A Supplementary Material

A.1 Relaxation on Local Polytope

The relaxation of (1) over the local polytope is given by:

$$\begin{align*}
\min_\mu \ & \sum_{u \in V} \sum_{i \in L} \mu_u(i) c(u, i) + \sum_{e = (u, v)} \sum_{i,j} \mu_e(ij) \theta_{(u,v)}(i,j) \\
\text{s.t.} \ & \sum_i \mu_u(i) = 1, \quad \forall i \in L. \\
& \sum_j \mu_e(ij) = \mu_u(i), \quad \forall e = (u, v) \in E, i \in L. \\
& \sum_i \mu_e(ij) = \mu_v(j), \quad \forall e = (u, v) \in E, j \in L. \\
& \mu_u(i) \geq 0, \quad \forall u \in V, i \in L. \\
& \mu_v(j) \geq 0, \quad \forall e \in E, i, j \in L.
\end{align*}$$

For a Ferromagnetic Potts Model, the objective becomes:

$$\begin{align*}
\min_\mu \ & \sum_{u \in V} \sum_{i \in L} \mu_u(i) c(u, i) + \sum_{e = (u, v)} \sum_{i,j} \mu_e(ij) \mathbb{1}(i \neq j) \\
\text{s.t.} \ & \sum_i \mu_u(i) = 1, \quad \forall i \in L.
\end{align*}$$

Fix the values $\mu_u(i)$. We want to minimize

$$\sum_{e = (u, v)} w(u, v) \sum_{i,j} \mu_e(ij) \mathbb{1}(i \neq j)$$

subject to the constraints

$$\begin{align*}
& \sum_j \mu_e(ij) = \mu_u(i), \quad \forall e = (u, v) \in E, i \in L. \\
& \sum_i \mu_e(ij) = \mu_v(j), \quad \forall e = (u, v) \in E, j \in L. \\
& \mu_u(i) \geq 0, \quad \forall u \in V, i \in L. \\
& \mu_v(j) \geq 0, \quad \forall e \in E, i, j \in L.
\end{align*}$$

Because $w(u, v) \geq 0$ and $\mu_e(ij) \geq 0$, we want to put as much mass on $\mu_e(ii)$ as possible without violating a constraint, since those terms do not appear in the objective. To that end, we set $\mu_e(ii) = \min(\mu_u(i), \mu_v(i))$. Then using the first constraint, the objective becomes:

$$\begin{align*}
& \sum_{e = (u, v)} w(u, v) \sum_i \mu_u(i) - \min(\mu_u(i), \mu_v(i)) \\
= & \sum_{e = (u, v)} w(u, v) \left( 1 - \frac{1}{2} \sum_i \mu_u(i) + \mu_v(i) \right) \\
& + \sum \left[ \mu_u(i) - \mu_v(i) \right] \\
= & \sum_{e = (u, v)} w(u, v) \sum_i |\mu_u(i) - \mu_v(i)| \\
= & \sum_{e = (u, v)} w(u, v) \frac{\mu_u - \mu_v}{2}
\end{align*}$$

where we use multiple times that $\sum_i \mu_u(i) = 1$. The LP objective is thus:

$$\begin{align*}
\min_\mu \ & \sum_{u \in V} \sum_{i \in L} \mu_u(i) c(u, i) + \sum_{e = (u, v)} w(u, v) \frac{|\mu_u - \mu_v|}{2} \\
\text{s.t.} \ & \sum_i \mu_u(i) = 1, \quad \forall i \in L.
\end{align*}$$

Identifying $\mu_u$ with $\tilde{u}$ and $\mu_v$ with $\tilde{v}$, we obtain the LP (3).

A.2 Proof of Lemma 1

Proof. This argument is similar to the one in Angelidakis et al. (2017). First, we verify the last two conditions in Lemma 1. Let $\alpha = \frac{3}{2} = \frac{3}{2}$ and $\beta = k \theta = \frac{6}{5}$. The algorithm clearly returns a feasible solution (i.e., a valid labeling). Consider any two vertices $u$ and $v$, and let $\Delta = d(u, v)$. There are two cases: $j(u) = j(v)$ and $j(u) \neq j(v)$. In the first case, let $j = j(u) = j(v)$. We have $P(u) \neq P(v)$ exactly when $r \in \{\min(\tilde{u}_i, \tilde{v}_i), \max(\tilde{u}_i, \tilde{v}_i)\}$ and $i \neq j$. $r$ is uniformly distributed in $(0, \theta)$, so the probability of this occurring is

$$\begin{align*}
\mathbb{P}[P(u) \neq P(v)] = \frac{1}{k} \sum_{i \neq j} \frac{|\tilde{u}_i - \tilde{v}_i|}{\theta} \leq \frac{2}{k \theta} d(u, v) = \alpha \Delta.
\end{align*}$$

Note that we used $u_i \leq \varepsilon < \theta$ for $i \neq j$ and for all $u$. Now consider the case where $j(u) \neq j(v)$. Here $d(u, v) \geq d(e_{j(u)}), \ v_{j(v)}) - d(e_{j(u)}) - d(v, e_{j(v)})$ by the triangle inequality ($e_i$ is the $i$th standard basis vector in $\mathbb{R}^k$). So $d(u, v) \geq 1 - 2\varepsilon \geq 1 - 2/30$ for $k \geq 3$. So $d(u, v) \geq 14/15$, and $\alpha = 5/3$ so $\alpha \Delta > 1$ and the bound trivially applies.

Next we verify the “co-approximation” condition. First consider the case where $j(u) = j(v) = j$. Then $d(u, v) \leq d(u, e_j) + d(e_j, v) \leq 2 \varepsilon \leq 1/15$. As we showed, $\mathbb{P}[P(u) \neq P(v)] \leq \alpha \Delta$. So $\mathbb{P}[P(u) = P(v)] \geq 1 - \alpha \Delta \geq \beta^{-1}(1 - \Delta)$, where the last inequality is because $\frac{1 - \beta^{-1}}{\alpha - \beta^{-1}} = \frac{1/6}{5/3 - 5/6} = \frac{1}{5} \geq \Delta$. Now assume $j(u) \neq j(v)$. Note that if $\tilde{u}_i \geq r$ and $\tilde{v}_i \geq r$, $u$ and $v$ are both added to $P$. So

$$\begin{align*}
\mathbb{P}[P(u) = P(v)] \geq \mathbb{P}[u_i \geq r, v_i \geq r] \\
= \frac{1}{k} \sum_{i=1}^k \min(\tilde{u}_i, \tilde{v}_i).
\end{align*}$$

Here we used that for all $i$, $\min(\tilde{u}_i, \tilde{v}_i) \leq \varepsilon < \theta$ since $j(u) \neq j(v)$. Then

$$\begin{align*}
\mathbb{P}[P(u) = P(v)] \geq \frac{1}{k} \sum_{i=1}^k \tilde{u}_i + \tilde{v}_i - |\tilde{u}_i - \tilde{v}_i| \\
= \frac{1}{k \theta} (1 - d(u, v)) = \beta^{-1}(1 - d(u, v)).
\end{align*}$$

The approximation conditions hold.
Finally, we check the first two conditions of Lemma 1. First consider $\mathbb{P}[P(u) = i, i \neq j(u)]$. This can only occur when $i$ is selected and $u$ is assigned to $P_i$. So
\[
\mathbb{P}[P(u) = i, i \neq j(u)] = \frac{1}{k} \mathbb{P}[\tilde{u}_i \geq r] = \frac{1}{k} \mathbb{P}[\tilde{u}_i \geq \frac{5}{6} \tilde{u}_i].
\]
Now we compute $\mathbb{P}[P(u) \neq j(u)]$. A vertex $u$ clearly can only be assigned a label $i \neq j(u)$ if such an $i$ is selected and $u$ is assigned to it; namely,
\[
\mathbb{P}[P(u) \neq j(u)] = \frac{1}{k} \sum_{i : i \neq j(u)} \frac{\tilde{u}_i}{\theta} = \frac{1}{k \theta} (1 - \tilde{u}_{j(u)}) = \frac{5}{6} (1 - \tilde{u}_{j(u)}).
\]
This concludes the proof.

A.3 Full Proof of Theorem 1

Here we reproduce the proof of Theorem 1 in more detail.

Theorem. On a (2,1)-stable instance of Uniform Metric Labeling with optimal integer solution $g$, the LP relaxation (3) is tight.

Proof. Assume for a contradiction that the optimal LP solution $\bar{u}^L_P$ of (3) is fractional. To construct a stability-violating labeling, we will run Algorithm 2 on a fractional labeling $\bar{u}$ constructed from $\bar{u}^L_P$ and the optimal integer solution $g$. We then use Lemma 1 to show that in expectation, the output of $\mathcal{R}(\{\bar{u}\})$ must be better than the optimal integer solution in a particular (2, 1)-perturbation, which contradicts (2, 1)-stability.

Let $\{\bar{u}^\delta\}$ be the solution to (3) corresponding to $g$, and define the following $\varepsilon$-close solution $\bar{u}$: for every $u$ and every $i$, set $\bar{u}_i = (1 - \varepsilon)\bar{u}^\delta_i + \varepsilon \bar{u}^L_P$. Note that $\{\bar{u}\}$ is fractional and $j(u) = g(u)$ for all $u$.

Recall that $E_g$ is the set of edges cut by the optimal solution $g$. Define the following (2, 1)-perturbation $w'$ of the weights $w$: $w'(u, v) = \begin{cases} w(u, v) & (u, v) \in E_g \\ \frac{1}{2} w(u, v) & (u, v) \in E \setminus E_g. \end{cases}$

We refer to the objective with modified weights $w'$ as $Q'$ (that is, $Q'$ is the objective in the instance with weights $w'$ and costs $c$).

Now let $h = \mathcal{R}(\{\bar{u}\})$. To compare $g$ and $h$, we will compute $\mathbb{E}[Q'(g) - Q'(h)]$, where the expectation is over the randomness of the rounding algorithm. By definition,
\[
\mathbb{E}[Q'(g) - Q'(h)] = \mathbb{E}[Q'(g) - Q'(h)|h = g] \Pr(h = g) + \mathbb{E}[Q'(g) - Q'(h)|h \neq g] \Pr(h \neq g).
\]
The first term of the sum above is clearly zero. Further, as $\{\bar{u}\}$ is fractional, the guarantees in Lemma 1 imply that $\Pr(h \neq g) > 0$. By (2,1)-stability of the instance, any labeling $h \neq g$ must satisfy $Q'(h) > Q'(g)$. So stability and fractionality of the LP imply $\mathbb{E}[Q'(g) - Q'(h)] < 0$.

If we compute $\mathbb{E}[Q'(g) - Q'(h)]$ and simplify using Lemma 1 and the definition of $w'$ (see the appendix for a full derivation), we obtain:
\[
Q'(g) - Q'(h) = \sum_{u \in V} c(u, g(u)) + \sum_{(u, v) \in E \setminus E_h} w'(u, v) - \sum_{u \in V} c(u, h(u)) - \sum_{(u, v) \in E_h \setminus E_g} w'(u, v).
\]
Taking the expectation, we obtain:
\[
\mathbb{E}[Q'(g) - Q'(h)] = \sum_{u \in V} c(u, g(u)) \Pr(h(u) \neq g(u)) + \sum_{(u, v) \in E_h} w'(u, v) \Pr((u, v) \text{ not cut}) - \sum_{(u, v) \in E_h} \sum_{i \neq j(u)} c(u, i) \Pr(h(u) = i) - \sum_{(u, v) \in E_h \setminus E_g} w'(u, v) \Pr((u, v) \text{ cut}).
\]
Applying Lemma 1 with $j(u) = g(u)$,
\[
\mathbb{E}[Q'(g) - Q'(h)] \geq \frac{5}{6} \left( \sum_{u \in V} c(u, g(u))(1 - \bar{u}_g(u)) + \sum_{(u, v) \in E_h} w'(u, v)(1 - d(u, v)) - \sum_{(u, v) \in E_h} \sum_{i \neq j(u)} c(u, i) \bar{u}_i - \sum_{(u, v) \in E_h \setminus E_g} 2 w'(u, v) d(u, v) \right).
\]
Using the definition of $w'$,
\[
\mathbb{E}[Q'(g) - Q'(h)] \geq \frac{5}{6} \left( \sum_{u \in V} c(u, g(u)) + \sum_{(u, v) \in E_h} w'(u, v) - \sum_{u \in V} \sum_{i \in L} c(u, i) \bar{u}_i - \sum_{(u, v) \in E} w(u, v) d(u, v) \right).
\]
The first two terms are simply $Q(g)$, and the last two are the objective $Q(\{\bar{u}\})$ of the LP solution $\bar{u}$. Since $\bar{u} = (1 - \varepsilon)\bar{u}^\delta + \varepsilon \bar{u}^L_P$ and $Q(\{\bar{u}^L_P\}) \leq Q(\{\bar{u}^\delta\})$, the convexity of the LP objective implies $\mathbb{E}[Q(\{\bar{u}^\delta\})] \leq Q(\{\bar{u}\}) = Q(g)$. So $\mathbb{E}[Q'(g) - Q'(h)] \geq 0$. But stability of the instance and fractionality of the LP solution implied $\mathbb{E}[Q'(g) - Q'(h)] < 0$. ∀
A.4 Generating Counterexamples

The following procedure can be used to find \((\beta, \gamma)\)-stable instances.

1. Given a fixed number of nodes \(n\) and labels \(k\), randomly generate a graph \(G\) as follows:
   
   (a) Connect any two nodes \((u, v)\) with an edge with probability \(\text{connectProb}\).
   
   (b) When connecting two nodes, choose the edge weight \(w(u, v)\) uniformly at random from \(Z \cap [0, \text{weightMax}]\).

2. For each node \(u\), choose an index \(i\) uniformly at random from \(\{1 \ldots k\}\). Draw \(c(u, i)\) uniformly at random from \(Z \cap [0, \text{costMax}]\). Set \(c(u, j) = 0\) for \(j \neq i\).

3. Find the optimal solution \(g\) to the instance \((G, w, c, L)\).

4. Let \(E_g\) be the set of edges cut by \(g\), and consider the following adversarial perturbation \(w'\) of \(w\):

   \[
   w'(u, v) = \begin{cases} 
   \frac{1}{2}w(u, v) & (u, v) \in E \setminus E_g \\
   \gamma w(u, v) & (u, v) \in E_g
   \end{cases}
   \]

   Let \(Q'\) be the objective with these modified weights.

5. Enumerate the \(k^n - 1\) possible labelings not equal to \(g\). If any of them have \(Q'(h) \leq Q'(g)\), return to step 1. Otherwise, print \(V, E, w, c\).

Following this procedure, we can also enforce additional properties of the instance in step 5 before printing it out. For instance, we can enforce that the LP must be fractional on the instance, or that \(\alpha\)-expansion must not find the optimal solution. If these additional conditions fail to hold, we return to step 1.

The examples in Section 6 were found with \(\text{connectProb} = 0.5\), \(\text{weightMax} = 4\), \(\text{costMax} = 20\), and then modified for simplicity. Steps 3-5 were repeated for each modification to ensure the resulting instances satisfied the correct stability conditions. In Section 6, \(\beta = 1\) and \(\gamma = 2\); in Section 6, \(\beta = 2\) and \(\gamma = 1\).

The following lemma proves that steps 3-5 are sufficient to verify stability.

**Lemma A.1.** Let \(w^*\) be an arbitrary \((\beta, \gamma)\)-perturbation of the weights \(w\), and let \(w'\) be the adversarial perturbation for the optimal solution \(g\). Then for any labeling \(h\), \(Q^*(h) \leq Q^*(g)\) implies \(Q'(h) \leq Q'(g)\).

In other words, if a labeling \(h\) violates stability in any perturbation, it violates stability in the adversarial perturbation \(w'\).

**Proof.** We show that \(Q'(g) - Q'(h) \leq Q'(g) - Q'(h)\).

Let \(V_\Delta = \{u \in V \mid g(u) \neq h(u)\}\). Recall that \(E_g\) and \(E_h\) are the sets of edges cut by \(g\) and \(h\), respectively. We compute

\[
Q'(g) - Q'(h) = \sum_{u \in V_\Delta} c(u, g(u)) + \sum_{(u, v) \in E_g \setminus E_h} w'(u, v) - \sum_{u \in V_\Delta} c(u, h(u)) - \sum_{(u, v) \in E_h \setminus E_g} w'(u, v).
\]

Using the definition of \(w'\),

\[
Q'(g) - Q'(h) = \sum_{u \in V_\Delta} c(u, g(u)) + \sum_{(u, v) \in E_g \setminus E_h} \gamma w(u, v) - \sum_{u \in V_\Delta} c(u, h(u)) - \sum_{(u, v) \in E_h \setminus E_g} \beta w(u, v).
\]

Since \(w^*\) is a valid \((\beta, \gamma)\)-perturbation, \(\frac{1}{\beta} w(u, v) \leq w^*(u, v) \leq \gamma w(u, v)\). Then since all the \(c\)'s and \(w\)'s are nonnegative,

\[
Q'(g) - Q'(h) \geq \sum_{u \in V_\Delta} c(u, g(u)) + \sum_{(u, v) \in E_g \setminus E_h} w^*(u, v) - \sum_{u \in V_\Delta} c(u, h(u)) - \sum_{(u, v) \in E_h \setminus E_g} w^*(u, v) = Q^*(g) - Q^*(h).
\]

\(\square\)