Appendix of “No-regret algorithms for online $k$-submodular maximization”

A Omitted proofs in applications

A.1 Coupled feature selection

It suffices to show that $f$ satisfies the pairwise-monotonicity and orthant submodularity. The pairwise-monotonicity is immediate since $f$ is a monotone function. For the orthant submodularity, we note that $f$ decomposes into two functions: $g(X_1, \ldots, X_k) = H(X_1, \ldots, X_k)$ and $-\sum_{i \in [k]} \sum_{j \in X_i} H(Y_j | Z_i)$. The latter term is simply modular and we can ignore it. For the former term, let us take $X_i \subseteq X'_i (i \in [k])$ and $j \in [n] \setminus (X'_1 \cup \cdots \cup X'_k)$. Let $S = X_1 \cup \cdots \cup X_k$ and $S' = X'_1 \cup \cdots \cup X'_k$. Then, for any $i$,

$$
\Delta_{j,i}g(X_1, \ldots, X_k) + \Delta_{j,i}g(X'_1, \ldots, X'_k) = H(S \cup j) - H(S) - H(S' \cup j) + H(S') \geq 0,
$$

since the entropy function $H$ is submodular.

A.2 Sensor placement with $k$ different sensors

Again, we show the pairwise-monotonicity and orthant submodularity. For the pairwise-monotonicity, let us fix $(X_1, \ldots, X_k) \in (k+1)^V$ and $j \not\in \cup_i X_i$. Then, for arbitrary distinct $i$ and $i'$,

$$
\Delta_{j,i}f(X_1, \ldots, X_k) + \Delta_{j,i'}f(X_1, \ldots, X_k) = g(X_i \cup j) - g(X_i) + g(X_{i'} \cup j) - g(X_{i'}) = g(V \setminus (X_i \cup j)) + g(X_{i'} \cup j) - g(X_i) - g(X_{i'}) \geq 0,
$$

by the symmetricity and submodularity of $g$. For the orthant submodularity, let us take $X_i \subseteq Y_i (i \in [k])$ and $j \in [n] \setminus (Y_1 \cup \cdots \cup Y_k)$. Then, for any $i$,

$$
\Delta_{j,i}f(X_1, \ldots, X_k) + \Delta_{j,i}f(Y_1, \ldots, Y_k) = g(X_i \cup j) - g(X_i) - g(Y_i \cup j) + g(Y_i) \geq 0,
$$

by the submodularity of $g$. 