Supplement:
Gauged Mini-Bucket Elimination for Approximate Inference

A  Example of gauge transformations

$$f_a(x_1, x_2, x_3) = \begin{bmatrix} 2432 & 832 & 4864 & 384 \\ 4672 & 640 & 5120 & 4160 \end{bmatrix}$$

$$f_a(x_1, x_2, x_3; G_a) = \begin{bmatrix} 2837 & 1559 & 3631 & 2005 \\ 1591 & 2077 & 4261 & 2077 \end{bmatrix}$$

$$f_b(x_1, x_2, x_3) = \begin{bmatrix} 2142 & 1434 & 966 & 1490 \\ 4634 & 4558 & 1490 & 758 \end{bmatrix}$$

$$f_b(x_1, x_2, x_3; G_b) = \begin{bmatrix} 1088 & 128 & 4928 & 4608 \\ 768 & 1856 & 448 & 1664 \end{bmatrix}$$

$$f_b(x_1, x_4, x_5) = \begin{bmatrix} 1216 & 5440 & 768 & 1856 \\ 5568 & 896 & 640 & 512 \end{bmatrix}$$

$$f_b(x_1, x_4, x_5; G_b) = \begin{bmatrix} 3960 & 8808 & -1608 & -2520 \\ -328 & -3288 & 6520 & 8296 \end{bmatrix}$$

$$f_c(x_2, x_4, x_6) = \begin{bmatrix} 1088 & 128 & 4928 & 4608 \\ 768 & 1856 & 448 & 1664 \end{bmatrix}$$

$$f_c(x_2, x_4, x_6; G_c) = \begin{bmatrix} 2408 & 9160 & 10760 & 9192 \\ 14536 & 4558 & 1490 & 758 \end{bmatrix}$$

$$G_{1a}, G_{2a}, G_{3a}, G_{4b}, G_{5b}, G_{6c} = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

$$G_{1b}, G_{2c}, G_{3d}, G_{4c}, G_{5d}, G_{6d} = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}$$

Figure 5: Example of gauge transformations on the complete graph (with respect to factors) of size 4. Arrays follow row-column major indexing, e.g., $f_a(1, 1, 2) = 4864$ and $f_a(1, 2, 1) = 832$.

B  Proof of Theorem

We prove reparameterization with respect to $\theta = \{\theta_{v\alpha}(x_v) = 0, \forall (v, \alpha) \in E, x_v\}$ is optimal at GM with symmetric factors in the following optimization:

$$\min_{\theta} \sum_{x_1} \cdots \sum_{x_1} \prod_{\alpha \in F} f_{a}(\bar{x}_\alpha; \theta_\alpha),$$

subject to $\prod_{\alpha \in N(v)} \exp(\theta_{v\alpha}(x_v)) = 1 \ \forall v \in X, x_v.$

The optimization is convex, and assuming $\theta_{v\beta} + \theta_{v\alpha} = 0$ from the constraint, $\partial \log Z_{WMBE} / \partial \theta_{v\alpha} = 0$ implies optimality of the solution. To this end, the derivative is expressed as:

$$\frac{\partial \log Z_{WMBE}}{\partial \theta_\alpha(x_{v\alpha})} = \sum_{x_\alpha \setminus v} q(\bar{x}_\alpha) - \sum_{x_\beta \setminus v} q(\bar{x}_\beta).$$

When factors are symmetric, it immediately follows that

$$\sum_{x_\alpha \setminus v} q(\bar{x}_\alpha) = \sum_{x_\beta \setminus v} q(\bar{x}_\beta) = 0.5,$$

since $q$ is expressed via weighted absolute sum and normalization operation of factors, which both preserve symmetry. Hence marginals are also symmetric, implying uniform distribution. Hence the optimality condition is satisfied.