Improved Semi-Supervised Learning with Multiple Graphs: Supplementary Material

Krishnamurthy Viswanathan[1] Google Research
Sushant Sachdeva[1] University of Toronto
Sujith Ravi **Andrew Tomkins** Google Research

1 Additional Proofs

Lemma 1.1. *We can express $\mathbf{\Sigma}_{\widehat{\mathbf{y}\mathbf{s}}}$ equivalently as follows,*

$$\mathbf{\Sigma}_{\widehat{\mathbf{y}\mathbf{s}}} = I + \Pi_S \left(\textstyle\sum_{i=1}^n \alpha_i L_{G_i}\right)^{-1} \Pi_S^\top$$

Proof. Let

$$L_w = \sum_{i=1}^n \alpha_i L_{G_i}$$

denote the weighted sum of Laplacians. Then

$$\begin{aligned}
&\mathbf{\Sigma}_{\widehat{\mathbf{y}\mathbf{s}}}^{-1} \left(I + \Pi_S L_w^{-1} \Pi_S^\top\right) \\
&= \left(I - \Pi_S \left(I_S + L_w\right)^{-1} \Pi_S^\top\right) \\
&\quad \left(I + \Pi_S L_w^{-1} \Pi_S^\top\right) \\
&= I - \Pi_S \left(I_S + L_w\right)^{-1} \Pi_S^\top \\
&\quad + \Pi_S L_w^{-1} \Pi_S^\top \\
&\quad - \Pi_S \left(I_S + L_w\right)^{-1} I_S L_w^{-1} \Pi_S^\top \\
&= I - \Pi_S \left(I_S + L_w\right)^{-1} \Pi_S^\top \\
&\quad + \Pi_S \left(I_S + L_w\right)^{-1} \left(I_S + L_w\right) L_w^{-1} \Pi_S^\top \\
&\quad - \Pi_S \left(I_S + L_w\right)^{-1} I_S L_w^{-1} \Pi_S^\top \\
&= I - \Pi_S \left(I_S + L_w\right)^{-1} \Pi_S^\top \\
&\quad + \Pi_S \left(I_S + L_w\right)^{-1} L_w L_w^{-1} \Pi_S^\top \\
&= I.
\end{aligned}$$

□

Lemma 1.2. *Let B and W denote the edge-vertex incidence matrix, and the diagonal weight matrix of $\sum_i \alpha_i L_{G_i}$. Let M be the matrix $\left[I, \Pi_S (\sum_i \alpha_i L_{G_i})^{-1} B W^{\frac{1}{2}}\right]$. Then $MM^\top = \mathbf{\Sigma}_{\widehat{\mathbf{y}\mathbf{s}}}$, and thus,*

$$\begin{aligned}
&Tr(\mathbf{\Sigma}_{\widehat{\mathbf{y}\mathbf{s}}} \Pi_S L^{-1} L_{G_i} L^{-1} \Pi_S^\top) \\
&\quad = Tr((L^{-1} \Pi_S^\top M)^\top L_{G_i} (L^{-1} \Pi_S^\top M))
\end{aligned}$$

Proof. We have $BWB^\top = \sum_i \alpha_i L_{G_i}$. Thus,

$$\begin{aligned}
MM^\top &= I + \Pi_S (\sum_i \alpha_i L_{G_i})^{-1} BWB^\top (\sum_i \alpha_i L_{G_i}^{-1} \Pi_S^\top \\
&= I + \Pi_S (\sum_i \alpha_i L_{G_i})^{-1} \Pi_S^\top = \mathbf{\Sigma}_{\widehat{\mathbf{y}\mathbf{s}}}.
\end{aligned}$$

The claim about $\mathrm{Tr}(\cdot)$ follows due to the cyclic property of the trace operator. □

[1] Equal contribution