h(y_1)h(y_2)}$. Let $\Lambda(x,y) = f(y-x)g(x)$. We further define

$$\begin{aligned} H(x) &= \Lambda(x, \frac{y_1+y_2}{2}) = f(\frac{y_1+y_2}{2} - x)g(x) \\ F(x) &= \Lambda(x, y_1) = f(y_1 - x)g(x) \\ G(x) &= \Lambda(x, y_2) = f(y_2 - x)g(x) \end{aligned}$$

Because (non-strict) log-concavity is preserved under convolution, $\Lambda(x,y)$ is log-concave. We have that for any $x \in \mathbb{R}$, $H(x) \geq \sqrt{F(x)G(x)}$. The Prékopa-Leindler inequality asserts that

$$\int_{\mathbb{R}} H(x)dx \geq \sqrt{\int_{\mathbb{R}} F(x)dx \int_{\mathbb{R}} G(x)dx} \tag{19}$$

Because $h(\frac{y_1+y_2}{2}) = \int_{\mathbb{R}} H(x)dx$, $h(y_1) = \int_{\mathbb{R}} F(x)dx$, $h(y_2) = \int_{\mathbb{R}} G(x)dx$, and $h(\frac{y_1+y_2}{2}) = \sqrt{h(y_1)h(y_2)}$, (19) becomes an equation. It was proved by Dubuc (1977) that: there exist $a > 0$ and $b \in \mathbb{R}$ such that the following conditions hold almost everywhere for $x \in \mathbb{R}$ (see the translation of Dubuc's result in English by Ball & Böröczky (2010)). 1. $F(x) = aH(x+b)$, 2. $G(x) = a^{-1}H(x-b)$.

The first condition means that for almost every $x \in \mathbb{R}$,

$$\begin{aligned} f(y_1 - x)g(x) &= af(\frac{y_1+y_2}{2} - x - b)g(x+b) \\ \iff \frac{g(x)}{g(x+b)} &= a\frac{f(\frac{y_1+y_2}{2} - x - b)}{f(y_1 - x)} \end{aligned} \tag{20}$$

The second condition means that for almost all $x \in \mathbb{R}$, $f(y_2 - x)g(x) = a^{-1}f(\frac{y_1+y_2}{2} - x + b)g(x-b) \iff \frac{g(x-b)}{g(x)} = a\frac{f(y_2-x)}{f(\frac{y_1+y_2}{2}-x+b)}$. Therefore, for almost all $x \in \mathbb{R}$,

$$\frac{g(x)}{g(x+b)} = a\frac{f(y_2 - x - b)}{f(\frac{y_1+y_2}{2} - x)} \tag{21}$$

Combining (20) and (21), for almost every $x \in \mathbb{R}$ we have

$$\frac{g(x)}{g(x+b)} = a\frac{f(y_2 - x - b)}{f(\frac{y_1+y_2}{2} - x)} = a\frac{f(\frac{y_1+y_2}{2} - x - b)}{f(y_1 - x)} \tag{22}$$

Because $f(x)$ is strictly log-concave, for any fixed $c \neq 0$, $\frac{f(x+c)}{f(x)}$ is strictly monotonic. Because $y_1 \neq y_2$ and $y_2 - x - b - (\frac{y_1+y_2}{2} - x) = \frac{y_1+y_2}{2} - x - b - (y_1 - x) = \frac{y_2-y_1}{2} - b$, we must have that $\frac{y_2-y_1}{2} - b = 0$, namely $b = \frac{y_2-y_1}{2}$. Therefore, (22) becomes $\frac{g(x)}{g(x+\frac{y_2-y_1}{2})} = a$ for almost every $x \in \mathbb{R}$, which contradicts the strict log-concavity of g. This means that $h = f * g$ is strictly log-concave. ∎

Theorem 2 Let $h(x, y)$ be a strictly log-concave function on $\mathbb{R}^2$. Then $\int_{\mathbb{R}} h(x, y)dx$ is strictly log-concave on $\mathbb{R}$.

Proof: Again, the proof is done by examining the equality condition for the Prékopa-Leindler inequality. Let $h^*(y) = \int_{\mathbb{R}} h(x, y)dx$. It suffices to prove that for any different $y_1, y_2 \in \mathbb{R}$, $h^*(\frac{y_1+y_2}{2}) > \sqrt{h^*(y_1)h^*(y_2)}$.

Suppose for the sake of contradiction the claim is not true. Because (non-strict) log-concavity is preserved under marginalization, h^* is log-concave. Therefore, there exist $y_1 < y_2$ such that $h^*(\frac{y_1+y_2}{2}) = \sqrt{h^*(y_1)h^*(y_2)}$. We further define the following functions. $H(x) = h(x, \frac{y_1+y_2}{2})$, $F(x) = h(x, y_1)$, and $G(x) = h(x, y_2)$.

Because $h(x, y)$ is strictly log-concave, we have that for any $x \in \mathbb{R}$, $H(x) > \sqrt{F(x)G(x)}$. The Prékopa-Leindler inequality asserts that

$$\int_{\mathbb{R}} H(x)dx \geq \sqrt{\int_{\mathbb{R}} F(x)dx \int_{\mathbb{R}} G(x)dx} \tag{23}$$

Because $h^*(\frac{y_1+y_2}{2}) = \int_{\mathbb{R}} H(x)dx$, $h^*(y_1) = \int_{\mathbb{R}} F(x)dx$, $h^*(y_2) = \int_{\mathbb{R}} G(x)dx$, and $h^*(\frac{y_1+y_2}{2}) = \sqrt{h^*(y_1)h^*(y_2)}$, (23) becomes an equation. Following Dubuc (1977)'s result, we have that there exist $a > 0$ and $b \in \mathbb{R}$ such that $F(x) = aH(x+b)$ and $G(x) = a^{-1}H(x-b)$ hold almost everywhere for $x \in \mathbb{R}$.

$F(x) = aH(x+b)$ means that for almost every $x \in \mathbb{R}$, $ah(x+b, \frac{y_1+y_2}{2}) = h(x, y_1)$. $G(x) = a^{-1}H(x-b)$ means that for almost every $x \in \mathbb{R}$, $a^{-1}h(x-b, \frac{y_1+y_2}{2}) = h(x, y_2)$. This means that for almost every $x \in \mathbb{R}$, $a^{-1}h(x+b, \frac{y_1+y_2}{2}) = h(x+2b, y_2)$. Therefore, for almost every $x \in \mathbb{R}$, we have $h(x+b, \frac{y_1+y_2}{2}) \cdot h(x+b, \frac{y_1+y_2}{2}) = h(x, y_1) \cdot h(x+2b, y_2)$, which contradicts the strict log-concavity of h. ∎

Theorem 3 Given any profile P, the composite likelihood function for Plackett-Luce, i.e. $\text{CL}_{\text{PL}}(\vec{\theta}, P)$, is strictly log-concave if and only if $\mathcal{W}$ is weakly connected. $\arg\max_{\vec{\theta}} \text{CL}_{\text{PL}}(\vec{\theta}, P)$ is bounded if and only if $\mathcal{W} \otimes G(P)$ is strongly connected.

Proof: It is not hard to check that when $\mathcal{W}$ is not weakly connected, there exist $\vec{\theta}^{(1)}$ and $\vec{\theta}^{(2)}$ such that for any $0 < \lambda < 1$ we have $\text{CLL}_{\text{PL}}(\vec{\theta}^{(1)}, P) = \text{CLL}_{\text{PL}}(\vec{\theta}^{(2)}, P) = \lambda\text{CLL}_{\text{PL}}(\vec{\theta}^{(1)}, P) + (1-\lambda)\text{CLL}_{\text{PL}}(\vec{\theta}^{(2)}, P)$, which violates strict log-concavity.

Suppose $\mathcal{W}$ is weakly connected, we only need to show that

$$f(\vec{\theta}) = \sum_{i_1 \neq i_2} (-(\kappa_{i_1 i_2} w_{i_1 i_2} + \kappa_{i_2 i_1} w_{i_2 i_1}) \ln(e^{\theta_{i_1}} + e^{\theta_{i_2}})) \tag{24}$$

is concave. The proof is similar to the log-concavity of likelihood for BTL by (Hunter, 2004). Hölder's inequality shows that for positive $c_t, d_t > 0$, where $t = 1, \ldots, N$ and $0 < \lambda < 1$, we have

$$\ln \sum_{t=1}^{N} c_t^{\lambda} d_t^{1-\lambda} \leq \lambda \ln \sum_{t=1}^{N} c_t + (1-\lambda) \ln \sum_{t=1}^{N} d_t \tag{25}$$

with equality if and only if $\exists\zeta$ s.t. $c_t = \zeta d_t$ for all t.

Let $\vec{\theta}^{(1)}$ and $\vec{\theta}^{(2)}$ be two parameters. For any two alternatives a_{i_1} and a_{i_2}, by (25), we have

$$-\ln(e^{\lambda\theta_{i_1}^{(1)}+(1-\lambda)\theta_{i_1}^{(2)}}+e^{\lambda\theta_{i_2}^{(1)}+(1-\lambda)\theta_{i_2}^{(2)}}\geq -\lambda\ln(e^{\theta_{i_1}^{(1)}}+e^{\theta_{i_2}^{(1)}})-(1-\lambda)\ln(e^{\theta_{i_1}^{(2)}}+e^{\theta_{i_2}^{(2)}})$$

Multiplying both sides by $\kappa_{i_1 i_2}w_{i_1 i_2}+\kappa_{i_2 i_1}w_{i_2 i_1}$ and summing over all $i_i \neq i_2$ demonstrates the concavity of (24).

To prove strict concavity, we need to check the condition when the equality of (25) holds. For all $1 \leq i \leq m$, $e^{\theta_i^{(1)}} = \zeta e^{\theta_i^{(2)}}$. Namely $\theta_i^{(1)} = \theta_i^{(2)} + \ln\zeta$ holds for all i. Because random utility models are invariant under parameter shifts, it is exactly the same model. Thus, we proves the strict concavity of (24).

The proof for the condition of boundedness is also similar to that by Hunter (2004). ■

Theorem 4 Let $\mathcal{M}$ be an RUM where the CDF of each utility distribution is strictly log-concave. Given any profile P, the composite likelihood function for $\mathcal{M}$, i.e. $\text{CL}_{\mathcal{M}}(\vec{\theta}, P)$, is strictly log-concave if and only if $\mathcal{W}$ is weakly connected. $\arg\max_{\vec{\theta}}\text{CL}_{\mathcal{M}}(\vec{\theta}, P)$ is bounded if and only if $\mathcal{W}\otimes G(P)$ is strongly connected.

Proof: Similar to the proof for Plackett-Luce, the only hard part is to prove that when $\mathcal{W}$ is weakly connected, $\text{CL}_{\mathcal{M}}(\vec{\theta}, P)$ is strictly log-concave. It suffice to prove for any $i_1 \neq i_2$, $\Pr(a_{i_1} \succ a_{i_2}|\vec{\theta})$ is log-concave, namely $\Pr(u_{i_1} > u_{i_2}|\vec{\theta})$ is log-concave. We can write this probability as integral over $u_{i_2} - u_{i_1}$: $\Pr(u_{i_1} > u_{i_2}|\vec{\theta}) = \int_0^\infty \Pr(u_{i_2} - u_{i_1} = s|\vec{\theta})ds$.

Let $\pi_{i_2}^*(\cdot|\vec{\theta})$ denote the flipped distribution of $\pi_{i_2}(\cdot|\vec{\theta})$ around $x = s$, then we have $\pi_{i_2}^*(s-x|\vec{\theta}) = \pi_{i_2}(s+x|\vec{\theta})$. Therefore we have

$$\Pr(u_{i_1} > u_{i_2}|\vec{\theta}) = \int_0^\infty\int_{-\infty}^\infty \pi_{i_1}(x|\theta_{i_1})\pi_{i_2}(x+s|\theta_{i_2})dxds = \int_0^\infty\int_{-\infty}^\infty \pi_{i_1}(x|\theta_{i_1})\pi_{i_2}^*(s-x|\theta_{i_2})dxds = \int_0^\infty \pi_{i_1} * \pi_{i_2}^* ds$$

By Theorem 1 we know $\pi_{i_1} * \pi_{i_2}^*$ is strictly log-concave. We only need to prove that tail probability of a strictly log-concave distribution is also strictly log-concave, which is shown in Lemma 1. ■

Theorem 5 Given any RUM $\mathcal{M}$, any $\vec{\theta}_0$ and any profile P with n rankings. Let $\vec{\theta}^*$ be the output of $\text{RBCML}(\mathcal{G}, \mathcal{W})$. When $n \to \infty$, we have $\vec{\theta}^* \xrightarrow{p} \vec{\theta}_0$ and $\sqrt{n}(\vec{\theta}^* - \vec{\theta}_0) \xrightarrow{d} N(0, H_0^{-1}(\vec{\theta}_0)\text{Var}[\nabla\text{CLL}_{\mathcal{M}}(\vec{\theta}_0, R)]H_0^{-1}(\vec{\theta}_0))$ if and only if $\vec{\theta}_0$ is the only solution to

$$\nabla\text{ELL}_{\mathcal{M}}(\vec{\theta}) = \vec{0} \tag{26}$$

Proof: The "only if" direction is straightforward. The solution to (26) is unique because $\text{CLL}_{\mathcal{M}}(\vec{\theta}, P)$ is strictly concave. Suppose $\vec{\theta}_1$, other than $\vec{\theta}_0$, is the solution to (26), then when $n \to \infty$, $\vec{\theta}_1$ will be the estimate of $\text{RBCML}(\mathcal{G}, \mathcal{W})$, which means $\text{RBCML}(\mathcal{G}, \mathcal{W})$ is not consistent.

Now we prove the "if" direction. First we prove consistency. It is required by Xu & Reid (2011) that for different parameters, the probabilities for any composite likelihood event are different, which is not true in our case. A simple counterexample is $\theta_1^{(1)} = 1, \theta_1^{(2)} = 2, \theta_2^{(1)} = \theta_3^{(1)} = \theta_2^{(2)} = \theta_3^{(2)} = 0$. Then $\Pr(a_2 \succ a_3|\vec{\theta}^{(1)}) = \Pr(a_2 \succ a_3|\vec{\theta}^{(2)})$.

By the law of large numbers, we have for any ϵ, $\Pr(|\text{CLL}_{\mathcal{M}}(\vec{\theta}, P) - \text{ELL}_{\mathcal{M}}(\vec{\theta})| \leq \epsilon/2) \to 1$ as $n \to \infty$. This implies $\lim_{n\to\infty}\Pr(\text{CLL}_{\mathcal{M}}(\vec{\theta}^*, P) \leq \text{ELL}_{\mathcal{M}}(\vec{\theta}^*) + \epsilon/2) = 1$. Similarly we have $\lim_{n\to\infty}\Pr(\text{ELL}_{\mathcal{M}}(\vec{\theta}_0) \leq \text{CLL}_{\mathcal{M}}(\vec{\theta}_0, P) + \epsilon/2) = 1$. Since $\vec{\theta}^*$ maximize $\text{CLL}_{\mathcal{M}}(\vec{\theta}, P)$, we have $\Pr(\text{CLL}_{\mathcal{M}}(\vec{\theta}_0, P) \leq \text{CLL}_{\mathcal{M}}(\vec{\theta}^*, P)) = 1$. The above three equations imply that $\lim_{n\to\infty}\Pr(\text{ELL}_{\mathcal{M}}(\vec{\theta}_0) - \text{ELL}_{\mathcal{M}}(\vec{\theta}^*) \leq \epsilon) = 1$.

Let Θ_ϵ be the subset of parameter space s.t. $\forall\vec{\theta} \in \Theta_\epsilon$, $\text{ELL}_{\mathcal{M}}(\vec{\theta}_0) - \text{ELL}_{\mathcal{M}}(\vec{\theta}) \leq \epsilon$. Because $\text{ELL}_{\mathcal{M}}(\vec{\theta})$ is strictly concave, Θ_ϵ is compact and has a unique maximum at $\vec{\theta}_0$. Thus for any $\epsilon > 0$, $\lim_{n\to\infty}\Pr(\vec{\theta}^* \in \Theta_\epsilon) = 1$. This implies consistency, i.e., $\vec{\theta}^* \xrightarrow{p} \vec{\theta}_0$.

Now we prove asymptotic normality. By mean value theorem, we have $0 = \nabla\text{CLL}_{\mathcal{M}}(\vec{\theta}^*, P) = \nabla\text{CLL}_{\mathcal{M}}(\vec{\theta}_0, P) + H(\alpha\vec{\theta}^* + (1-\alpha)\vec{\theta}_0, P)(\vec{\theta}^* - \vec{\theta}_0)$, where $0 \leq \alpha \leq 1$. Therefore, we have $\sqrt{n}(\vec{\theta}^* - \vec{\theta}) = -H^{-1}(\alpha\vec{\theta}^* + (1-\alpha)\vec{\theta}_0, P)(\sqrt{n}\nabla\text{CLL}_{\mathcal{M}}(\vec{\theta}_0, P))$. Since $\nabla\text{CLL}_{\mathcal{M}}(\vec{\theta}_0, P) = \frac{1}{n}\sum_{j=1}^n \nabla\text{CLL}_{\mathcal{M}}(\vec{\theta}_0, R_j)$, by the central limit theorem, we have

$$\sqrt{n}\nabla \text{CLL}_{\mathcal{M}}(\vec{\theta}_0, P) \xrightarrow{d} N(0, \text{Var}[\nabla \text{CLL}_{\mathcal{M}}(\vec{\theta}_0, R)])$$

Because $\vec{\theta}^* \xrightarrow{p} \vec{\theta}_0$ and H is continuous, we have $H(\alpha\vec{\theta}^* + (1-\alpha)\vec{\theta}_0, P) \xrightarrow{p} H(\vec{\theta}_0, P)$. Since $H(\vec{\theta}, P) = \frac{1}{n}\sum_{j=1}^{n} H(\vec{\theta}, R_j)$, by law of large numbers, we have $H(\vec{\theta}, P) \xrightarrow{p} H_0(\vec{\theta}_0)$. Therefore, we have

$$\sqrt{n}(\vec{\theta}^* - \vec{\theta}) = -H_0^{-1}(\vec{\theta}_0)(\sqrt{n}\nabla \text{CLL}_{\mathcal{M}}(\vec{\theta}_0, P)),$$

which implies that $\text{Var}[\sqrt{n}(\vec{\theta}^* - \vec{\theta})] = H_0^{-1}(\vec{\theta}_0)\text{Var}[\nabla \text{CLL}_{\mathcal{M}}(\vec{\theta}_0, R)]H_0^{-1}(\vec{\theta}_0)$. ■

Theorem 6 RBCML$(\mathcal{G}, \mathcal{W}_{\text{u}})$ is consistent for Plackett-Luce if and only if the breaking is weighted union of position-k breaking.

Proof: The "if" direction is proved in (Khetan & Oh, 2016b). We only prove the "only if" direction.

We will prove this theorem by induction on m. When $m = 2$, the only breaking is the comparison between the two alternatives. The conclusion holds. Suppose it holds for $m = l$, then when $m = l + 1$, we first apply Lemma 2 to $\mathcal{G}_{[2,m]}$, which must be a weighted union of position-k breaking. Then apply Lemma 2 to $\mathcal{G}_{[1,m-1]}$. For all $i \leq m - 1$, g_{1i} are the same, denoted by g_0. We claim that $g_{1m} = g_0$. The reason is as follows.

For the purpose of contradiction suppose $g_{1m} \neq g_0$. If $g_{1m} > g_0$. We split this edge into two parts, one with weight g_0 and the other $g_{1m} - g_0$. Let $\mathcal{G}_1 = \{g_{1m} = g_0\} \cup (\mathcal{G} - g_{1m})$, and $\mathcal{G}_2 = \{g_{1m} = g - g_0\}$. So we have $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$. Because RBCML$(\mathcal{G}_1, \mathcal{W}_{\text{u}})$ is consistent and RBCML$(\mathcal{G}_2, \mathcal{W}_{\text{u}})$ is not (Lemma 7). By Lemma 9, RBCML$(\mathcal{G}, \mathcal{W}_{\text{u}})$ is not consistent, which is a contradiction. The case where $g < g_0$ is similar. ■

Theorem 7 Let $\pi_1, \pi_2, \ldots, \pi_m$ denote the utility distributions for a symmetric RUM. Suppose there exists π_i s.t. $(\ln \pi_i(x))'$ is monotonically decreasing and $\lim_{x \to -\infty}(\ln \pi_i(x))' \to \infty$. RBCML$(\mathcal{G}, \mathcal{W})$ is consistent if and only if $\mathcal{G}$ is uniform.

Proof: We prove the theorem by induction on m. $m = 2$ is trivial because the only breaking is uniform. For $m = 3$ we know the uniform breaking is consistent and the one-edge breaking $\mathcal{G} = \{g_{13} = C > 0\}$ is not consistent by Lemma 8. Suppose the breaking is $\mathcal{G} = \{g_{12} = x, g_{23} = y, g_{13} = z\}$.

Case 1: $x + y \neq 2z$. For the sake of contradiction suppose RBCML$(\mathcal{G}, \mathcal{W}_{\text{u}})$ is consistent. By Lemma 5, RBCML$(\mathcal{G}^*, \mathcal{W}_{\text{u}})$ is consistent for $\mathcal{M}^*$, which is $\mathcal{M}$ due to the symmetry of utility distributions. Applying Lemma 9 we have RBCML$(\mathcal{G} + \mathcal{G}^*, \mathcal{W}_{\text{u}})$ is consistent, where $\mathcal{G} + \mathcal{G}^* = \{g_{12} = x + y, g_{23} = x + y, g_{13} = 2z\}$. If $x + y < 2z$, we have RBCML$(\mathcal{G} + \mathcal{G}^* - (x + y)\mathcal{G}_{\text{u}}, \mathcal{W}_{\text{u}})$ is consistent, where $\mathcal{G} + \mathcal{G}^* - (x + y)\mathcal{G}_{\text{u}} = \{g_{1m} = 2z - x - y\}$. This contradicts Lemma 8. The case with $x + y > 2z$ is similar.

Case 2: x+y=2z. Lemma 10 states that RBCML$(\mathcal{G}_{210}, \mathcal{W}_{\text{u}})$ is not consistent where $\mathcal{G}_{210} = \{g_{12} = 2, g_{13} = 1\}$. We have $\mathcal{G} = y\mathcal{G}_{\text{u}} + (z - y)\mathcal{G}_{210}$. Since any $\mathcal{G}_{\text{u}}$ is consistent, RBCML$(\mathcal{G}, \mathcal{W}_{\text{u}})$ is not consistent.

Suppose the theorem holds for $m = k$. When $m = k + 1$, W.l.o.g. we let π_2 satisfy the conditions that $(\ln \pi_i(x))'$ is monotonically decreasing and $\lim_{x \to -\infty}(\ln \pi_i(x))' \to \infty$. Let $\theta_1 = L$, $\theta_m = -L$, and $\theta_2 = \ldots = \theta_{m-1} = 0$. So when $L \to \infty$, with probability that goes to 1, a_1 is ranked at the top and a_m is ranked at the bottom. Let $\mathcal{G}_{\{1m\}} = \{g_{1m} = 1\}$. We apply Lemma 6 to $\mathcal{G}_{[2,m]}$ and $\mathcal{G}_{[1,m-1]}$. By induction hypothesis $\mathcal{G}_{[2,m]}$ (or $\mathcal{G}_{[1,m-1]}$) is uniform breaking graph or empty. If $\mathcal{G}_{[2,m]}$ is empty, then $\mathcal{G}_{[1,m-1]}$ is also empty. As $\mathcal{G}$ is nonempty, $\mathcal{G} = C\mathcal{G}_{\{1m\}}$, which contradicts Lemma 8. If $\mathcal{G}_{[2,m]}$ is uniform. We denote the weight as g_0. Then $\mathcal{G}_{[1,m-1]}$ is also uniform with weight g_0. Then the only consistent breaking is uniform. The reason is as follows. We can write $\mathcal{G} = g_0\mathcal{G}_{\text{u}} + (g_{1m} - g_0)\mathcal{G}_{\{1m\}}$. By Lemma 8 and Lemma 9, RBCML$(\mathcal{G}, \mathcal{W})$ is not consistent, which is a contradiction. ■

Theorem 8 RBCML$(\mathcal{G}, \mathcal{W})$ for Plackett-Luce is consistent if and only if $\mathcal{G}$ is the weighted union of position-k breakings and $\mathcal{W}$ is connected and symmetric.

Proof: The "only if" direction: 2(c) part of the Lemma 11 states that if RBCML$(\mathcal{G}, \mathcal{W})$ is consistent then RBCML$(\mathcal{G}, \mathcal{W}_{\text{u}})$ is consistent, which means that $\mathcal{G}$ is the weighted union of position-k breakings by Theorem 6. Then following 1(c) part of the Lemma 11, $\mathcal{W}$ must be connected and symmetric.

The "if" direction: $\mathcal{G}$ is the weighted union of position-k breakings. For any a_i, $a_{i'}$, we have $\sum_{i' \neq i}(\bar{\kappa}_{ii'} - (\bar{\kappa}_{ii'} + \bar{\kappa}_{i'i})\frac{e^{\theta_i}}{e^{\theta_i} + e^{\theta_{i'}}}) = 0$. Because $w_{ii'} = w_{i'i}$, we have $\nabla_i \text{ELL}_{\mathcal{M}}(\vec{\theta}) = \sum_{i' \neq i}(\bar{\kappa}_{ii'}w_{ii'} - (\kappa_{ii'}w_{ii'} + \bar{\kappa}_{i'i}w_{i'i})\frac{e^{\theta_i}}{e^{\theta_i} + e^{\theta_{i'}}}) = 0$. This means the ground truth is the solution to $\nabla \text{ELL}_{\mathcal{M}}(\vec{\theta}) = \vec{0}$. As $\mathcal{W}$ is connected and symmetric, it is strongly connected.

Thus CLL_{PL} is strictly concave, which means the ground truth is the only solution. Further by Theorem 5, RBCML$(\mathcal{G}, \mathcal{W})$ is consistent. ■

Theorem 9 Let π be any symmetric distribution that satisfies the condition in Theorem 7. Then RBCML$(\mathcal{G}, \mathcal{W})$ is consistent for RUM(π) if and only if $\mathcal{G}$ is uniform and $\mathcal{W}$ is connected and symmetric.

Proof: The “only if" direction: 2(c) part of Lemma 11 states that RBCML$(\mathcal{G}, \mathcal{W})$ is consistent with uniform $\mathcal{W}$, which implies $\mathcal{G}$ must be uniform by Theorem 7. Then 1(c) of Lemma 11 implies that RBCML$(\mathcal{G}, \mathcal{W})$ is consistent for any connected and symmetric $\mathcal{W}$.

The “if" direction: Since $\mathcal{G}$ is uniform breaking, we have $\sum_{i' \neq i}(\frac{\bar{\kappa}_{ii'}}{p_{ii'}(\vec{\theta})}\frac{\partial p_{ii'}(\vec{\theta})}{\partial \theta_i} + \frac{\bar{\kappa}_{i'i}}{p_{i'i}(\vec{\theta})}\frac{\partial p_{i'i}(\vec{\theta})}{\partial \theta_i}) = 0$ Because $w_{ii'} = w_{i'i}$, we have

$$\nabla_i \text{ELL}_{\mathcal{M}}(\vec{\theta}) = \sum_{i' \neq i}(\frac{\bar{\kappa}_{ii'} w_{ii'}}{p_{ii'}(\vec{\theta})}\frac{\partial p_{ii'}(\vec{\theta})}{\partial \theta_i} + \frac{\bar{\kappa}_{i'i} w_{i'i}}{p_{i'i}(\vec{\theta})}\frac{\partial p_{i'i}(\vec{\theta})}{\partial \theta_i}) = 0$$

holds for all i. This means the ground truth is the solution to $\nabla \text{ELL}_{\mathcal{M}}(\vec{\theta}) = \vec{0}$. As $\mathcal{W}$ is connected and symmetric, it is strongly connected. Thus $\text{CLL}_{\mathcal{M}}$ is strictly concave, which means the ground truth is the only solution. Further by Theorem 5, RBCML$(\mathcal{G}, \mathcal{W})$ is consistent. ■