A Proof of Theorem 1

First, let’s review the lower bound of the linear bandit setting. The linear bandit setting is almost identical to ours except that the $\theta_t$’s do not vary across rounds, and are equal to the same (unknown) $\theta$, i.e., $\forall t \in [T] \; \theta_t = \theta$.

Lemma 4 ([20]). For any $T_0 \geq \sqrt{d}/2$ and let $D = \{ x \in \mathbb{R}^d : \| x \| \leq 1 \}$, then there exists a $\theta \in \{ \pm \sqrt{d}/4T_0 \}^d$, such that the worst case regret of any algorithm for linear bandits with unknown parameter $\theta$ is $\Omega(d\sqrt{T_0})$.

Going back to the non-stationary environment, suppose nature divides the whole time horizon into $[T/H]$ blocks of equal length $H$ rounds (the last block can possibly have less than $H$ rounds), and each block is a decoupled linear bandit instance so that the knowledge of previous blocks cannot help the decision within the current block.

Following Lemma 4, we restrict the sequence of $\theta_t$’s are drawn from the set $\{ \pm \sqrt{d}/4H \}^d$. Moreover, $\theta_t$’s remain fixed within a block, and can vary across different blocks, i.e.,

$$\forall i \in \left[\frac{T}{H}\right] \forall t_1, t_2 \in [(i-1)H+1, i \cdot H \land T] \; \theta_{t_1} = \theta_{t_2}. \quad (27)$$

We argue that even if the learner knows this additional information, it still incur a regret $\Omega(d^{2/3} B_T^{1/3} T^{2/3})$. Note that different blocks are completely decoupled, and information is thus not passed across blocks. Therefore, the regret of each block is $\Omega\left(d\sqrt{H}\right)$, and the total regret is at least

$$\left(\left\lfloor \frac{T}{H} \right\rfloor - 1 \right) \Omega\left(d\sqrt{H}\right) = \Omega\left(dTH^{-3/2}\right). \quad (28)$$

Intuitively, if $H$, the number of length of each block, is smaller, the worst case regret lower bound becomes larger. But too small a block length can result in a violation of the variation budget. So we work on the total variation of $\theta_t$’s to see how small can $H$ be. The total variation of the $\theta_t$’s can be seen as the total variation across consecutive blocks as $\theta_t$ remains unchanged within a single block. Observe that for any pair of $\theta, \theta' \in \{ \pm \sqrt{d}/4H \}^d$, the $\ell_2$ difference between $\theta$ and $\theta'$ is upper bounded as

$$\sum_{i=1}^{d} \frac{4d}{4H} = \frac{d}{\sqrt{H}} \quad (29)$$

and there are at most $\lfloor T/H \rfloor$ changes across the whole time horizon, the total variation is at most

$$B = \frac{T}{H} \cdot \frac{d}{\sqrt{H}} = dTH^{-3/2}. \quad (30)$$

By definition, we require that $B \leq B_T$, and this indicates that

$$H \geq (dT)^{3/2} B_T^{-3/2}. \quad (31)$$

Taking $H = \left\lfloor (dT)^{3/2} B_T^{-3/2} \right\rfloor$, the worst case regret is

$$\Omega\left(dT \left((dT)^{3/2} B_T^{-3/2}\right)^{-3/2}\right) = \Omega\left(d^{3/2} B_T^{-1/2} T^{3/2}\right). \quad (32)$$

B Proof of Lemma 1

In the proof, we denote $B(1)$ as the unit Euclidean ball, and $\lambda_{\max}(M)$ as the maximum eigenvalue of a square matrix $M$. By folklore, we know that $\lambda_{\max}(M) = \max_{z \in B(1)} z^\top M z$. In addition, recall the definition that $V_{t-1}^{-1} = M + \sum_{s=1}^{t-1} X_s X_s^\top$. We prove the Lemma as follows:

$$\left\| V_{t-1}^{-1} \sum_{s=1 \land (t-w)}^{t-1} X_s X_s^\top (\theta_s - \theta_t) \right\| = \left\| V_{t-1}^{-1} \sum_{s=1 \land (t-w)}^{t-1} X_s X_s^\top \sum_{p=s}^{t-1} (\theta_p - \theta_{p+1}) \right\|$$
\[
\begin{align*}
&= \left\| V_{t-1}^{-1} \sum_{p=1}^{t-1} \sum_{s=1}^{\sqrt{(t-w)}} X_s X_s^T (\theta_p - \theta_{p+1}) \right\| \\
&\leq \sum_{p=1}^{t-1} \left\| V_{t-1}^{-1} \left( \sum_{s=1}^{\sqrt{(t-w)}} X_s X_s^T \right) (\theta_p - \theta_{p+1}) \right\| \\
&\leq \sum_{p=1}^{t-1} \lambda_{\max} \left( V_{t-1}^{-1} \left( \sum_{s=1}^{\sqrt{(t-w)}} X_s X_s^T \right) \right) \| \theta_p - \theta_{p+1} \| \\
&\leq \sum_{p=1}^{t-1} \| \theta_p - \theta_{p+1} \|. 
\end{align*}
\] (33)

Equality (33) is by the observation that both sides of the equation is summing over the terms \( X_s X_s^T (\theta_p - \theta_{p+1}) \) with indexes \((s,p)\) ranging over \( \{(s,p) : 1 \leq s \leq p \leq t - 1\} \). Inequality (34) is by the triangle inequality.

To proceed with the remaining steps, we argue that, for any index subset \( S \subseteq \{ 1 \leq (t-w), \ldots, t-1 \} \), the matrix \( V_{t-1}^{-1} \left( \sum_{s \in S} X_s X_s^T \right) \) is positive semi-definite (PSD). Now, let’s denote \( A = \sum_{s \in S} X_s X_s^T \). Evidently, matrix \( A \) is PSD, while matrix \( V_{t-1}^{-1} \) is positive definite, and both matrices \( A, V_{t-1}^{-1} \) are symmetric. Matrices \( V_{t-1}^{-1} A \) and \( V_{t-1}^{-1/2} A V_{t-1}^{-1/2} \) have the same sets of eigenvalues, since these matrices have the same characteristics polynomial (with the variable denoted as \( \eta \) below):

\[
\det(\eta I - V_{t-1}^{-1} A) = \det(V_{t-1}^{-1/2} \det(\eta V_{t-1}^{-1/2} - V_{t-1}^{-1/2} A) \\
= \det(\eta V_{t-1}^{-1/2} - V_{t-1}^{-1/2} A) \det(\eta I) = \det(\eta I - V_{t-1}^{-1/2} AV_{t-1}^{-1/2}).
\]

Evidently, \( V_{t-1}^{-1/2} AV_{t-1}^{-1/2} \) is PSD, since for any \( y \in \mathbb{R}^d \) we clearly have \( y^T V_{t-1}^{-1/2} AV_{t-1}^{-1/2} y = \| A^{1/2} V_{t-1}^{-1/2} y \|^2 \geq 0 \) (Matrices \( A^{1/2}, V_{t-1}^{-1/2} \) are symmetric). Altogether, we have shown that \( V_{t-1}^{-1} \left( \sum_{s=1}^{\sqrt{(t-w)}} X_s X_s^T \right) \) is PSD.

Inequality (35) is by the fact that, for any matrix \( M \in \mathbb{R}^{d \times d} \) with \( \lambda_{\max}(M) \geq 0 \) and any vector \( y \in \mathbb{R}^d \), we have \( \| M y \| \leq \lambda_{\max}(M) \| y \| \). Without loss of generality, assume \( y \neq 0 \). Now, it is evident that

\[
\| M y \| = \left\| M \frac{y}{\| y \|} \right\| \cdot \| y \| \leq \max_{z \in B(1)} \| M z \| \cdot \| y \| = \| \lambda_{\max}(M) \cdot y \| = \lambda_{\max}(M) \| y \|.
\]

Applying the above claim with \( M = V_{t-1}^{-1} \left( \sum_{s=1}^{\sqrt{(t-w)}} X_s X_s^T \right) \), which is PSD, and \( y = \theta_p - \theta_{p+1} \) demonstrates inequality (35).

Finally, inequality (36) is by the inequality \( \lambda_{\max} \left( V_{t-1}^{-1} \left( \sum_{s=1}^{\sqrt{(t-w)}} X_s X_s^T \right) \right) \leq 1 \). Indeed,

\[
\begin{align*}
\lambda_{\max} \left( V_{t-1}^{-1} \left( \sum_{s=1}^{\sqrt{(t-w)}} X_s X_s^T \right) \right) &= \max_{z \in B(1)} z^T V_{t-1}^{-1} \left( \sum_{s=1}^{\sqrt{(t-w)}} X_s X_s^T \right) z \\
&\leq \max_{z \in B(1)} \left\{ z^T V_{t-1}^{-1} \left( \sum_{s=1}^{\sqrt{(t-w)}} X_s X_s^T \right) z + z^T V_{t-1}^{-1} \left( \sum_{s=p+1}^{t-1} X_s X_s^T \right) z + \lambda z^T V_{t-1}^{-1} z \right\} \\
&= \max_{z \in B(1)} z^T V_{t-1}^{-1} V_{t-1} z = 1,
\end{align*}
\] (37)

where inequality (37) is by the property that both matrices \( V_{t-1}^{-1} \left( \sum_{s=p+1}^{t-1} X_s X_s^T \right), V_{t-1}^{-1} \) are PSD, as we establish previously. Altogether, the Lemma is proved.
C Proof of Theorem 2

Fixed any $\delta \in [0, 1]$, we have that for any $t \in [T]$ and any $x \in D_t$,

$$\left| x^\top (\hat{\theta}_t - \theta_t) \right| = \left| x^\top \left( V_{t-1}^{-1} \sum_{s=1 \vee (t-w)}^{t-1} X_s X_s^\top (\theta_s - \theta_t) \right) + x^\top V_{t-1}^{-1} \left( \sum_{s=1 \vee (t-w)}^{t-1} \eta_s X_s - \lambda \theta_t \right) \right|$$

$$\leq \left| x^\top \left( V_{t-1}^{-1} \sum_{s=1 \vee (t-w)}^{t-1} X_s X_s^\top (\theta_s - \theta_t) \right) \right| + \left| x^\top V_{t-1}^{-1} \left( \sum_{s=1 \vee (t-w)}^{t-1} \eta_s X_s - \lambda \theta_t \right) \right|$$

$$\leq \left\| x \right\| \cdot \left| \sum_{s=1 \vee (t-w)}^{t-1} X_s X_s^\top (\theta_s - \theta_t) \right| + \left\| x \right\| \left| \sum_{s=1 \vee (t-w)}^{t-1} \eta_s X_s - \lambda \theta_t \right|$$

$$\leq L \sum_{s=1 \vee (t-w)}^{t-1} \left\| \theta_s - \theta_{s+1} \right\| + \left\| x \right\| \left| \sum_{s=1 \vee (t-w)}^{t-1} \eta_s X_s - \lambda \theta_t \right|$$

$$\leq \sum_{s=1 \vee (t-w)}^{t-1} \left\| \theta_s - \theta_{s+1} \right\| + \left\| x \right\| \left[ R \left\| \left( 1 + w L^2 / \lambda \right) + \sqrt{S} \right\| \right],$$

where inequality (38) uses triangular inequality, inequality (39) follows from Cauchy-Schwarz inequality, and inequality (40) are consequences of Lemmas 1, 2.

D Proof of Theorem 3

In the proof, we choose $\lambda$ so that $\beta \geq 1$, for example by choosing $\lambda \geq 1/S^2$. By virtue of UCB, the regret in any round $t \in [T]$ is

$$\left\langle x^*_t - X_t, \theta_t \right\rangle \leq L \sum_{s=1 \vee (t-w)}^{t-1} \left\| \theta_s - \theta_{s+1} \right\| + \left\langle X_t, \hat{\theta}_t \right\rangle + \beta \left\| X_t \right\| \left| \sum_{s=1 \vee (t-w)}^{t-1} \eta_s X_s - \lambda \theta_t \right|$$

$$\leq 2L \sum_{s=1 \vee (t-w)}^{t-1} \left\| \theta_s - \theta_{s+1} \right\| + 2\beta \left\| X_t \right\| \left| \sum_{s=1 \vee (t-w)}^{t-1} \eta_s X_s - \lambda \theta_t \right|.$$ 

Inequality (41) is by an application of our SW-UCB algorithm established in equation (10). Inequality (42) is by an application of inequality (40), which bounds the difference $\left| \left\langle X_t, \hat{\theta}_t - \theta_t \right\rangle \right|$ from above. By the evident fact that $\left\langle X_t, \hat{\theta}_t - \theta_t \right\rangle \leq 2$, we have

$$\left\langle x^*_t - X_t, \theta_t \right\rangle \leq 2L \sum_{s=1 \vee (t-w)}^{t-1} \left\| \theta_s - \theta_{s+1} \right\| + 2\beta \left( \left\| X_t \right\| \left| \sum_{s=1 \vee (t-w)}^{t-1} \eta_s X_s - \lambda \theta_t \right| \right).$$

Summing equation (43) over $1 \leq t \leq T$, the regret of the SW-UCB algorithm is upper bounded as

$$E \left[ \text{Regret}_T \left( \text{SW-UCB algorithm} \right) \right] = \sum_{t=1}^{T} \left\langle x^*_t - X_t, \theta_t \right\rangle$$

$$\leq 2L \left[ \sum_{t=1}^{T} \sum_{s=1 \vee (t-w)}^{t-1} \left\| \theta_s - \theta_{s+1} \right\| \right] + 2\beta \sum_{t=1}^{T} \left( \left\| X_t \right\| \left| \sum_{s=1 \vee (t-w)}^{t-1} \eta_s X_s - \lambda \theta_t \right| \right)$$

$$= 2L \left[ \sum_{t=1}^{T} \sum_{s=s+1}^{t \wedge T} \left\| \theta_s - \theta_{s+1} \right\| \right] + 2\beta \sum_{t=1}^{T} \left( \left\| X_t \right\| \left| \sum_{s=s+1}^{t \wedge T} \eta_s X_s - \lambda \theta_t \right| \right)$$

$$\leq 2LwB_t + 2\beta \sum_{t=1}^{T} \left( \left\| X_t \right\| \left| \sum_{s=s+1}^{t \wedge T} \eta_s X_s - \lambda \theta_t \right| \right).$$
What’s left is to upper bound the quantity $2\beta \sum_{t \in [T]} \left(1 \wedge \|X_t\|_{V_{t-1}}^2\right)$. Following the trick introduced by the authors of [1], we apply Cauchy-Schwarz inequality to the term $\sum_{t \in [T]} \left(1 \wedge \|X_t\|_{V_{t-1}}^2\right)$.

$$\sum_{t \in [T]} \left(1 \wedge \|X_t\|_{V_{t-1}}^2\right) \leq \sqrt{\sum_{t \in [T]} 1 \wedge \|X_t\|_{V_{t-1}}^2}.$$  \hfill (45)

By dividing the whole time horizon into consecutive pieces of length $w$, we have

$$\sqrt{\sum_{t \in [T]} 1 \wedge \|X_t\|_{V_{t-1}}^2} \leq \sqrt{\sum_{i=0}^{\lfloor T/w \rfloor - 1} \sum_{t=i \cdot w + 1}^{(i+1)w} 1 \wedge \|X_t\|_{V_{t-1}}^2}.$$  \hfill (46)

While a similar quantity has been analyzed by Lemma 11 of [1], we note that due to the fact that $V_t$’s are accumulated according to the sliding window principle, the key eq. (6) in Lemma 11’s proof breaks, and thus the analysis of [1] cannot be applied here. To this end, we state a technical lemma based on a novel use of the Sherman-Morrison formula.

**Lemma 5.** For any $i \leq \lfloor T/w \rfloor - 1$,

$$\sum_{t=i \cdot w + 1}^{(i+1)w} 1 \wedge \|X_t\|_{V_{t-1}}^2 \leq \sum_{t=i \cdot w + 1}^{(i+1)w} 1 \wedge \|X_t\|_{V_{t-1}}^2,$$

where

$$V_{t-1} = \sum_{s=i \cdot w + 1}^{t-1} X_sX_s^T + \lambda I.$$ \hfill (47)

**Proof.** Proof of Lemma 5. For a fixed $i \leq \lfloor T/w \rfloor - 1$,

$$\sum_{t=i \cdot w + 1}^{(i+1)w} 1 \wedge \|X_t\|_{V_{t-1}}^2 = \sum_{t=i \cdot w + 1}^{(i+1)w} 1 \wedge X_t^T V_{t-1}^{-1} X_t$$

$$= \sum_{t=i \cdot w + 1}^{(i+1)w} 1 \wedge X_t^T \left( \sum_{s=1 \vee (t-w)}^{t-1} X_sX_s^T + \lambda I \right)^{-1} X_t.$$ \hfill (48)

Note that $i \cdot w + 1 \geq 1$ and $i \cdot w + 1 \geq t - w \forall t \leq (i+1)w$, we have

$$i \cdot w + 1 \geq 1 \vee (t-w).$$ \hfill (49)

Consider any $d$-by-$d$ positive definite matrix $A$ and $d$-dimensional vector $y$, then by the Sherman-Morrison formula, the matrix

$$B = A^{-1} - (A + yy^T)^{-1} = A^{-1} - A^{-1} + A^{-1}yy^T A^{-1} = A^{-1}yy^T A^{-1} \frac{1}{1 + y^T A^{-1} y}$$

is positive semi-definite. Therefore, for a given $t$, we can iteratively apply this fact to obtain

$$X_t^T \left( \sum_{s=i \cdot w + 1}^{t-1} X_sX_s^T + \lambda I \right)^{-1} X_t$$

$$= X_t^T \left( \sum_{s=i \cdot w}^{t-1} X_sX_s^T + \lambda I \right)^{-1} X_t + X_t^T \left( \sum_{s=i \cdot w + 1}^{t-1} X_sX_s^T + \lambda I \right)^{-1} \left( \sum_{s=i \cdot w + 1}^{t-1} X_sX_s^T + \lambda I \right)^{-1} X_t$$

$$= X_t^T \left( \sum_{s=i \cdot w}^{t-1} X_sX_s^T + \lambda I \right)^{-1} X_t + X_t^T \left( \sum_{s=i \cdot w}^{t-1} X_sX_s^T + \lambda I \right)^{-1} X_t.$$

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Now putting these two parts to eq. (44), we have

\[ \sum_{s=1}^{t-1} X_s X_s^T + \lambda I \]  

which concludes the proof.

From Lemma 5 and eq. (46), we know that

\[ 2\beta \sum_{t \in [T]} (1 \wedge \|X_t\|_{V_{t-1}}^2) \leq 2\beta \sqrt{T} \cdot \sum_{i=0}^{[T/w]-1} \sum_{t=i \cdot w+1}^{(i+1), w} 1 \wedge \|X_t\|_{V_{t-1}}^2 \]

\[ \leq 2\beta \sqrt{T} \cdot \sum_{i=0}^{[T/w]-1} 2d \ln \left( \frac{d\lambda + wL^2}{d\lambda} \right) \]

\[ \leq 2\beta T \frac{2d}{w} \ln \left( \frac{d\lambda + wL^2}{d\lambda} \right). \]

Here, eq. (53) follows from Lemma 11 of [1].

Now putting these two parts to eq. (44), we have

\[ \mathbb{E} [\text{Regret}_T (SW-UCB algorithm)] \leq 2LwB_T + 2\beta \sqrt{T} \left( \frac{2d}{w} \ln \left( \frac{d\lambda + wL^2}{d\lambda} \right) + 2T \delta \right) \]

\[ = 2LwB_T + \frac{2T}{\sqrt{w}} \left( R \sqrt{d \ln \left( \frac{1 + wL^2/\lambda}{\delta} \right) + \sqrt{\lambda} S} \right) \left( 2d \ln \left( \frac{d\lambda + wL^2}{d\lambda} \right) + 2T \delta. \right) \]

Now if \( B_T \) is known, we can take \( w = O \left( (dT)^{2/3} B_T^{-2/3} \right) \) and \( \delta = 1/T, \) we have

\[ \mathbb{E} [\text{Regret}_T (SW-UCB algorithm)] = \tilde{O} \left( d^2 B_T^4 T^{2/3} \right); \]

while if \( B_T \) is not unknown taking \( w = O \left( (dT)^{2/3} \right) \) and \( \delta = 1/T, \) we have

\[ \mathbb{E} [\text{Regret}_T (SW-UCB algorithm)] = \tilde{O} \left( d^2 (B_T + 1)^{2/3} \right). \]

### E Proof of Lemma 3

For any block \( i, \) the absolute sum of rewards can be written as

\[ \sum_{t=(i-1)H+1}^{t \leq (i-1)H+1} \langle X_t, \theta_t \rangle + \eta_t \leq \sum_{t=(i-1)H+1}^{i \wedge T} \langle X_t, \theta_t \rangle + \sum_{t=(i-1)H+1}^{i \wedge T} \eta_t \leq H + \sum_{t=(i-1)H+1}^{i \wedge T} \eta_t, \]
where we have iteratively applied the triangular inequality as well as the fact that $|\langle X_t, \theta_i \rangle| \leq 1$ for all $t$.

Now by property of the $R$-sub-Gaussian [22], we have the absolute value of the noise term $\eta_t$ exceeds $2R\sqrt{\ln T}$ for a fixed $t$ with probability at most $1/T^2$, i.e.,

$$\Pr \left( \left| \sum_{t=(i-1)T+1}^{iT} \eta_t \right| \geq 2R \sqrt{\ln \frac{T}{\sqrt{H}}} \right) \leq \frac{2H}{T^2}. \tag{55}$$

Applying a simple union bound, we have

$$\Pr \left( \exists i \in \left[ \frac{T}{H} \right] : \left| \sum_{t=(i-1)T+1}^{iT} \eta_t \right| \geq 2R \sqrt{\ln \frac{T}{\sqrt{H}}} \right) \leq \sum_{i=1}^{\left\lceil \frac{T}{H} \right\rceil} \Pr \left( \left| \sum_{t=(i-1)T+1}^{iT} \eta_t \right| \geq 2R \sqrt{\ln \frac{T}{\sqrt{H}}} \right) \leq \frac{2}{T}. \tag{56}$$

Therefore, we have

$$\Pr \left( Q \geq H + 2R \sqrt{\ln \frac{T}{\sqrt{H}}} \right) \leq \Pr \left( \exists i \in \left[ \frac{T}{H} \right] : \left| \sum_{t=(i-1)T+1}^{iT} \eta_t \right| \geq 2R \sqrt{\ln \frac{T}{\sqrt{H}}} \right) \leq \frac{2}{T}. \tag{57}$$

The statement then follows.