6 Appendix

Proof of modified Proposition 3.1. In this version, we assume \((w,x,y)\) is a trajectory of (1) rather than being a trajectory of (8).

All we need to show is that for any pair of \((x,y)\), there exist another pair \((\tilde{x}, R_y)\), such that they give the same update. In particular, we set \(\tilde{x} = ax\) and show that there always exists an \(a \in [-1,1]\) such that

\[
(y - w^T x) = (R_y - w^T ax)ax.
\]

This simplifies to

\[
g(a) := (w^T x)a^2 - R_ya + (y - w^T x) = 0. \quad (15)
\]

The discriminant of the quadratic (15) is

\[
R_y^2 - 4w^T x(y - w^T x) \geq R_y^2 - 4w^T x (R_y + |w^T x|) = (R_y - 2|w^T x|)^2 \geq 0
\]

So there always exists a solution \(a \in \mathbb{R}\). Moreover, \(g(-1) = R_y + y \geq 0\) and \(g(1) = -R_y + y \leq 0\), so there must be a real root in \([-1,1]\).

Proof of Theorem 3.2. We showed in Section 3 that Regime V trajectories are 2D. We also argued that solutions that reach \(w_*\) via Regime III–IV are not unique and need not be 2D. We will now show that it’s always possible to construct a 2D solution.

We begin by characterizing the set of \(w_*\) reachable via Regime III–IV. Recall from Section 3 that the transition between III and IV occurs when \(|w| = R := \frac{R_y}{2}\).

If \(t_0\) is the time at which this transition occurs, then for \(0 \leq t \leq t_0\), the solution is \(x = \frac{R_y}{2||w||}w\), which leads to a straight-line trajectory from \(w_0\) to \(w(t_0)\).

Now consider the part of the trajectory in Regime IV, where \(t_0 \leq t \leq t_f\). As derived in Section 3, Regime IV trajectories satisfy \(\dot{w} = w^T x = \frac{R_y}{2}\). These lead to

\[
\frac{d||w||^2}{dt} = \frac{R_y^2}{2},
\]

which means that \(||w||\) grows at the same rate regardless of \(x\). If our trajectory reaches \(w(t_f) = w_*\), then we can deduce via integration that

\[
||w_*||^2 - ||w(t_0)||^2 = \frac{R_y^2}{2} (t_f - t_0), \quad (16)
\]

Suppose \((w(t), x(t))\) for \(t_0 \leq t \leq t_f\) is a trajectory that reaches \(w_*\). Refer to Figure 8. The reachable set at time \(t_f\) is a spherical sector whose boundary requires a trajectory that maximizes curvature. We will now derive this fact.

Let \(\theta_{\max}\) be the largest possible angle between \(w(t_0)\) and any reachable \(w(t_f) = w_*\), where we have fixed \(t_f\).

Define \(\theta(t)\) to be the angle between \(w(t)\) and \(w(t_f)\).

\[
\theta(t) = \int_{t_0}^{t_f} \dot{\theta} dt \leq \int_{t_0}^{t_f} |\dot{\theta}| dt
\]

Figure 8: If a reachable \(w_*\) is contained in the concave funnel shape, which is the reachable set in Regime IV, it can be reached by some trajectory \((w(t), x(t))\) lying entirely in the 2D subspace defined by \(\text{span} \{w_0, w_*\}\): follow the max-curvature solution until \(t_1\) and then transition to a radial solution until \(t_f\).

An alternative expression for this rate of change is the projection of \(\dot{w}\) onto the orthogonal complement of \(w\):

\[
|\dot{\theta}| = \frac{||\dot{w} - (w^T x) w||}{||w||} = \frac{R_y||x - \frac{R_y}{2||w||}w||}{2||w||}
\]

Where we used the fact that \(\dot{w} = w^T x = \frac{R_y}{2}\) in Regime IV. Now,

\[
\theta_{\max} = \max_{x: w^T x \leq \frac{R_y}{2}} \theta(t_0)
\]

\[
\leq \max_{x: w^T x \leq \frac{R_y}{2}} \int_{t_0}^{t_f} \frac{R_y}{2||w||} ||x - \frac{R_y}{2||w||}w|| dt
\]

\[
\leq \int_{t_0}^{t_f} \frac{R_y^2 - (\frac{R_y}{2||w||})^2}{2||w||} dt
\]

In the final step, we maximized over \(x\). Notice that the integrand (17) is an upper bound that only depends on \(t_0\) and \(||w_*||\) but not on \(x\). One can also verify that this upper bound is achieved by the choice

\[
\hat{x} = \frac{R_y}{2||w||}w + \sqrt{R_y^2 - \left(\frac{R_y}{2||w||}\right)^2} \frac{w_* - (w^T w_*) \hat{w}}{||w_* - (w^T w_*) \hat{w}||},
\]

where \(\hat{w} := w/||w||\) and \(w_*\) is any vector that satisfies (16) with angle \(\theta_{\max}\) with \(w(t_0)\). Any \(w_*\) with this norm but angle \(\theta < \theta_{\max}\) can also be reached by using the max-curvature control until \(t_1\), where \(t_1\) is chosen such that \(\theta_f = \int_{t_0}^{t_1} \sqrt{R_y^2 - (\frac{R_y}{2||w||})^2} dt\), and then using \(x = \frac{R_y}{2||w||}w\) for \(t_1 \leq t \leq t_f\). This piecewise path is illustrated in Figure 8.

Our constructed optimal trajectory lies in the 2D span of \(w_*\) and \(w_0\). This shows that all reachable \(w_*\) can be reached via a 2D trajectory. ■