A PROOFS OF LEMMAS AND FACTS

A.1 Proof of Lemma 4

The proof is very similar to the proof of Lemma 2 of Heckel et al. (2018). There are several cases of \( q_1 \) and \( b_2 \) to consider. We will show each by contradiction, starting with the assumption that the terminating condition is false and both \( \mathcal{E}_{\text{bad}}(q_1) \) and \( \mathcal{E}_{\text{bad}}(b_2) \) do not occur, all under the event \( \mathcal{E}_a \). Let \( \mathcal{E}_{\text{good}}(i) \) denote the complement of \( \mathcal{E}_{\text{bad}}(i) \). It also will be useful to define the quantity

\[
m_2 = \arg \max_{i \in \{(k+1),\ldots,(k+h)\}} \alpha_i
\]

such that \( b_2 = \arg \max_{i \in \{m_2, q_2\}} \alpha_i \).

i. When \( q_1 \leq k \) and \( b_2 > k + h \), we have by \( \mathcal{E}_{\text{good}}(q_1) \) that

\[
\hat{d}_{q_1} + \alpha_{q_1} < \hat{d}_{q_1} + 3\alpha_{q_1} \leq \gamma
\]

and similarly that \( \hat{d}_{b_2} - \alpha_{b_2} > \gamma \) by \( \mathcal{E}_{\text{good}}(b_2) \). Since \( \hat{d}_{q_2} - \alpha_{q_2} \geq \hat{d}_{m_2} - \alpha_{m_2} \), we have that \( \hat{d}_{q_2} - \alpha_{q_2} > \gamma \) in both the case that \( b_2 = m_2 \) and \( b_2 = q_2 \). Together, this implies that the termination condition (17) is true, which violates our assumption.

ii. When \( q_1 \leq k \) and \( k < b_2 \leq k + h \), we have first by \( \mathcal{E}_{\text{good}}(q_1) \) that \( \hat{d}_{q_1} + 3\alpha_{q_1} \leq \gamma \). Starting from here, and using the definition of \( q_1 \), we have for all \( i \in \hat{S}_{\text{close}} \),

\[
\gamma \geq \hat{d}_{q_1} + \alpha_{q_1} + 2\alpha_{q_1} \\
\geq \hat{d}_i + \alpha_i + 2\alpha_{q_1} \\
\geq \hat{d}_i + 2\alpha_{q_1} \\
> \hat{d}_i.
\]

Now we let \( \Delta \) denote \( d_{k+1+h} - d_k \). By definition of \( b_2 \), using \( \mathcal{E}_{\text{good}}(b_2) \), we have that \( \alpha_j \leq \Delta/4 \) for all \( j \in \hat{S}_{\text{middle}} \cup \{q_1\} \). Then we can start from \( \gamma > \hat{d}_{q_1} + \alpha_{q_1} \) to conclude that for all \( j \in \hat{S}_{\text{middle}} \cup \{q_2\} \),

\[
\gamma > \hat{d}_{q_1} + \alpha_{q_1} \\
\geq \hat{d}_{q_2} - \alpha_{q_2} \\
\geq \hat{d}_{q_2} - \Delta/4 \\
\geq \hat{d}_j - \Delta/4 \\
\geq d_j - \alpha_j - \Delta/4
\]

where (i) comes from our assumption that the terminating condition (17) is false. Combining (17) and (18) along with \( \gamma + \Delta/2 = d_{k+1+h} \), we obtain that \( d_{k+1+h} > d_i \) for all \( i \in \hat{S} \cup \{q_2\} \), which is a contradiction, since there can be at most \( k + h \) values of \( d_i \) that are smaller than \( d_{k+1+h} \).

iii. When \( k < q_1 \leq k + h \) and \( b_2 > k + h \), the case is similar to the previous case, except that we need to bound \( \alpha_i \) for \( i \in \hat{S}_{\text{middle}} \) in a different way. By \( \mathcal{E}_{\text{good}}(b_2) \), \( \hat{d}_{b_2} \geq d_{q_2} \), and \( \alpha_{b_2} \geq \alpha_{q_2} \), we have analogously to (17), for all \( i \in \hat{S}_{\text{far}} \),

\[
\gamma \leq \hat{d}_{b_2} - 3\alpha_{b_2} \\
\leq \hat{d}_{b_2} - \alpha_{q_2} - 2\alpha_{b_2} \\
\leq d_i - 2\alpha_{b_2}.
\]

Equivalently, \( d_i \geq \gamma + 2\alpha_{b_2} \). Since there are \( n - k - h \) values of \( i \) for which this inequality holds, it must hold for \( d_{k+1+h} \), so we obtain

\[
\alpha_{b_2} \leq \frac{d_{k+1+h} - \gamma}{2} = \frac{\Delta}{4}.
\]

By definition of \( b_2 \), \( \alpha_i \leq \Delta/4 \) for all \( i \in \hat{S}_{\text{middle}} \cup \{q_2\} \), and a contradiction can be reached similarly as in case ii.

iv. For the case when both \( q_1, b_2 \in \{k+1, \ldots, k+h\} \), we first show that at least one of \( \gamma < \hat{d}_{q_1} + \alpha_{q_1} \) or \( \gamma > \hat{d}_{q_2} - \alpha_{q_2} \) is true. To see this, first suppose the former is false. Then using that the terminating condition (17) is false, we have

\[
\gamma \geq \hat{d}_{q_1} + \alpha_{q_1} > \hat{d}_{q_2} - \alpha_{q_2}.
\]

Now that we know that at least one of these inequalities holds, and we proceed similarly for each. First suppose that the former inequality, \( \gamma < \hat{d}_{q_1} + \alpha_{q_1} \), holds. Using that by \( \mathcal{E}_{\text{good}}(q_1) \) and \( \mathcal{E}_{\text{good}}(b_2) \) we have \( \alpha_i \leq \Delta/4 \) for all \( i \in \{q_1, q_2\} \cup \hat{S}_{\text{middle}} \), we have that, for all \( i \in \{q_1, q_2\} \cup \hat{S}_{\text{middle}} \),

\[
\gamma < \hat{d}_{q_1} + \alpha_{q_1} \\
\leq \hat{d}_i + \alpha_{q_1} \\
\leq d_i + \alpha_i + \alpha_{q_1} \\
\leq d_i + \frac{\Delta}{2}
\]

We also have for all \( j \in \hat{S}_{\text{far}} \) that

\[
\gamma < \hat{d}_{q_1} + \alpha_{q_1} \\
\leq \hat{d}_{q_2} - \alpha_{q_2} + \alpha_{q_2} + \alpha_{q_1}.
\]
\[ \leq \hat{d}_j - \alpha_j + \alpha_{q_2} + \alpha_{q_4} \]
\[ \leq d_j + \alpha_{q_2} + \alpha_{q_4} \]
\[ \leq d_j + \Delta / 2. \]  

Combining (22) and (23), we have that \( d_i > d_k \) for all \( i \in \{q_1\} \cup \mathcal{S}_{\text{middle}} \cup \mathcal{S}_{\text{tar}} \), which is a contradiction, since at most \( n - k \) values of \( i \) can satisfy this inequality.

The case that \( \gamma > \hat{d}_{q_2} - \alpha_{q_2} \) is entirely analogous.

v. When \( q_1 > k + h \) or \( b_2 \leq k \), we can make similar arguments to the previous cases to reach a contradiction.

### A.2 Proof of Fact 3

First, when \( i \leq k \), we have
\[ \hat{d}_i + 3\alpha_i \leq d_i + 4\alpha_i \]
\[ \leq d_i + \Delta / 2 \]
\[ \leq \frac{d_{k+1} + d_i}{2} \leq \gamma, \]  
where the last inequality uses \( d_i \leq d_k \), so \( \mathcal{E}_{\text{bad}}(i) \) does not occur. This is similarly shown for \( i > k + h \). For \( k < i \leq k + h \), that \( \mathcal{E}_{\text{bad}}(i) \) does not occur follows immediately from \( \alpha_i \leq \Delta_i / 8 \leq \Delta_i / 4 \).

### A.3 Proof of Fact 4

Recalling that \( \alpha_i(u) = \sqrt{\frac{2\beta(u,\delta/n)}{u}} \), at \( \alpha_i(u) = \Delta_i / 8 \) we have that \( u = 2(\Delta_i / 8)^{-2} \beta(u, \delta') \), so we need to bound the greatest fixed point \( u^* \) of
\[ f(u) = 2(\Delta_i / 8)^{-2} \beta(u, \delta'). \]

Let \( u_0 = 2(\Delta_i / 8)^{-2} \), and note that for all \( u \geq u_0 \),
\[ f'(u) = \frac{2(\Delta_i / 8)^{-2}(2)}{u \log((1.12)2(\Delta_i / 8)^{-2})} \]
\[ \leq \frac{2(\Delta_i / 8)^{-2} \log((1.12)2(\Delta_i / 8)^{-2})}{2} \]
\[ < 1. \]  
Then
\[ u^* = \frac{f(u_0) - u_0 f'(z)}{1 - f'(z)} \]
\[ \leq \frac{f(u_0)}{1 - f'(u_0)}. \]  

So, we can bound the greatest fixed point of \( f \) as
\[ u^* \leq \max \left\{ u_0, \frac{f(u_0)}{1 - f'(u_0)} \right\} \]
\[ = 2(\Delta_i / 8)^{2 - 2} \beta(2(\Delta_i / 8)^{-2}, \delta') \]
\[ = c_1 \Delta_i^{-2} \beta(2(\Delta_i / 8)^{-2}, \delta'), \]  
where \( c_1 = 128/(1 - 2 \log((1.12)32)) \). Since \( \bar{T}_i \leq u^* + 1 \), letting \( c_2 = c_1 + 1 \),
\[ \bar{T}_i \leq \frac{c_2}{\Delta_i^{2}} \log \left( 125 \frac{\sqrt{n}}{\delta} \log \left( \frac{(1.12)128}{\Delta_i^{2}} \right) \right). \]

Then for \( c \) sufficiently large,
\[ \bar{T}_i \leq c \log \left( \frac{n}{\delta} \right) \log \left( \frac{2 \log(2/\Delta_i)}{\Delta_i^{2}} \right). \]

### B ADDITIONAL THEOREM PROOF DETAILS

In this section we provide details on bounding \( \sum_{i \in \mathcal{S}(\nu)} E_{\nu}[N_i] \) that we omitted in the proof of Theorem 2. We consider the set \( \mathcal{M} = \{\ell_1, \ldots, \ell_{h+1}\} \subseteq \mathcal{S}_1(\nu) \) and construct an alternative distribution \( \nu' \) such that under that distribution \( \mathcal{M} \subseteq \mathcal{S}_2(\nu') \). Then under \( \nu' \), if \( \mathcal{A} \) succeeds, then at most \( h \) elements of \( \mathcal{S}_2(\nu') \) can be in \( \mathcal{S} \), meaning that at least one element of \( \mathcal{M} \) is not in \( \mathcal{S} \) and that \( \mathcal{E} \) does not occur. So, if \( \mathcal{A} \) succeeds with probability at least \( 1 - \delta \), then both \( P_{\nu'}[\mathcal{E}] \geq 1 - \delta \) and \( P_{\nu'}[\mathcal{F}] \leq \delta \).

Our alternative distribution \( \nu' \) is defined as
\[ \nu'_i = \begin{cases} \nu_{k+1+h}, & i \in \mathcal{M} \\ \nu_i, & \text{otherwise}. \end{cases} \]

Again, to avoid ties, for \( \ell \in \mathcal{M} \), one should take \( \nu' = \nu_{k+1+h} + \varepsilon \) and let \( \varepsilon \to 0 \), but we omit this detail. The remainder of the arguments are entirely analogous to the case shown previously, giving us the bound
\[ \sum_{i \in \mathcal{S}_1(\nu)} E_{\nu}[N_i] \geq \log \frac{1}{2\delta} \sum_{i=1}^{k-h} \frac{d_{k+1+h}(1 - d_{k+1+h})}{d_i - d_{k+1+h}}. \]