Supplement

In Section A we prove our remark on the validity of the Bernstein condition for higher-order derivatives in the case of kernels with faster spectral decay. The result extends the example of Gaussian kernels detailed in the main part of the paper.

A BERNSTEIN CONDITION FOR HIGHER-ORDER DERIVATIVES

We prove that in the case of kernels with spectral density decaying as \( f_A(\omega) \propto e^{-\omega^2} \) (\( \ell \in \mathbb{N}^+ \)), the Bernstein condition (15) holds for \( r \leq 2\ell \)-order derivatives. This example extends the case of Gaussian kernels where \( \ell = 1 \) and \( r \leq 2 \). Let \( \ell \in \mathbb{N}^+ \) and the spectral measure associated with kernel \( k \) be absolutely continuous w.r.t. the Lebesgue measure with density

\[
f_A(\omega) = c_\ell e^{-\omega^{2\ell}}
\]

for some \( c_\ell > 0 \). \( f_A \) is positive and we determine \( c_\ell \) as:

\[
1 = \int_\mathbb{R} f_A(\omega) d\omega = \int_\mathbb{R} c_\ell e^{-\omega^{2\ell}} d\omega = 2c_\ell \int_0^\infty e^{-\omega^{2\ell}} d\omega
\]

\[
= \frac{c_\ell}{\ell} \int_0^\infty e^{-y^{2/\ell}} y^{2/\ell - 1} dy = \frac{c_\ell}{\ell} \Gamma \left( \frac{1}{2\ell} \right) \Rightarrow
\]

\[
c_\ell = \frac{\ell}{\Gamma \left( \frac{1}{2\ell} \right)}
\]

where we used \( y = \omega^{2\ell} \), \( \omega = y^{1/\ell} \), \( d\omega = \frac{1}{\ell} y^{1/\ell - 1} dy \) and the pdf of the Gamma distribution \( (b = 1, a = \frac{1}{2\ell}) \)

\[
g(y; a, b) = \frac{y^{a-1} e^{-by}}{\Gamma(a)}
\]

\( (y > 0, a > 0, b > 0) \) from which it follows that

\[
\int_0^\infty y^{a-1} e^{-by} dy = \frac{\Gamma(a)}{b^a}.
\] (24)

Consequently, one obtains

\[
A_{r,n} = A_{r,n}(\Lambda) = \sqrt{\int_\mathbb{R} |\omega|^r d\Lambda(\omega)} = \frac{\Gamma \left( \frac{r+1}{2\ell} \right)}{\Gamma \left( \frac{1}{2\ell} \right)}
\]

by using (24) with \( b = 1, a = \frac{r+1}{2\ell} \) and the value of \( c_\ell \):

\[
\int_\mathbb{R} |\omega|^r d\Lambda(\omega) = \int_\mathbb{R} |\omega|^r c_\ell e^{-\omega^{2\ell}} d\omega
\]

\[
= 2c_\ell \int_0^\infty \omega^r e^{-\omega^{2\ell}} d\omega = \frac{c_\ell}{\ell} \int_0^\infty e^{-y^{2/\ell}} y^{r/\ell - 1} dy
\]

\[
= \frac{c_\ell}{\ell} \Gamma \left( \frac{r+1}{2\ell} \right) = \frac{\Gamma \left( \frac{r+1}{2\ell} \right)}{\Gamma \left( \frac{1}{2\ell} \right)}
\]

Next we assume that \( r \leq 2\ell \) is fixed and apply induction to prove (15).

- For \( n = 2 \), by definition \( A_{r,2} = 1 \) \((\forall r \in \mathbb{N}^+)\).
- The induction argument is as follows. By the inductive assumption it is sufficient to show the existence of \( K_r \geq 1 \) such that

\[
B_{r,n} := \frac{A_{r,n+1}}{A_{r,n}} \leq (n+1)K_r
\] (25)

since \( A_{r,n} \leq \frac{n!}{2^n} K_r^{n-2} \) and \( \frac{A_{r,n+1}}{A_{r,n}} \leq (n+1)K_r \) imply

\[
A_{r,n+1} \leq \frac{(n+1)!}{2^{n+1}} K_r^{n-1}.
\]

By defining \( c_r := \frac{\Gamma \left( \frac{r+1}{2\ell} \right)}{\Gamma \left( \frac{1}{2\ell} \right)} \), we obtain

\[
B_{r,n} = \frac{\Gamma \left( \frac{(r+1)+1}{2\ell} \right)}{\Gamma \left( \frac{1}{2\ell} \right)} \left( c_r \right)^{\frac{r}{2\ell}} = \frac{\Gamma \left( \frac{(r+1)+1}{2\ell} \right)}{\Gamma \left( \frac{1}{2\ell} \right)} \Gamma \left( \frac{r+1}{2\ell} \right)
\]

\[
= \frac{\Gamma \left( \frac{r+1}{2\ell} + \frac{r}{2\ell} \right)}{\Gamma \left( \frac{r}{2\ell} \right)} \leq D_{r,n} \frac{\Gamma \left( \frac{r+1}{2\ell} + \frac{2r}{2\ell} \right)}{\Gamma \left( \frac{r}{2\ell} + \frac{2r}{2\ell} \right)}
\]

\[
 \leq D_{r,n} \frac{1}{\sqrt{c_r}} \frac{r+1}{2\ell} \leq D_{r,n} \frac{1}{\sqrt{c_r}} \frac{2\ell n + 1}{2\ell} = \frac{n+1}{\sqrt{c_r}}
\]

\[
< D_{r,n} \frac{n+1}{\sqrt{c_r}}.
\]

Indeed,

- (a): The Gamma function has a global minima on the positive real line at \( z_{\min} \approx 1.46163 \), it is strictly monotonically decreasing on \((0, z_{\min})\) and strictly monotonically increasing on \((z_{\min}, \infty)\).

The latter implies

\[
\Gamma(z_1) \leq \Gamma(z_2) \text{ for } z_{\min} \leq z_1 \leq z_2.
\] (26)

Let us choose \( z_1 = \frac{r+1}{2\ell} + \frac{r}{2\ell} \) and \( z_2 = \frac{r+1}{2\ell} + \frac{2r}{2\ell} \). \( z_1 \leq z_2 \) since \( r \leq 2\ell \). With this choice (26) guarantees (a) with \( D_{r,n} = 1 \) if

\[
z_{\min} \leq \frac{n+2}{2\ell} \leq \frac{r+1}{2\ell} + \frac{r}{2\ell} \Rightarrow \leq \frac{r+1}{2\ell} + \frac{r}{2\ell} = z_1.
\]

If \( n_s := \left[ 2\ell z_{\min} - 2 \right] \leq n \), then (d) holds. This means that (a) holds with

\[
D_{r,n} = 1 \text{ if } n_s \leq n.
\]

For the remaining \( n = 2, \ldots, n_s - 1 \) values, (a) is fulfilled with equality using \( D_{r,n} := \frac{\Gamma \left( \frac{r+1}{2\ell} + \frac{r}{2\ell} \right)}{\Gamma \left( \frac{r+1}{2\ell} \right)} \).

- (b): We applied the Gamma function property.

- (c): It follows from \( r \leq 2\ell \).

To sum up, we got that

\[
B_{r,n} \leq D_{r,n} \frac{n+1}{\sqrt{c_r}} (n+1), \text{ with}
\]

\[
D_{r,n} = \begin{cases} 1 & \text{if } n_s \leq n \\ \frac{\Gamma \left( \frac{r+1}{2\ell} + \frac{r}{2\ell} \right)}{\Gamma \left( \frac{r+1}{2\ell} \right)} & \text{if } n = 2, \ldots, n_s - 1. \end{cases}
\]
Thus, one can choose

\[
K_r = \max \left( \frac{D_{r,2}}{\sqrt{c_r}}, \ldots, \frac{D_{r,n_r-1}}{\sqrt{c_r}}, 1 \right)
\]

in (25).