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Construction of Incoherent Dictionaries via Direct Babel Function Minimization: Supplementary Material

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1. Proof in Section 2

Theorem 1 *Assume that* $\{\mathbf{X}^k, \mathbf{Y}^k, \mathbf{W}^k\}$ *is bounded,* $\epsilon_k \to 0$ *and* $\mathbf{W}^*\mathbf{V}_{i,:}^T\mathbf{V}_{i,:}$, $i = 1, \cdots, n$, *are linear dependent. Let* $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{W}^*)$ *be an accumulation point of* $(\mathbf{X}^k, \mathbf{Y}^k, \mathbf{W}^k)$*, then* $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{W}^*)$ *is a KKT point of problem (7)*

Proof We first prove that $\mathbf{X}^*, \mathbf{Y}^*, \mathbf{W}^*$ is feasible. Since $\{\mathbf{X}^k\}$, $\{\mathbf{Y}^k\}$ and $\{\mathbf{W}^k\}$ are bounded, then there exists $\mathbf{X}^*$, $\mathbf{Y}^*$, $\mathbf{W}^*$ and infinite subsequence $\mathbf{K}$ such that $\lim_{k\in\mathbf{K}} \mathbf{X}^{k+1} = \mathbf{X}^*$, $\lim_{k\in\mathbf{K}} \mathbf{Y}^{k+1} = \mathbf{Y}^*$ and $\lim_{k\in\mathbf{K}} \mathbf{W}^{k+1} = \mathbf{W}^*$.

If $\{\rho^k\}$ is bounded, then ρ is not updated from some iteration. So $\lim_{k\to\infty} \|\mathbf{X}^k - \mathbf{Y}^k\|_F$ and $\lim_{k\to\infty} \|\mathbf{Y}^k - \mathbf{V}\mathbf{W}^k\mathbf{V}^T + \mathbf{I}\|_F = 0$. We can have $\mathbf{X}^* = \mathbf{Y}^*$ and $\mathbf{Y}^* = \mathbf{V}\mathbf{W}^*\mathbf{V}^T - \mathbf{I}$.

Now we consider the case that $\{\rho^k\}$ is unbounded. From step 1, we have

$$\boldsymbol{\sigma}_1^k \in \partial f(\mathbf{X}^{k+1}) + \boldsymbol{\Lambda}_1^k + \rho^k(\mathbf{X}^{k+1} - \mathbf{Y}^{k+1}), \tag{1a}$$

$$\boldsymbol{\sigma}_2^k + \boldsymbol{\Lambda}_1^k - \boldsymbol{\Lambda}_2^k + \rho^k(\mathbf{X}^{k+1} - \mathbf{Y}^{k+1}) - \rho^k(\mathbf{Y}^{k+1} - \mathbf{V}\mathbf{W}^{k+1}\mathbf{V}^T + \mathbf{I}) \in \sum_{i=1}^n N_{\Pi_i}(\mathbf{Y}^{k+1}), \tag{1b}$$

$$\boldsymbol{\sigma}_3^k + \mathbf{V}^T\boldsymbol{\Lambda}_2^k\mathbf{V} + \rho^k\mathbf{V}^T(\mathbf{Y}^{k+1} - \mathbf{V}\mathbf{W}^{k+1}\mathbf{V}^T + \mathbf{I})\mathbf{V} \in N_{\mathbf{S}_+}(\mathbf{W}^{k+1}) + N_{\mathbf{S}_m}(\mathbf{W}^{k+1}), \tag{1c}$$

where $N_{\mathbf{S}}(\mathbf{W})$ is the normal cone of $\mathbf{S}$ at $\mathbf{W} \in \mathbf{S}$, $\mathbf{S}_+ = \{\mathbf{W} \in \mathbf{R}^{r\times r} : \mathbf{W} = \mathbf{W}^T, \mathbf{W} \succeq 0\}$, $\mathbf{S}_m = \{\mathbf{W} \in \mathbf{R}^{r\times r} : \text{rank}(\mathbf{W}) \leq m\}$, $\Pi_i = \{\mathbf{Y} \in \mathbf{R}^{n\times n} : e_i^T\mathbf{Y}e_i = 0\}$ and we use $\partial\delta_{\mathbf{S}}(\mathbf{W}) = N_{\mathbf{S}}(\mathbf{W})$. Here we replace Ω with $\mathbf{S}_+ \cap \mathbf{S}_m$, Π with $\Pi_1 \cap \cdots \Pi_n$.

Divide both sides by ρ^k in (1a) and let $k \to \infty$, $k \in \mathbf{K}$. From $\boldsymbol{\sigma}_1^k \to 0$, the boundedness of $\partial f(\mathbf{X}^{k+1})$ and $\boldsymbol{\Lambda}_1^k$, we have $\mathbf{X}^* - \mathbf{Y}^* = 0$.

Divide both sides by ρ^k in (1b) and let $k \to \infty$, $k \in \mathbf{K}$. From $\boldsymbol{\sigma}_2^k \to 0$, $\mathbf{X}^* = \mathbf{Y}^*$, the boundedness of $\boldsymbol{\Lambda}_1^k$ and $\boldsymbol{\Lambda}_2^k$, we have $-(\mathbf{Y}^* - \mathbf{V}\mathbf{W}^*\mathbf{V}^T + \mathbf{I}) \in \sum_{i=1}^n N_{\Pi_i}(\mathbf{Y}^*)$. Since $N_{\Pi_i}(\mathbf{Y}^*) = \{\lambda e_i e_i^T : \lambda \in \mathbf{R}\}$, thus there exists $\lambda_i^*, i = 1, \cdots, n$ such that $\mathbf{Y}^* - \mathbf{V}\mathbf{W}^*\mathbf{V}^T + \mathbf{I} = \sum_{i=1}^n \lambda_i^* e_i e_i^T$.

Divide both sides by ρ^k in (1c) and let $k \to \infty$, $k \in \mathbf{K}$. From $\boldsymbol{\sigma}^k \to 0$, $\mathbf{Y}^* - \mathbf{V}\mathbf{W}^*\mathbf{V}^T + \mathbf{I} = \sum_{i=1}^n \lambda_i^* e_i e_i^T$ and the boundedness of $\boldsymbol{\Lambda}_2^k$, we have $\sum_{i=1}^n \lambda_i^*\mathbf{V}^T e_i e_i^T\mathbf{V} = \sum_{i=1}^n \lambda_i^*\mathbf{V}_{i,:}^T\mathbf{V}_{i,:} \in N_{\mathbf{S}_+}(\mathbf{W}^*) + N_{\mathbf{S}_m}(\mathbf{W}^*)$. Since $N_{\mathbf{S}_+}(\mathbf{W}) = \{\hat{\mathbf{W}} \in \mathbf{S}_-^r : \hat{\mathbf{W}}\mathbf{W}^T = 0\}$ (Fletcher, 1985) and $N_{\mathbf{S}_m}(\mathbf{W}) = \{\hat{\mathbf{W}} \in \mathbf{R}^{r\times r} : ker(\hat{\mathbf{W}})^\perp \cap ker(\mathbf{W})^\perp = \{0\}, \text{rank}(\hat{\mathbf{W}}) \leq r - m\} = \{\hat{\mathbf{W}} \in \mathbf{R}^{r\times r} : \hat{\mathbf{W}}\mathbf{W}^T = 0, \text{rank}(\hat{\mathbf{W}}) \leq r - m\}$ (Luke, 2013), then $(\hat{\mathbf{W}}_1 + \hat{\mathbf{W}}_2)\mathbf{W}^T = 0$ if $\hat{\mathbf{W}}_1 \in N_{\mathbf{S}_+}(\mathbf{W})$ and $\hat{\mathbf{W}}_2 \in N_{\mathbf{S}_m}(\mathbf{W})$. So we can have $0 = \sum_{i=1}^n \lambda_i^*\mathbf{V}_{i,:}^T\mathbf{V}_{i,:}(\mathbf{W}^*)^T = \sum_{i=1}^n \lambda_i^*\mathbf{W}^*\mathbf{V}_{i,:}^T\mathbf{V}_{i,:}$. From the assumption, we have $\lambda_i^* = 0, i = 1, \cdots, n$. So $\mathbf{Y}^* - \mathbf{V}\mathbf{W}^*\mathbf{V}^T + \mathbf{I} = 0$.

Now we prove that $\mathbf{X}^*, \mathbf{Y}^*\mathbf{W}^*$ is a KKT point. From (1a)-(1c), the definition of $\hat{\mathbf{\Lambda}}_1^{k+1}$ and $\hat{\mathbf{\Lambda}}_2^{k+1}$, we have

$$\boldsymbol{\sigma}_1^k \in \partial f(\mathbf{X}^{k+1}) + \hat{\mathbf{\Lambda}}_1^{k+1}, \tag{2}$$

$$\boldsymbol{\sigma}_2^k + \hat{\mathbf{\Lambda}}_1^{k+1} - \hat{\mathbf{\Lambda}}_2^{k+1} \in \sum_{i=1}^{n} N_{\Pi_i}(\mathbf{Y}^{k+1}), \tag{3}$$

$$\boldsymbol{\sigma}_3^k + \mathbf{V}^T\hat{\mathbf{\Lambda}}_2^{k+1}\mathbf{V} \in N_{\mathbf{S}_+}(\mathbf{W}^{k+1}) + N_{\mathbf{S}_m}(\mathbf{W}^{k+1}). \tag{4}$$

Since $\partial f(\mathbf{X}^{k+1})$ is bounded, thus $\hat{\mathbf{\Lambda}}_1^{k+1}$ must be bounded. There exists $\hat{\mathbf{\Lambda}}_1^*$ and infinite subsequence $\mathbf{K}_1 \in \mathbf{K}$ such that $\lim_{k\in\mathbf{K}_1} \hat{\mathbf{\Lambda}}_1^{k+1} = \hat{\mathbf{\Lambda}}_1^*$. From $\delta^k \to 0$ we have $-\hat{\mathbf{\Lambda}}_1^* \in \partial f(\mathbf{X}^*)$.

Now we consider two cases of $\{\hat{\mathbf{\Lambda}}_2^{k+1}\}$.

If $\{\|\hat{\mathbf{\Lambda}}_2^{k+1}\|_\infty\}$ is bounded, then there exists $\hat{\mathbf{\Lambda}}_2^*$ and infinite subsequence $\mathbf{K}_2 \in \mathbf{K}_1$ such that $\lim_{k\in\mathbf{K}_2} \hat{\mathbf{\Lambda}}_2^{k+1} = \hat{\mathbf{\Lambda}}_2^*$, $\hat{\mathbf{\Lambda}}_1^* - \hat{\mathbf{\Lambda}}_2^* \in \sum_{i=1}^n N_{\Pi_i}(\mathbf{Y}^*)$ and $\mathbf{V}^T\hat{\mathbf{\Lambda}}_2^*\mathbf{V} \in N_{\mathbf{S}_+}(\mathbf{W}^*) + N_{\mathbf{S}_m}(\mathbf{W}^*)$, which together with $-\hat{\mathbf{\Lambda}}_1^* \in \partial f(\mathbf{X}^*)$ and the feasibility, is the KKT condition.

If $\{\|\hat{\mathbf{\Lambda}}_2^{k+1}\|_\infty\}$ is unbounded, divide both sides of (3) and (4) by $\|\hat{\mathbf{\Lambda}}_2^{k+1}\|_\infty$, we have

$$\frac{\boldsymbol{\sigma}_2^k}{\|\hat{\mathbf{\Lambda}}_2^{k+1}\|_\infty} + \frac{\hat{\mathbf{\Lambda}}_1^{k+1}}{\|\hat{\mathbf{\Lambda}}_2^{k+1}\|_\infty} - \frac{\hat{\mathbf{\Lambda}}_2^{k+1}}{\|\hat{\mathbf{\Lambda}}_2^{k+1}\|_\infty} \in \sum_{i=1}^{n} N_{\Pi_i}(\mathbf{Y}^{k+1}),$$
$$\frac{\boldsymbol{\sigma}_3^k}{\|\hat{\mathbf{\Lambda}}_2^{k+1}\|_\infty} + \frac{\mathbf{V}^T\hat{\mathbf{\Lambda}}_2^{k+1}\mathbf{V}}{\|\hat{\mathbf{\Lambda}}_2^{k+1}\|_\infty} \in N_{\mathbf{S}_+}(\mathbf{W}^{k+1}) + N_{\mathbf{S}_m}(\mathbf{W}^{k+1}).$$

Since $\frac{\hat{\mathbf{\Lambda}}_2^{k+1}}{\|\hat{\mathbf{\Lambda}}_2^{k+1}\|_\infty}$ is bounded, then there exists $\mathbf{K}_3 \in \mathbf{K}_1$ such that $\lim_{k\in\mathbf{K}_3} \frac{\hat{\mathbf{\Lambda}}_2^{k+1}}{\|\hat{\mathbf{\Lambda}}_2^{k+1}\|_\infty} = \overline{\mathbf{\Lambda}}_2^*$ and $\|\overline{\mathbf{\Lambda}}_2^*\|_\infty = 1$. So there exists λ_i such that $\overline{\mathbf{\Lambda}}_2^* = \sum_{i=1}^n \lambda_i e_i e_i^T$ and $\mathbf{V}^T\overline{\mathbf{\Lambda}}_2^*\mathbf{V} \in N_{\mathbf{S}_+}(\mathbf{W}^*) + N_{\mathbf{S}_m}(\mathbf{W}^*)$, which leads to $\sum_{i=1}^n \lambda_i \mathbf{W}^*\mathbf{V}_{i,:}^T\mathbf{V}_{i,:} = 0$. From the assumption we have $\lambda_i = 0, i = 1, \cdots, n$ and $\overline{\mathbf{\Lambda}}_2^* = 0$, which contradicts with $\|\overline{\mathbf{\Lambda}}_2^*\|_\infty = 1$. ■

1.1. Details of Step 1 in ALM-BF

We can use the Proximal Alternating Minimization method Bolte et al. (2014) to solve the following subproblem in step 1 of ALM-BF:

$$\min_{\mathbf{X},\mathbf{Y},\mathbf{W}} L(\mathbf{X}, \mathbf{Y}, \mathbf{W}, \mathbf{\Lambda}_1^k, \mathbf{\Lambda}_2^k) \tag{5}$$

which consists of three steps in each iteration:

$$\begin{aligned}
\mathbf{X}^{k,t+1} &= \operatorname*{argmin}_{\mathbf{X}} L(\mathbf{X}, \mathbf{Y}^{k,t}, \mathbf{W}^{k,t}, \mathbf{\Lambda}_1^k, \mathbf{\Lambda}_2^k) + \frac{\tau}{2}\|\mathbf{X} - \mathbf{X}^{k,t}\|_F^2 \\
&= \text{Prox}_{\frac{1}{\rho^k}\|\cdot\|_{\infty,max_p}} \left((\rho^k \mathbf{Y}^{k,t} - \mathbf{\Lambda}_1^k + \tau \mathbf{X}^{k,t})/(\rho^k + \tau) \right), \\
\mathbf{Y}^{k,t+1} &= \operatorname*{argmin}_{\mathbf{Y}} L(\mathbf{X}^{k,t+1}, \mathbf{Y}, \mathbf{W}^{k,t}, \mathbf{\Lambda}_1^k, \mathbf{\Lambda}_2^k) + \frac{\tau}{2}\|\mathbf{Y} - \mathbf{Y}^{k,t}\|_F^2 \\
&= \text{Proj}_{\Pi} \left((\rho^k \mathbf{X}^{k,t+1} + \mathbf{\Lambda}_1^k + \rho^k \mathbf{V}\mathbf{W}^{k,t}\mathbf{V}^T - \rho^k \mathbf{I} - \mathbf{\Lambda}_2^k + \tau \mathbf{Y}^{k,t})/(2\rho^k + \tau) \right), \\
\mathbf{W}^{k,t+1} &= \operatorname*{argmin}_{\mathbf{W}} L(\mathbf{X}^{k,t+1}, \mathbf{Y}^{k,t+1}, \mathbf{W}, \mathbf{\Lambda}_1^k, \mathbf{\Lambda}_2^k) + \frac{\tau}{2}\|\mathbf{W} - \mathbf{W}^{k,t}\|_F^2 \\
&= \operatorname*{argmin}_{\mathbf{W}} \delta_\Omega(\mathbf{W}) + \frac{\rho}{2}\left\| \mathbf{V}\mathbf{W}\mathbf{V}^T - \left(\mathbf{Y}^{k,t+1} + \mathbf{I} + \frac{\mathbf{\Lambda}_2^k}{\rho} \right) \right\|_F^2 + \frac{\tau}{2}\|\mathbf{W} - \mathbf{W}^{k,t}\|_F^2 \\
&= \operatorname*{argmin}_{\mathbf{W}} \delta_\Omega(\mathbf{W}) + \frac{\rho}{2}\left\| \mathbf{W} - \mathbf{V}^T \left(\mathbf{Y}^{k,t+1} + \mathbf{I} + \frac{\mathbf{\Lambda}_2^k}{\rho} \right) \mathbf{V} \right\|_F^2 + \frac{\tau}{2}\|\mathbf{W} - \mathbf{W}^{k,t}\|_F^2 \\
&= \text{Proj}_{\Omega} \left((\mathbf{V}^T(\rho^k \mathbf{Y}^{k,t+1} + \rho^k \mathbf{I} + \mathbf{\Lambda}_2^k)\mathbf{V} + \tau \mathbf{W}^{k,t})/(\rho^k + \tau) \right),
\end{aligned}$$

where we use

$$\begin{aligned}
\|\mathbf{V}\mathbf{W}\mathbf{V}^T - \mathbf{Z}\|_F^2 &= \text{trace}((\mathbf{V}\mathbf{W}^T\mathbf{V}^T - \mathbf{Z}^T)(\mathbf{V}\mathbf{W}\mathbf{V}^T - \mathbf{Z})) \\
&= \text{trace}(\mathbf{V}\mathbf{W}^T\mathbf{V}^T\mathbf{V}\mathbf{W}\mathbf{V}^T) - 2\text{trace}(\mathbf{V}\mathbf{W}^T\mathbf{V}^T\mathbf{Z}) + \text{trace}(\mathbf{Z}^T\mathbf{Z}) \\
&= \text{trace}(\mathbf{V}\mathbf{W}^T\mathbf{W}\mathbf{V}^T) - 2\text{trace}(\mathbf{V}\mathbf{W}^T\mathbf{V}^T\mathbf{Z}) + \text{trace}(\mathbf{Z}^T\mathbf{Z}) \\
&= \text{trace}(\mathbf{V}^T\mathbf{V}\mathbf{W}^T\mathbf{W}) - 2\text{trace}(\mathbf{W}^T\mathbf{V}^T\mathbf{Z}\mathbf{V}) + \text{trace}(\mathbf{Z}^T\mathbf{Z}) \\
&= \text{trace}(\mathbf{W}^T\mathbf{W}) - 2\text{trace}(\mathbf{W}^T\mathbf{V}^T\mathbf{Z}\mathbf{V}) + \text{trace}(\mathbf{Z}^T\mathbf{Z}) \\
&= \|\mathbf{W} - \mathbf{V}^T\mathbf{Z}\mathbf{V}\|_F^2 - \|\mathbf{V}^T\mathbf{Z}\mathbf{V}\|_F^2 + \|\mathbf{Z}\|_F^2
\end{aligned}$$

and $\mathbf{V}^T\mathbf{V} = \mathbf{I}$ in the $\mathbf{W}$ update step. For an Arbitrary matrix $\mathbf{Z}$,

$$\begin{aligned}
\text{Proj}_\Omega(\mathbf{Z}) &= \operatorname*{argmin}_{\mathbf{W}\in\Omega} \|\mathbf{W} - \mathbf{Z}\|_F^2 = \operatorname*{argmin}_{\mathbf{W}\in\Omega} \left\| \mathbf{W} - \frac{\mathbf{Z} + \mathbf{Z}^T}{2} - \frac{\mathbf{Z} - \mathbf{Z}^T}{2} \right\|_F^2 \\
&= \operatorname*{argmin}_{\mathbf{W}\in\Omega} \left\| \mathbf{W} - \frac{\mathbf{Z} + \mathbf{Z}^T}{2} \right\|_F^2 + \left\| \frac{\mathbf{Z} - \mathbf{Z}^T}{2} \right\|_F^2,
\end{aligned}$$

where we use $\text{trace}(\mathbf{A}\mathbf{B}) = 0$ if $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{B} = -\mathbf{B}^T$. Let $\mathbf{U}\Sigma\mathbf{U}^T$ be the eigenvalue decomposition of $\frac{\mathbf{Z}+\mathbf{Z}^T}{2}$ with an non-increasing order of the diagonal of Σ and $\hat{\Sigma} = \text{diag}([\max\{0, \Sigma_{1,1}\}, \cdots, \max\{0, \Sigma_{m,m}\}, 0, \cdots, 0])$. Then $\text{Proj}_\Omega(\mathbf{Z}) = \mathbf{U}\hat{\Sigma}\mathbf{U}^T$.

2. Proof in Section 3

Lemma 2 *Let* $\mathbf{X}^* = Proj_{\|\cdot\|^*_{\infty,max_p}\le 1}(\rho\mathbf{Y})$, *then we have* $Prox_{\frac{1}{\rho}\|\cdot\|_{\infty,max_p}}(\mathbf{Y}) = \mathbf{Y} - \frac{\mathbf{X}^*}{\rho}$.

Proof From the definition of Fenchel dual, we have

$$\|\mathbf{Z}\|_{\infty,max_p} = \max_{\|\mathbf{X}\|^*_{\infty,max_p}\le 1} \langle \mathbf{Z}, \mathbf{X} \rangle .$$

Then we have

$$
\begin{aligned}
& \min_{\mathbf{Z}} \|\mathbf{Z}\|_{\infty,max_p} + \frac{\rho}{2}\|\mathbf{Z}-\mathbf{Y}\|_F^2 \\
= & \min_{\mathbf{Z}} \max_{\|\mathbf{X}\|^*_{\infty,max_p}\le 1} \langle \mathbf{Z},\mathbf{X}\rangle + \frac{\rho}{2}\|\mathbf{Z}-\mathbf{Y}\|_F^2 \\
= & \min_{\mathbf{Z}} \max_{\|\mathbf{X}\|^*_{\infty,max_p}\le 1} \frac{\rho}{2}\left\|\mathbf{Z}-\mathbf{Y}+\frac{\mathbf{X}}{\rho}\right\|_F^2 + \langle \mathbf{Y},\mathbf{X}\rangle - \frac{\|\mathbf{X}\|_F^2}{2\rho} \\
= & \max_{\|\mathbf{X}\|^*_{\infty,max_p}\le 1} \min_{\mathbf{Z}} \frac{\rho}{2}\left\|\mathbf{Z}-\mathbf{Y}+\frac{\mathbf{X}}{\rho}\right\|_F^2 + \langle \mathbf{Y},\mathbf{X}\rangle - \frac{\|\mathbf{X}\|_F^2}{2\rho} \\
= & \max_{\|\mathbf{X}\|^*_{\infty,max_p}\le 1} \langle \mathbf{Y},\mathbf{X}\rangle - \frac{\|\mathbf{X}\|_F^2}{2\rho}.
\end{aligned}
$$

Let $\mathbf{X}^* = \mathrm{Proj}_{\|\mathbf{X}\|^*_{\infty,max_p}\le 1}(\rho\mathbf{Y})$, then we have $\mathrm{Prox}_{\frac{1}{\rho}\|\mathbf{Z}\|_{\infty,max_p}}(\mathbf{Y}) = \mathbf{Y} - \frac{\mathbf{X}^*}{\rho}$. ■

Theorem 3 *Let $\|\mathbf{x}\|^*_{max_p}$ and $\|\mathbf{X}\|^*_{\infty,max_p}$ be the Fenchel dual norm of $\|\mathbf{x}\|_{max_p}$ and $\|\mathbf{X}\|_{\infty,max_p}$, respectively, then*

$$
\begin{aligned}
\|\mathbf{x}\|^*_{max_p} &= \max\left\{\|\mathbf{x}\|_\infty, \frac{1}{p}\|\mathbf{x}\|_1\right\} \equiv \|\mathbf{x}\|_{max\left\{l_\infty,\frac{1}{p}l_1\right\}}, \\
\|\mathbf{X}\|^*_{\infty,max_p} &= \sum_{i=1}^n \|\mathbf{X}_{i,:}\|^*_{max_p} \equiv \|\mathbf{X}\|_{1,max\left\{l_\infty,\frac{1}{p}l_1\right\}}.
\end{aligned}
$$

Proof From the definition, we have $\|\mathbf{x}\|^*_{max_p} = \max_{\|\mathbf{z}\|_{max_p}\le 1} \mathbf{x}^T\mathbf{z}$.

$$
\begin{aligned}
\mathbf{x}^T\mathbf{z} &\le \sum_{i=1}^n |\mathbf{x}_i||\mathbf{z}_i| \le \sum_{i=1}^n |\mathbf{x}_{\delta(i)}||\mathbf{z}_{\delta(i)}| = \sum_{i=1}^{p-1} |\mathbf{x}_{\delta(i)}||\mathbf{z}_{\delta(i)}| + \sum_{i=p}^n |\mathbf{x}_{\delta(i)}||\mathbf{z}_{\delta(i)}| \\
&\le \sum_{i=1}^{p-1} |\mathbf{x}_{\delta(i)}||\mathbf{z}_{\delta(i)}| + |\mathbf{z}_{\delta(p)}|\sum_{i=p}^n |\mathbf{x}_{\delta(i)}|.
\end{aligned}
$$

From $(|\mathbf{x}_{\delta(1)}| - |\mathbf{x}_{\delta(i)}|)(|\mathbf{z}_{\delta(i)}| - |\mathbf{z}_{\delta(p)}|) \ge 0, \forall i \le p$, we have

$$
|\mathbf{x}_{\delta(1)}|(|\mathbf{z}_{\delta(i)}| - |\mathbf{z}_{\delta(p)}|) + |\mathbf{x}_{\delta(i)}||\mathbf{z}_{\delta(p)}| \ge |\mathbf{x}_{\delta(i)}||\mathbf{z}_{\delta(i)}|, \quad \forall i \le p.
$$

Do this operation for $i = 1, \cdots, p-1$ and sum, we have

$$
\begin{aligned}
\sum_{i=1}^{p-1} |\mathbf{x}_{\delta(i)}||\mathbf{z}_{\delta(i)}| &\le |\mathbf{x}_{\delta(1)}|(\sum_{i=1}^{p-1}|\mathbf{z}_{\delta(i)}| - (p-1)|\mathbf{z}_{\delta(p)}|) + |\mathbf{z}_{\delta(p)}|\sum_{i=1}^{p-1}|\mathbf{x}_{\delta(i)}| \\
&\le |\mathbf{x}_{\delta(1)}|(1 - |\mathbf{z}_{\delta(p)}| - (p-1)|\mathbf{z}_{\delta(p)}|) + |\mathbf{z}_{\delta(p)}|\sum_{i=1}^{p-1}|\mathbf{x}_{\delta(i)}|,
\end{aligned}
$$

where we use the constraint of $\|\mathbf{z}\|_{max_p} \leq 1$. So

$$\begin{aligned}\mathbf{x}^T\mathbf{z} &\leq |\mathbf{x}_{\delta(1)}|(1 - p|\mathbf{z}_{\delta(p)}|) + |\mathbf{z}_{\delta(p)}| \sum_{i=1}^{p-1} |\mathbf{x}_{\delta(i)}| + |\mathbf{z}_{\delta(p)}| \sum_{i=p}^{n} |\mathbf{x}_{\delta(i)}| \\ &= |\mathbf{x}_{\delta(1)}| + |\mathbf{z}_{\delta(p)}| \left(\sum_{i=1}^{n} |\mathbf{x}_{\delta(i)}| - p|\mathbf{x}_{\delta(1)}| \right) \\ &= \|\mathbf{x}\|_\infty + |\mathbf{z}_{\delta(p)}| \left(\|\mathbf{x}\|_1 - p\|\mathbf{x}\|_\infty \right).\end{aligned}$$

From $\|\mathbf{z}\|_{max_p} \leq 1$, we have $0 \leq |\mathbf{z}_{\delta(p)}| \leq \frac{1}{p}$.

If $\|\mathbf{x}\|_1 \geq p\|\mathbf{x}\|_\infty$, the maximal value is obtained at $|\mathbf{z}_{\delta(p)}| = \frac{1}{p}$ and $\mathbf{x}^T\mathbf{z} \leq \frac{1}{p}\|\mathbf{x}\|_1$. When $\mathbf{z}_i = \frac{1}{p}\mathrm{sgn}(\mathbf{x}_i), \forall i = 1, \cdots, n$, the equality holds.

If $\|\mathbf{x}\|_1 < p\|\mathbf{x}\|_\infty$, the maximal value is obtained at $\mathbf{z}_{\delta(p)} = 0$ and $\mathbf{x}^T\mathbf{z} \leq \|\mathbf{x}\|_\infty$. When $\mathbf{z}_{\delta(1)} = \mathrm{sgn}(\mathbf{x}_{\delta(1)})$ and $\mathbf{z}_{\delta(i)} = 0, \forall i = 2, \cdots, n$, the equality holds.

So we have $\|\mathbf{x}\|^*_{max_p} = \max\left\{ \|\mathbf{x}\|_\infty, \frac{1}{p}\|\mathbf{x}\|_1 \right\}$.

Now consider $\|\mathbf{X}\|^*_{\infty,max_p}$, where $\|\mathbf{X}\|^*_{\infty,max_p} = \max_{\|\mathbf{Z}\|_{\infty,max_p} \leq 1} tr(\mathbf{X}^T\mathbf{Z})$ from the definition of Fenchel dual.

$$tr(\mathbf{X}^T\mathbf{Z}) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\mathbf{X}_{i,j}||\mathbf{Z}_{i,j}| \leq \sum_{i=1}^{n} \|\mathbf{X}_{i,:}\|^*_{max_p}.$$

When $\|\mathbf{Z}_{i,:}\|_{max_p} = 1, \forall i = 1, \cdots, n$, the equality holds. ■

3. Proof in Section 4

The KKT conditions:

$$\mathbf{x}_i - \mathbf{z}_i + \alpha_i + \theta - \beta_i = 0, \tag{8}$$

$$\alpha_i \geq 0, \quad \mathbf{x}_i \leq t, \quad \langle \alpha_i, \mathbf{x}_i - t \rangle = 0, \tag{9}$$

$$\theta \geq 0, \quad \textstyle\sum_{i=1}^n \mathbf{x}_i \leq pt, \quad \langle \theta, \sum_{i=1}^n \mathbf{x}_i - pt \rangle = 0, \tag{10}$$

$$\beta_i \geq 0, \quad \mathbf{x}_i \geq 0, \quad \langle \beta_i, \mathbf{x}_i \rangle = 0. \tag{11}$$

Theorem 4 *Let $\{\mathbf{x}, \alpha, \theta, \beta\}$ be the KKT point, $s = num(\mathbf{z}_i \geq t)$, then we have*

1. *If $\|\mathbf{z}\|_\infty \leq t$ and $\|\mathbf{z}\|_1 \leq pt$, then $\mathbf{x} = \mathbf{z}$.*

2. *If $\|\mathbf{z}\|_\infty > t$ and $\|\mathbf{z}\|_1 \leq pt$, then $\mathbf{x}_j = t$ if $\mathbf{z}_j > t$; $\mathbf{x}_j = \mathbf{z}_i$ if $\mathbf{z}_j \leq t$. And we have $p > s$.*

3. *If $\|\mathbf{z}\|_\infty \leq t$ and $\|\mathbf{z}\|_1 > pt$, then $\mathbf{x}_j = \mathbf{z}_j - \theta$ if $\mathbf{z}_j > \theta$; $\mathbf{x}_j = 0$ if $\mathbf{z}_j \leq \theta$. $\sum_{\mathbf{z}_j > \theta}(\mathbf{z}_j - \theta) = pt$ and $p \leq num(\mathbf{z}_j > \theta)$.*

4. *If $\|\mathbf{z}\|_\infty > t$ and $\|\mathbf{z}\|_1 > pt$, then $\mathbf{x}_j = t$ if $\mathbf{z}_j - \theta \geq t$; $\mathbf{x}_j = \mathbf{z}_j - \theta$ if $0 < \mathbf{z}_j - \theta < t$; $\mathbf{x}_j = 0$ if $\mathbf{z}_j \leq \theta$. Specially,*

(a) $\mathbf{z}_p - \mathbf{z}_{p+1} \geq t$, *then* $\mathbf{x}_j = t, \forall j \in [1, p]$; $\mathbf{x}_j = 0, \forall j \in [p+1, n]$.

(b) $\mathbf{z}_p - \mathbf{z}_{p+1} < t$ *and* $st + \sum_{\mathbf{z}_i < t} \mathbf{z}_i \leq pt$, *then* $\theta = 0$. $\mathbf{x}_j = t$ *if* $\mathbf{z}_j \geq t$; $\mathbf{x}_j = \mathbf{z}_j$ *if* $\mathbf{z}_j < t$. *And we have* $p > s$.

(c) $\mathbf{z}_p - \mathbf{z}_{p+1} < t$ *and* $st + \sum_{\mathbf{z}_i < t} \mathbf{z}_i > pt$, *then* $\theta > 0$. $\mathbf{x}_j = t$ *if* $\mathbf{z}_j - \theta \geq t$; $\mathbf{x}_j = \mathbf{z}_j - \theta$ *if* $0 < \mathbf{z}_j - \theta < t$; $\mathbf{x}_j = 0$ *if* $\mathbf{z}_j \leq \theta$. $num(\mathbf{z}_i - \theta \geq t) \times t + \sum_{0 < \mathbf{z}_i - \theta < t} (\mathbf{z}_i - \theta) = pt$, $num(\mathbf{z}_i - \theta \geq t) < p < num(\mathbf{z}_i > \theta)$.

Moreover, $\sum_{i=1}^n \alpha_i + p\theta = \sum_{i=1}^p (\mathbf{z}_i - \mathbf{x}_i)$.

Proof If $\mathbf{x}_i > 0$, then $\beta_i = 0$ and $\mathbf{x}_i = \mathbf{z}_i - \alpha_i - \theta \leq \mathbf{z}_i$ from (11) and (8). If $\mathbf{x}_i = 0$, we also have $\mathbf{x}_i \leq \mathbf{z}_i$. So $\mathbf{x}_i \leq \mathbf{z}_i, \forall i$.

Case 1: $\|\mathbf{z}\|_\infty \leq t$ and $\|\mathbf{z}\|_1 \leq pt$.

If there exists j such that $\mathbf{x}_j < \mathbf{z}_j$, consider two cases: (1). If $\mathbf{x}_j > 0$, then $\beta_j = 0$ and $\mathbf{x}_j = \mathbf{z}_j - \alpha_j - \theta$. So we have $\alpha_j > 0$ or $\theta > 0$. If $\alpha_j > 0$, then $\mathbf{x}_j = t$ from (9). So $t < \mathbf{z}_j$, which contradicts with $\|\mathbf{z}\|_\infty \leq t$. If $\theta > 0$, then $\sum_{i=1}^n \mathbf{x}_i = pt$ from (10). Since $\mathbf{x}_i \leq \mathbf{z}_i, \forall i$ and $\mathbf{x}_j < \mathbf{z}_j$, we have $\sum_{i=1}^n \mathbf{x}_i < \sum_{i=1}^n \mathbf{z}_i$. So $pt < \sum_{i=1}^n \mathbf{z}_i$, which contradicts with $\|\mathbf{z}\|_1 \leq pt$. (2). If $\mathbf{x}_j = 0$, then $\alpha_j = 0$ from (9) and $\theta = \mathbf{z}_j + \beta_j$ from (8). Since $\mathbf{z}_j > \mathbf{x}_j = 0$, so $\theta > 0$ and $\sum_{i=1}^n \mathbf{x}_i = pt$ from (10). So $pt < \sum_{i=1}^n \mathbf{z}_i$, which contradicts with $\|\mathbf{z}\|_1 \leq pt$. Thus we have $\mathbf{x}_i = \mathbf{z}_i, \forall i$.

Then we prove $\sum_{i=1}^n \alpha_i + p\theta = \sum_{i=1}^p (\mathbf{z}_i - \mathbf{x}_i)$. Since $\mathbf{x}_i = \mathbf{z}_i, \forall i$, we only need to prove $\theta = 0$ and $\alpha_i = 0, \forall i$.

If there exists some $\mathbf{x}_j$ such that $\mathbf{x}_j > 0$, then $\beta_j = 0$ and $\alpha_j + \theta = \mathbf{z}_j - \mathbf{x}_j = 0$ from (11) and (8). So $\theta = 0$ and $\alpha_j = 0$. For $\mathbf{x}_i = 0$, if exits, then $\alpha_i = 0$ from (9). So we have $\theta = 0$ and $\alpha_i = 0, \forall i$.

If $\mathbf{x}_i = 0, \forall i$, then $\alpha_i = 0$ and $0 = \sum_{i=1}^n \mathbf{x}_i < pt$, so $\theta = 0$.

Case 2: $\|\mathbf{z}\|_\infty > t$ and $\|\mathbf{z}\|_1 \leq pt$.

(1). If $\mathbf{x}_j = 0$, then $\alpha_j = 0$ and $\theta = \mathbf{z}_j + \beta_j$. If $\mathbf{z}_j > 0$, then $\theta > 0$ and $\sum_{i=1}^n \mathbf{x}_i = pt$. So $pt < \sum_{i=1}^n \mathbf{z}_i$, which contradicts with $\|\mathbf{z}\|_1 \leq pt$. Thus we have $\mathbf{z}_j = 0$.

(2). If $\mathbf{x}_j > 0$, then $\beta_j = 0$ and $\mathbf{x}_j = \mathbf{z}_j - \alpha_j - \theta \leq \mathbf{z}_j$.

If $\theta > 0$, then $\sum_{i=1}^n \mathbf{x}_i = pt$ and $\mathbf{x}_j < \mathbf{z}_j$. Since $\mathbf{x}_i \leq \mathbf{z}_i, \forall i$, so $pt = \sum_{i=1}^n \mathbf{x}_i < \sum_{i=1}^n \mathbf{z}_i$, which contradicts with $\|\mathbf{z}\|_1 \leq pt$. So $\theta = 0$. Then $\mathbf{x}_j = \mathbf{z}_j - \alpha_j$. (a). Consider case $\mathbf{z}_j \leq t$. If $\mathbf{x}_j \neq \mathbf{z}_j$, then $\mathbf{x}_j < \mathbf{z}_j$ and $\alpha_j > 0$, so $\mathbf{x}_j = t$, which contradicts with $\mathbf{x}_j < \mathbf{z}_j \leq t$. So $\mathbf{x}_j = \mathbf{z}_j$. (b). Consider case $\mathbf{z}_j > t$. If $\mathbf{x}_j \neq t$, then $\alpha_j = 0$ and $\mathbf{x}_j = \mathbf{z}_j > t$, which contradicts with $\mathbf{x}_j \leq t$. So $\mathbf{x}_j = t$.

Since $\|\mathbf{z}\|_\infty > t$, then there exists $\mathbf{x}_j = t < \mathbf{z}_j$. So $pt \geq \|\mathbf{z}\|_1 > \sum_{i=1}^n \mathbf{x}_i \geq \sum_{\mathbf{z}_i \geq t} t = st$. So $p > s$.

Since $\mathbf{x}_j = t > 0, \forall j \in [1, s]$, then from the above analysis we have $\theta = 0$ and $\alpha_j = \mathbf{z}_j - \mathbf{x}_j, \forall j \in [1, s]$. So we have $\sum_{i=1}^n \alpha_i + p\theta = \sum_{i=1}^n \alpha_i = \sum_{i=1}^s \alpha_i = \sum_{i=1}^s (\mathbf{z}_i - \mathbf{x}_i) = \sum_{i=1}^p (\mathbf{z}_i - \mathbf{x}_i)$, where we use $\alpha_i = 0, \forall i \in [s+1, n]$ since $\mathbf{x}_i = \mathbf{z}_i < t, \forall i \in [s+1, n]$. Specially, $\mathbf{x}_i = \mathbf{z}_i, \forall i \in [s+1, p]$.

Case 3: $\|\mathbf{z}\|_\infty \leq t$ and $\|\mathbf{z}\|_1 > pt$.

(1). If $\mathbf{x}_j > 0$, then $\beta_j = 0$ and $\mathbf{x}_j = \mathbf{z}_j - \alpha_j - \theta \leq \mathbf{z}_j$. If $\alpha_j > 0$, then $\mathbf{x}_j < \mathbf{z}_j$ and $\mathbf{x}_j = t$, which contradicts with $\|\mathbf{z}\|_\infty \leq t$. So $\alpha_j = 0$ and $\mathbf{x}_j = \mathbf{z}_j - \theta$. Moreover, $\mathbf{z}_j - \theta = \mathbf{x}_j > 0$.

(2). If $\mathbf{x}_j = 0$, then $\alpha_j = 0$ and $\mathbf{z}_j - \theta = -\beta_j \leq 0$.

So $\mathbf{x}_j = \mathbf{z}_j - \theta$ if $\mathbf{z}_j - \theta > 0$; $\mathbf{x}_j = 0$ if $\mathbf{z}_j - \theta \leq 0$.

If $\sum_{i=1}^n \mathbf{x}_i < pt$, then $\theta = 0$. From the above analysis we have $\mathbf{x}_j = \mathbf{z}_j$ if $\mathbf{z}_j > 0$ and $\mathbf{x}_j = 0$ if $\mathbf{z}_j = 0$. So $pt > \sum_{i=1}^n \mathbf{x}_i = \sum_{i=1}^n \mathbf{z}_i$, which contradicts with $\|\mathbf{z}\|_1 > pt$. Thus $\sum_{i=1}^n \mathbf{x}_i = pt$.

Let $d = \mathrm{num}(\mathbf{z}_i > \theta)$. Since $\mathbf{z}_1 \geq \mathbf{z}_2 \cdots \geq \mathbf{z}_n$, then $\mathbf{x}_j = \mathbf{z}_j - \theta$ and $\mathbf{z}_j - \theta > 0, \forall 1 \leq j \leq d$; $\mathbf{x}_j = 0, \forall j > d$.

$pt = \sum_{i=1}^n \mathbf{x}_i = \sum_{i=1}^d \mathbf{x}_i = \sum_{i=1}^d (\mathbf{z}_i - \theta) \leq \sum_{i=1}^d \mathbf{z}_i \leq \sum_{i=1}^d t = dt$, where we use $\|\mathbf{z}\|_\infty \leq t$. So $p \leq d$ and $\mathbf{x}_j = \mathbf{z}_j - \theta, \forall j \in [1, p]$. So $\sum_{i=1}^n \alpha_i + p\theta = p\theta = \sum_{i=1}^p (\mathbf{z}_i - \mathbf{x}_i)$, where we use $\alpha_i = 0, \forall i$ from the above analysis.

Case 4:$\|\mathbf{z}\|_\infty > t$ and $\|\mathbf{z}\|_1 > pt$.

(1). If $\mathbf{x}_j > 0$, then $\beta_j = 0$ and $\mathbf{x}_j = \mathbf{z}_j - \alpha_j - \theta \leq \mathbf{z}_j$.

Consider case $\mathbf{z}_j - \theta > t$. Since $\mathbf{x}_j \leq t$, then $\alpha_j > 0$ and $\mathbf{x}_j = t$.

Consider case $\mathbf{z}_j - \theta = t$. If $\mathbf{x}_j < t$, then from $\mathbf{x}_j = \mathbf{z}_j - \alpha_j - \theta$ we have $\alpha_j > 0$, so $\mathbf{x}_j = t$, which contradicts with $\mathbf{x}_j < t$. So we have $\mathbf{x}_j = t$.

Consider case $0 < \mathbf{z}_j - \theta < t$, then $\mathbf{x}_j = \mathbf{z}_j - \theta - \alpha_j < t$, so $\alpha_j = 0$ and $\mathbf{x}_j = c_j - \theta$.

Consider case $\mathbf{z}_j \leq \theta$, then $\mathbf{x}_j = \mathbf{z}_j - \theta - \alpha_j \leq 0$, which contradicts with the case of $\mathbf{x}_j > 0$.

(2). If $\mathbf{x}_j = 0$, then $\alpha_j = 0$ and $\mathbf{z}_j - \theta = -\beta_j \leq 0$.

So $\mathbf{x}_j = t$ for $\mathbf{z}_j - \theta \geq t$, $\mathbf{x}_j = \mathbf{z}_j - \theta$ for $0 < \mathbf{z}_j - \theta < t$, $\mathbf{x}_j = 0$ for $\mathbf{z}_j \leq \theta$.

Then we consider three subcases in details.

Subcase 1: $\mathbf{z}_p - \mathbf{z}_{p+1} \geq t$.

Since $pt \geq \sum_{i=1}^n \mathbf{x}_i$, $\mathbf{x}_1 \geq \mathbf{x}_2 \geq \cdots \geq \mathbf{x}_n$ and $\mathbf{x}_j$ can only take t, $\mathbf{z}_j - \theta$ and 0, then the values of $\mathbf{x}_p$ and $\mathbf{x}_{p+1}$ have only four cases: (a) $\mathbf{x}_p = t$ and $\mathbf{x}_{p+1} = 0$. (b) $\mathbf{x}_p = \mathbf{z}_p - \theta$ and $\mathbf{x}_{p+1} = \mathbf{z}_{p+1} - \theta$. (c) $\mathbf{x}_p = \mathbf{z}_p - \theta$ and $\mathbf{x}_{p+1} = 0$ (d) $\mathbf{x}_p = 0$ and $\mathbf{x}_{p+1} = 0$. The following two cases cannot happen since $pt \geq \sum_{i=1}^n \mathbf{x}_i$: (e) $\mathbf{x}_{p+1} = t$ and (f) $\mathbf{x}_p = t$, $\mathbf{x}_{p+1} = \mathbf{z}_{p+1} - \theta > 0$.

For the first case, we have $\mathbf{z}_p - \theta \geq t$ and $\mathbf{z}_{p+1} \leq \theta$, so $\mathbf{z}_p - \mathbf{z}_{p+1} \geq t$.

For the second case, we have $0 < \mathbf{z}_p - \theta < t$ and $0 < \mathbf{z}_{p+1} - \theta < t$, so we have $\mathbf{z}_p - \mathbf{z}_{p+1} < t$, which contradicts with the assumption.

For the third and forth case, since $\mathbf{x}_i \leq t, \forall i$, $\mathbf{x}_p < t$ and $\mathbf{x}_i = 0, \forall j \geq p+1$, then $\sum_{i=1}^n \mathbf{x}_i < pt$ and $\theta = 0$. For the third case, we have $c < \mathbf{z}_p = \mathbf{z}_p - \theta < t$ and $0 \leq \mathbf{z}_{p+1} \leq \theta = 0$, which contradicts with $\mathbf{z}_p - \mathbf{z}_{p+1} \geq t$. For the forth case, we have $0 \leq \mathbf{z}_p \leq \theta = 0$ and $0 \leq \mathbf{z}_{p+1} \leq \theta = 0$, which contradicts with the $\mathbf{z}_p - \mathbf{z}_{p+1} \geq t$.

So we have $\mathbf{x}_i = t, \mathbf{z}_i - \theta \geq t, \forall i \in [1, p]$, $\mathbf{x}_i = 0, \mathbf{z}_i \leq \theta, \forall i \in [p+1, n]$. So $\alpha_i = 0, \forall i \in [p+1, n]$ and $\beta_i = 0, \mathbf{x}_i - \mathbf{z}_i + \alpha_i + \theta = 0, \forall i \in [1, p]$. So $\sum_{i=1}^n \alpha_i + p\theta = \sum_{i=1}^p (\alpha_i + \theta) = \sum_{i=1}^p (\mathbf{z}_i - \mathbf{x}_i)$.

Subcase 2: $\mathbf{z}_p - \mathbf{z}_{p+1} < t$ and $st + \sum_{\mathbf{z}_i < t} \mathbf{z}_i \leq pt$.

From $st + \sum_{\mathbf{z}_i < t} \mathbf{z}_i \leq pt$ we know $s \leq p$. If $s = p$, then there exists no $\mathbf{z}_i$ such that $0 < \mathbf{z}_i < t$. Since $\mathbf{z}_s \geq t$ and $\mathbf{z}_{s+1} < t$ from the definition of s, then $\mathbf{z}_{s+1} = 0$. So $\mathbf{z}_p \geq t$ and $\mathbf{z}_{p+1} = 0$, which contradicts with $\mathbf{z}_p - \mathbf{z}_{p+1} < t$. So $s < p$.

If $\theta > 0$, then $\sum_{i=1}^n \mathbf{x}_i = pt$. Since $\mathbf{x}_i \leq \mathbf{z}_i$ and $\mathbf{x}_i \leq t$, then $pt \geq st + \sum_{\mathbf{z}_i < t} \mathbf{z}_i = st + \sum_{i=s+1}^n \mathbf{z}_i \geq \sum_{i=1}^s \mathbf{x}_i + \sum_{i=s+1}^n \mathbf{x}_i = \sum_{i=1}^n \mathbf{x}_i = pt$. So the equalities hold and $st + \sum_{\mathbf{z}_i < t} \mathbf{z}_i = pt$, $\mathbf{x}_i = t, \forall i \leq s$, $\mathbf{x}_i = \mathbf{z}_i, \forall i > s$. Since $s < p$ and $pt = \sum_{i=1}^n \mathbf{x}_i = st + \sum_{i=s+1}^n \mathbf{z}_i$, then $\mathbf{z}_{s+1} > 0$. So $\mathbf{x}_{s+1} = \mathbf{z}_{s+1} \in (0, t)$, then $\alpha_{s+1} = 0$ and $\beta_{s+1} = 0$. So $\theta = 0$ from (8), which contradicts with the assumption $\theta > 0$. So $\theta = 0$. Then $\mathbf{x}_i = t$ for $\mathbf{z}_i \geq t$, $\mathbf{x}_i = \mathbf{z}_i$ for $\mathbf{z}_i < t$. That is, $\mathbf{x}_i = t, \forall i \in [1, s]$, $\mathbf{x}_i = \mathbf{z}_i < t, \forall i \in [s+1, n]$.

Since $\alpha_i = 0, \forall i \in [s+1, n]$, $\beta_i = 0, \mathbf{x}_i - \mathbf{z}_i + \alpha_i = 0, \forall i \in [1, s]$, $p > s$ and $\mathbf{x}_i = \mathbf{z}_i, \forall i \in [s+1, n]$, then $\sum_{i=1}^n \alpha_i + p\theta = \sum_{i=1}^s \alpha_i = \sum_{i=1}^s (\mathbf{z}_i - \mathbf{x}_i) = \sum_{i=1}^p (\mathbf{z}_i - \mathbf{x}_i)$.

Subcase 3: $\mathbf{z}_p - \mathbf{z}_{p+1} < t$ and $st + \sum_{\mathbf{z}_i < t} \mathbf{z}_i > pt$.

If $\theta = 0$, then $\mathbf{x}_i = t$ if $\mathbf{z}_i \geq t$, $\mathbf{x}_i = \mathbf{z}_i$ if $0 < \mathbf{z}_i < t$, $\mathbf{x}_i = 0$ if $\mathbf{z}_i = 0$. So $pt \geq \sum_{i=1}^n \mathbf{x}_i = st + \sum_{\mathbf{z}_i < t} \mathbf{z}_i$, which contradicts with the assumption. So $\theta > 0$ and $pt = \sum_{i=1}^n \mathbf{x}_i = \text{num}(\mathbf{z}_i - \theta \geq t) \times t + \sum_{0 < \mathbf{z}_i - \theta < t} (\mathbf{z}_i - \theta)$.

Let $d = \text{num}(\mathbf{x}_i > 0) = \text{num}(\mathbf{z}_i > \theta)$ and $r = \text{num}(\mathbf{z}_i - \theta \geq t)$. Then $\mathbf{x}_i = t$, $\mathbf{z}_i - \theta \geq t, \forall i \leq r$; $\mathbf{x}_i = \mathbf{z}_i - \theta$, $0 < \mathbf{z}_i - \theta < t, \forall r < i \leq d$; $\mathbf{x}_i = 0$, $\mathbf{z}_i \leq \theta, \forall i > d$. Notice that in this case r can be 0, d can be n.

If $r = d$, then $\mathbf{x}_r = t$ and $\mathbf{x}_{r+1} = 0$. Since $pt = \sum_{i=1}^n \mathbf{x}_i$, then $p = r$, $\mathbf{x}_p = t$ and $\mathbf{x}_{p+1} = 0$. So $\mathbf{z}_p - \theta \geq t$ and $\mathbf{z}_{p+1} \leq \theta$. So $\mathbf{z}_p - \mathbf{z}_{p+1} \geq t$, which contradicts with the assumption. So $r < d$.

Since $rt < rt + \sum_{i=r+1}^d (\mathbf{z}_i - \theta) < rt + \sum_{i=r+1}^d t = dt$, then $r < p < d$.

Since $\alpha_i = 0, \forall i \in [r+1, n]$ and $\beta_i = 0, \forall i \in [1, d]$, then $\sum_{i=1}^n \alpha_i + p\theta = \sum_{i=1}^r \alpha_i + p\theta = \sum_{i=1}^r (\mathbf{z}_i - \mathbf{x}_i - \theta) + p\theta = \sum_{i=1}^r (\mathbf{z}_i - \mathbf{x}_i) + (p-r)\theta = \sum_{i=1}^r (\mathbf{z}_i - \mathbf{x}_i) + \sum_{r+1}^p \theta = \sum_{i=1}^r (\mathbf{z}_i - \mathbf{x}_i) + \sum_{r+1}^p (\mathbf{z}_i - \mathbf{x}_i) = \sum_{i=1}^p (\mathbf{z}_i - \mathbf{x}_i)$, where we use $p < d$ and $\mathbf{x}_i = \mathbf{z}_i - \theta$ for $i \in [r+1, p]$. ■

Lemma 5 *In case 3, let* $h(\theta) = \sum_{\mathbf{z}_i > \theta} (\mathbf{z}_i - \theta)$, $\theta \in [0, \mathbf{z}_1)$, $\mathbf{z}_{n+1} = 0$ *then*

$$h(\theta) = \sum_{i=1}^k \mathbf{z}_i - k\theta, \quad \theta \in [\mathbf{z}_{k+1}, \mathbf{z}_k), \forall k = n, \cdots, 1.$$

and $h(\theta) \in (0, \|\mathbf{z}\|_1]$ *is continuous, piecewise linear and strictly decreasing. Thus there is a unique solution for* $h(\theta) = pt$.

Proof Since $\mathbf{z}_1 \geq \mathbf{z}_2 \geq \cdots \geq \mathbf{z}_n$, then $h(\theta) = \sum_{\mathbf{z}_i > \theta} (\mathbf{z}_i - \theta) = \sum_{i=1}^k (\mathbf{z}_i - \theta)$ if $\theta \in [\mathbf{z}_{k+1}, \mathbf{z}_k)$. So $\lim_{\theta \xrightarrow{+} \mathbf{z}_k} = \sum_{i=1}^k (\mathbf{z}_i - \mathbf{z}_k) = \sum_{i=1}^{k-1} (\mathbf{z}_i - \mathbf{z}_k) = h(\mathbf{z}_k)$. Thus $h(\theta) \in (0, \|\mathbf{z}\|_1]$ is continuous, piecewise linear and strictly decreasing. ■

Lemma 6 *Let* $d + k = \max\{i : \mathbf{z}_i = \mathbf{z}_{d+1}\}$, $r + j = \max\{i : \mathbf{z}_i = \mathbf{z}_{r+1}\}$, $k^* = \max\{i : \mathbf{z}_i = \mathbf{z}_1\}$, $\mathbf{z}_{n+1} = 0$, $\mathbf{z}_0 = \infty$. *Define interval*

$$S(r, d) = (\max\{\mathbf{z}_{d+1}, \mathbf{z}_{r+1} - t\}, \min\{\mathbf{z}_d, \mathbf{z}_r - t\}].$$

Go left from nonempty $S(0, k^*) = (\max\{\mathbf{z}_{k^*+1}, \mathbf{z}_1 - t\}, \mathbf{z}_1]$ *and end when* $S(r, d)$ *reaches 0. For nonempty* $S(r, d)$,

1. *If* $\mathbf{z}_{d+1} < \mathbf{z}_{r+1} - t < \min\{\mathbf{z}_d, \mathbf{z}_r - t\}$, *then* $S(r + j, d)$ *is on the left hand side of* $S(r, d)$ *and* $S(r + j, d)$ *is nonempty.*

2. *If* $\mathbf{z}_{r+1} - t < \mathbf{z}_{d+1} < \min\{\mathbf{z}_d, \mathbf{z}_r - t\}$, *then* $S(r, d + k)$ *is on the left hand side of* $S(r, d)$ *and* $S(r, d + k)$ *is nonempty.*

3. *If* $\mathbf{z}_{r+1} - t = \mathbf{z}_{d+1} < \min\{\mathbf{z}_d, \mathbf{z}_r - t\}$*, then* $S(r+j, d+k)$ *is on the left hand side of* $S(r,d)$ *and* $S(r+j, d+k)$ *is nonempty.*

The union of the constructed disjoint intervals is $[0, \mathbf{z}_1]$*.*

Proof $S(r,d)$ is nonempty, so we can consider three cases:

If $\mathbf{z}_{d+1} < \mathbf{z}_{r+1} - t < \min\{\mathbf{z}_d, \mathbf{z}_r - t\}$, then $S(r,d) = (\mathbf{z}_{r+1} - t, \min\{\mathbf{z}_d, \mathbf{z}_r - t\}]$,

$$\begin{aligned} S(r+j, d) &= (\max\{\mathbf{z}_{d+1}, \mathbf{z}_{r+j+1} - t\}, \min\{\mathbf{z}_d, \mathbf{z}_{r+j} - t\}] \\ &= (\max\{\mathbf{z}_{d+1}, \mathbf{z}_{r+j+1} - t\}, \min\{\mathbf{z}_d, \mathbf{z}_{r+1} - t\}] \\ &= (\max\{\mathbf{z}_{d+1}, \mathbf{z}_{r+j+1} - t\}, \mathbf{z}_{r+1} - t], \end{aligned}$$

and $S(r+j, d)$ is nonempty from the definition of $r+j$.

If $\mathbf{z}_{r+1} - t < \mathbf{z}_{d+1} < \min\{\mathbf{z}_d, \mathbf{z}_r - t\}$, then $S(r,d) = (\mathbf{z}_{d+1}, \min\{\mathbf{z}_d, \mathbf{z}_r - t\}]$,

$$\begin{aligned} S(r, d+k) &= (\max\{\mathbf{z}_{d+k+1}, \mathbf{z}_{r+1} - t\}, \min\{\mathbf{z}_{d+k}, \mathbf{z}_r - t\}] \\ &= (\max\{\mathbf{z}_{d+k+1}, \mathbf{z}_{r+1} - t\}, \min\{\mathbf{z}_{d+1}, \mathbf{z}_r - t\}] \\ &= (\max\{\mathbf{z}_{d+k+1}, \mathbf{z}_{r+1} - t\}, \mathbf{z}_{d+1}], \end{aligned}$$

and $S(r, d+k)$ is nonempty from the definition of $d+k$.

If $\mathbf{z}_{r+1} - t = \mathbf{z}_{d+1} < \min\{\mathbf{z}_d, \mathbf{z}_r - t\}$, then $S(r,d) = (\mathbf{z}_{r+1} - t, \min\{\mathbf{z}_d, \mathbf{z}_r - t\}] = (\mathbf{z}_{d+1}, \min\{\mathbf{z}_d, \mathbf{z}_r - t\}]$,

$$\begin{aligned} S(r+j, d+k) &= (\max\{\mathbf{z}_{d+k+1}, \mathbf{z}_{r+j+1} - t\}, \min\{\mathbf{z}_{d+k}, \mathbf{z}_{r+j} - t\}] \\ &= (\max\{\mathbf{z}_{d+k+1}, \mathbf{z}_{r+j+1} - t\}, \min\{\mathbf{z}_{d+1}, \mathbf{z}_{r+1} - t\}] \\ &= (\max\{\mathbf{z}_{d+k+1}, \mathbf{z}_{r+j+1} - t\}, \mathbf{z}_{r+1} - t] \\ &= (\max\{\mathbf{z}_{d+k+1}, \mathbf{z}_{r+j+1} - t\}, \mathbf{z}_{d+1}], \end{aligned}$$

and $S(r+j, d+k)$ is nonempty.

As $\mathbf{z}_{n+1} = 0$, so $S(r,d)$ can reach 0. ■

Lemma 7 *In case 4.3, let* $\mathbf{z}_{n+1} = 0$*,* $\mathbf{z}_{n+2} < 0$*,* $\mathbf{z}_0 = \infty$*,* $h(\theta) = num(\mathbf{z}_i - \theta \geq t) \times t + \sum_{0<\mathbf{z}_i - \theta < t}(\mathbf{z}_i - \theta)$*. Consider* $S(r,d)$ *constructed in Lemma 6, then*

$$h(\theta) = rt + \sum_{i=r+1}^{d} \mathbf{z}_i - (d-r)\theta, \quad \theta \in S(r,d).$$

$h(\theta), \theta \in [0, \mathbf{z}_1]$ *is continuous, piecewise linear, non-increasing and there is a unique solution for* $h(\theta) = pt$*.*

Proof

$\theta \in S(r,d) \Rightarrow \theta \in (\mathbf{z}_{d+1}, \mathbf{z}_d]$, $\theta \in (\mathbf{z}_{r+1} - t, \mathbf{z}_r - t]$, so

$$\begin{aligned} h(\theta) &= \text{num}(\mathbf{z}_i - \theta \geq t) \times t + \sum_{0<\mathbf{z}_i-\theta<t} (\mathbf{z}_i - \theta) \\ &= \text{num}(\mathbf{z}_i - \theta \geq t) \times t + \sum_{0\leq\mathbf{z}_i-\theta<t} (\mathbf{z}_i - \theta) \\ &= rt + \sum_{i=r+1}^{d} (\mathbf{z}_i - \theta) = rt + \sum_{i=r+1}^{d} \mathbf{z}_i - (d-r)\theta. \end{aligned}$$

Since S is nonempty, then $\mathbf{z}_{d+1} \leq \mathbf{z}_r - t < \mathbf{z}_r \Rightarrow r \leq d$. Thus $h(\theta), \theta \in S(r,d)$ is a linear strictly decreasing function if $d > r$ and the constant rt if $r = d$.

Now we prove that $h(\theta)$ is continuous when $\theta \in [0, \mathbf{z}_1]$.

If $\mathbf{z}_{d+1} < \mathbf{z}_{r+1} - t < \min\{\mathbf{z}_d, \mathbf{z}_r - t\}$, then $S(r,d) = (\mathbf{z}_{r+1} - t, \min\{\mathbf{z}_d, \mathbf{z}_r - t\}]$ and $S(r+j,d) = (\max\{\mathbf{z}_{d+1}, \mathbf{z}_{r+j+1} - t\}, \mathbf{z}_{r+1} - t]$ is on the left hand side of $S(r,d)$.

$$\begin{aligned} \lim_{\theta \xrightarrow{-} \mathbf{z}_{r+1}-t} h(\theta) &= rt + \sum_{i=r+1}^{d} (\mathbf{z}_i - (\mathbf{z}_{r+1} - t)) \\ &= (r+j)t + \sum_{i=r+j+1}^{d} (\mathbf{z}_i - (\mathbf{z}_{r+1} - t)) - jt + \sum_{i=r+1}^{r+j} (\mathbf{z}_i - (\mathbf{z}_{r+1} - t)) \\ &= (r+j)t + \sum_{i=r+j+1}^{d} (\mathbf{z}_i - (\mathbf{z}_{r+1} - t)) = h(\mathbf{z}_{r+1} - t). \end{aligned}$$

where $\mathbf{z}_i = \mathbf{z}_{r+1}, \forall i \in [r+1, r+j]$ from the definition of $r+j$.

If $\mathbf{z}_{r+1} - t < \mathbf{z}_{d+1} < \min\{\mathbf{z}_d, \mathbf{z}_r - t\}$, then $S(r,d) = (\mathbf{z}_{d+1}, \min\{\mathbf{z}_d, \mathbf{z}_r - t\}]$ and $S(r, d+k) = (\max\{\mathbf{z}_{d+k+1}, \mathbf{z}_{r+1} - t\}, \mathbf{z}_{d+1}]$ is on the left hand side of $S(r,d)$.

$$\begin{aligned} \lim_{\theta \xrightarrow{-} \mathbf{z}_{d+1}} h(\theta) &= rt + \sum_{i=r+1}^{d} (\mathbf{z}_i - \mathbf{z}_{d+1}) = rt + \sum_{i=r+1}^{d+k} (\mathbf{z}_i - \mathbf{z}_{d+1}) - \sum_{i=d+1}^{d+k} (\mathbf{z}_i - \mathbf{z}_{d+1}) \\ &= rt + \sum_{i=r+1}^{d+k} (\mathbf{z}_i - \mathbf{z}_{d+1}) = h(\mathbf{z}_{d+1}). \end{aligned}$$

where $\mathbf{z}_i = \mathbf{z}_{d+1}, \forall i \in [d+1, d+k]$ from the definition of $d+k$.

If $\mathbf{z}_{r+1} - t = \mathbf{z}_{d+1} < \min\{\mathbf{z}_d, \mathbf{z}_r - t\}$, then $S(r,d) = (\mathbf{z}_{r+1} - t, \min\{\mathbf{z}_d, \mathbf{z}_r - t\}] = (\mathbf{z}_{d+1}, \min\{\mathbf{z}_d, \mathbf{z}_r - t\}]$ and $S(r+j, d+k) = (\max\{\mathbf{z}_{d+k+1}, \mathbf{z}_{r+j+1} - t\}, \mathbf{z}_{r+1} - t] = (\max\{\mathbf{z}_{d+k+1}, \mathbf{z}_{r+j+1} -$

$t\}, \mathbf{z}_{d+1}]$ is on the left hand side of $S(r, d)$.

$$\begin{aligned}
\lim_{\theta \bar{\rightarrow} \mathbf{z}_{r+1}-t} h(\theta) &= \lim_{\theta \bar{\rightarrow} \mathbf{z}_{d+1}} h(\theta) = rt + \sum_{i=r+1}^{d} (\mathbf{z}_i - \mathbf{z}_{d+1}) \\
&= (r+j)t + \sum_{i=r+j+1}^{d+k} (\mathbf{z}_i - \mathbf{z}_{d+1}) - \sum_{i=d+1}^{d+k} (\mathbf{z}_i - \mathbf{z}_{d+1}) - jt + \sum_{i=r+1}^{r+j} (\mathbf{z}_i - \mathbf{z}_{d+1}) \\
&= (r+j)t + \sum_{i=r+j+1}^{d+k} (\mathbf{z}_i - \mathbf{z}_{d+1}) - \sum_{i=d+1}^{d+k} (\mathbf{z}_i - \mathbf{z}_{d+1}) - jt + \sum_{i=r+1}^{r+j} (\mathbf{z}_i - \mathbf{z}_{r+1} + t) \\
&= (r+j)t + \sum_{i=r+j+1}^{d+k} (\mathbf{z}_i - \mathbf{z}_{d+1}) = h(\mathbf{z}_{d+1}).
\end{aligned}$$

Thus $h(\theta)$ is continuous when $\theta \in [0, \mathbf{z}_1]$.

Now we claim that if $h(\theta)$ is a constant at some interval, then $h(\theta) = rt \neq pt$. Otherwise, $r = p = d$. Since S is nonempty, then $\mathbf{z}_{p+1} \leq \mathbf{z}_p - t$, which contradicts with the assumption $\mathbf{z}_p - \mathbf{z}_{p+1} < t$.

From $0 \in S(r, d) \Rightarrow \mathbf{z}_{d+1} < 0 \leq \mathbf{z}_d, \mathbf{z}_{r+1} - t < 0 \leq \mathbf{z}_r - t$, we have $r = s \equiv \text{num}(\mathbf{z}_i \geq t)$ and $d = n + 1$. So $h(0) = st + \sum_{\mathbf{z}_i < t} \mathbf{z}_i$. From $\mathbf{z}_1 \in S(r, d) \Rightarrow \mathbf{z}_{d+1} < \mathbf{z}_1 \leq \mathbf{z}_d, \mathbf{z}_{r+1} - t < \mathbf{z}_1 \leq \mathbf{z}_r - t$, we have $d = k^* = \max\{i : \mathbf{z}_i = \mathbf{z}_1\}$ and $r = 0$. So $h(\mathbf{z}_1) = \sum_{i=1}^{k^*} (\mathbf{z}_i - \mathbf{z}_1) = 0$. Thus $h(\theta) \in [0, st + \sum_{\mathbf{z}_i < t} \mathbf{z}_i]$. Since $st + \sum_{\mathbf{z}_i < t} \mathbf{z}_i > pt$, then there is a unique solution for $h(\theta) = pt$. ■

4. Proof in Section 5

The Lagrangian function is:

$$\begin{aligned}
L(\mathbf{X}, \mathbf{g}, \alpha, \theta, \beta, \lambda) = & \frac{1}{2} \sum_{i,j} |\mathbf{Z}_{i,j} - \mathbf{X}_{i,j}|^2 + \sum_{i,j} \langle \alpha_{i,j}, \mathbf{X}_{i,j} - \mathbf{g}_i \rangle \\
& + \sum_{i=1}^{n} \left\langle \theta_i, \sum_{j=1}^{n} \mathbf{X}_{i,j} - p\mathbf{g}_i \right\rangle + \left\langle \lambda, \sum_{i=1}^{n} \mathbf{g}_i - T \right\rangle - \sum_{i,j} \langle \beta_{i,j}, \mathbf{X}_{i,j} \rangle .
\end{aligned}$$

and its KKT conditions are:

$$\mathbf{X}_{i,j} - \mathbf{Z}_{i,j} + \alpha_{i,j} - \beta_{i,j} + \theta_i = 0, \tag{12}$$

$$-\sum_j \alpha_{i,j} - p\theta_i + \lambda = 0, \tag{13}$$

$$\sum_{i=1}^{n} \mathbf{g}_i = T, \tag{14}$$

$$\mathbf{X}_{i,j} \leq \mathbf{g}_i, \quad \alpha_{i,j} \geq 0, \quad \langle \alpha_{i,j}, \mathbf{X}_{i,j} - \mathbf{g}_i \rangle = 0, \tag{15}$$

$$\theta_i \geq 0, \quad \sum_{j=1}^{n} \mathbf{X}_{i,j} \leq p\mathbf{g}_i, \quad \left\langle \theta_i, \sum_{j=1}^{n} \mathbf{X}_{i,j} - p\mathbf{g}_i \right\rangle = 0, \tag{16}$$

$$\mathbf{X}_{i,j} \geq 0, \quad \beta_{i,j} \geq 0, \quad \langle \beta_{i,j}, \mathbf{X}_{i,j} \rangle = 0. \tag{17}$$

Lemma 8 *At the optimal solution, either (1)* $\mathbf{g}_i > 0$ *and* $\sum_{j=1}^{p}(\mathbf{Z}_{i,j} - \mathbf{X}_{i,j}) = \lambda$*; or (2)* $\mathbf{g}_i = 0$ *and* $\sum_{j=1}^{p} \mathbf{Z}_{i,j} \leq \lambda$*.*

Proof If $\mathbf{g}_i = 0$, then $\mathbf{X}_{i,j} = 0, \forall j$, so $\alpha_{i,j} + \theta_i = \mathbf{Z}_{i,j} + \beta_{i,j} \geq \mathbf{Z}_{i,j}$.

$$\lambda = \sum_{j=1}^{n} \alpha_{i,j} + p\theta_i \geq \sum_{j=1}^{p} \alpha_{i,j} + p\theta_i = \sum_{j=1}^{p} (\alpha_{i,j} + \theta_i) \geq \sum_{j=1}^{p} \mathbf{Z}_{i,j}.$$

If $\mathbf{g}_i > 0$. Consider (12), (15), (16) and (17), the four conditions are equivalent to minimizing the following problem with fixed $\mathbf{g}_i$:

$$\begin{aligned} &\min_{\mathbf{X}_i} \frac{1}{2} \sum_j |\mathbf{Z}_{i,j} - \mathbf{X}_{i,j}|^2 \\ &s.t. \mathbf{X}_{i,j} \leq \mathbf{g}_i, \forall j, \quad \frac{1}{p} \sum_{j=1}^{n} \mathbf{X}_{i,j} \leq \mathbf{g}_i, \quad \mathbf{X}_{i,j} \geq 0, \forall j. \end{aligned} \tag{18}$$

From Theorem 4, we have $\sum_{j=1}^{n} \alpha_{i,j} + p\theta_i = \sum_{j=1}^{p} (\mathbf{Z}_{i,j} - \mathbf{X}_{i,j})$. So from (13) we have $\lambda = \sum_{j=1}^{p} (\mathbf{Z}_{i,j} - \mathbf{X}_{i,j})$. ■

Lemma 9 *Let* $s = num(\mathbf{z}_i \geq t)$*,* $r = num(\mathbf{z}_i - \theta \geq t)$ *and* $d = num(\mathbf{z}_i > \theta)$ *in case 4.3 of Theorem 4.*

If $\|\mathbf{z}\|_\infty \geq \frac{1}{p}\|\mathbf{z}\|_1$*, then*

$$g(t) = \begin{cases} 0, & t \geq \|\mathbf{z}\|_\infty \\ \sum_{i=1}^{s} \mathbf{z}_i - st, & t^* \leq t < \|\mathbf{z}\|_\infty \\ \sum_{i=1}^{r} \mathbf{z}_i - rt + (p-r)\theta, & \mathbf{z}_p - \mathbf{z}_{p+1} < t < t^* \\ \sum_{i=1}^{p} \mathbf{z}_i - pt, & t \leq \mathbf{z}_p - \mathbf{z}_{p+1} \end{cases}$$

where $t^* \in [\mathbf{z}_p - \mathbf{z}_{p+1}, \|\mathbf{z}\|_1/p]$ *is the unique solution satisfying* $num(\mathbf{z}_i \geq t) + \frac{\sum_{\mathbf{z}_i < t} \mathbf{z}_i}{t} = p$*.*

If $\|\mathbf{z}\|_\infty < \frac{1}{p}\|\mathbf{z}\|_1$, *then*

$$g(t) = \begin{cases} 0, & t \ge \frac{1}{p}\|\mathbf{z}\|_1 \\ p\theta, & \|\mathbf{z}\|_\infty \le t < \frac{1}{p}\|\mathbf{z}\|_1 \\ \sum_{i=1}^{r}\mathbf{z}_i - rt + (p-r)\theta, & \mathbf{z}_p - \mathbf{z}_{p+1} < t < \|\mathbf{z}\|_\infty \\ \sum_{i=1}^{p}\mathbf{z}_i - pt, & t \le \mathbf{z}_p - \mathbf{z}_{p+1} \end{cases}$$

Proof Similar to Theorem 4, we consider four cases.

(1). If $t \ge \|\mathbf{z}\|_\infty$ and $t \ge \frac{1}{p}\|\mathbf{z}\|_1$, then $\mathbf{z} = \mathbf{x}$ and $g(t) = 0$.

(2). If $t < \|\mathbf{z}\|_\infty$ and $t \ge \frac{1}{p}\|\mathbf{z}\|_1$, then $\mathbf{x}_i = t, \forall i \le s$, $\mathbf{x}_i = \mathbf{z}_i, \forall i > s$, and $p > s$. So $g(t) = \sum_{i=1}^{s}\mathbf{z}_i - st$.

(3). If $t \ge \|\mathbf{z}\|_\infty$ and $t < \frac{1}{p}\|\mathbf{z}\|_1$, then $\mathbf{x}_i = \mathbf{z}_i - \theta$ if $\mathbf{z}_i - \theta > 0$, $\mathbf{x}_i = 0$ if $\mathbf{z}_i - \theta \le 0$, And $p \le \mathrm{num}(\mathbf{z}_i > \theta)$. So $g(t) = p\theta$.

(4). If $t < \|\mathbf{z}\|_\infty$ and $t < \frac{1}{p}\|\mathbf{z}\|_1$, consider three cases:

(4a). If $\mathbf{z}_p - \mathbf{z}_{p+1} \ge t$, then $\mathbf{x}_i = t, \forall i \in [1,p]$, $\mathbf{x}_i = 0, \forall i \in [p+1,n]$. So $g(t) = \sum_{i=1}^{p}\mathbf{z}_i - pt$.

(4b). If $\mathbf{z}_p - \mathbf{z}_{p+1} < t$ and $st + \sum_{\mathbf{z}_i < t}\mathbf{z}_i \le pt$, then $\mathbf{x}_i = t, \forall i \le s$, $\mathbf{x}_i = \mathbf{z}_i, \forall i > s$, $p \ge s$. So $g(t) = \sum_{i=1}^{s}\mathbf{z}_i - st$.

(4c). If $\mathbf{z}_p - \mathbf{z}_{p+1} < t$ and $st + \sum_{\mathbf{z}_i < t}\mathbf{z}_i > pt$, then $\mathbf{x}_i = t, \forall i \le r$, $\mathbf{x}_i = \mathbf{z}_i - \theta, \forall r < i \le d$, $\mathbf{x}_i = 0, \forall i > d$, $r < p < d$. So $g(t) = \sum_{i=1}^{r}\mathbf{z}_i - rt + (p-r)\theta$.

Let $h(t) = \mathrm{num}(\mathbf{z}_i \ge t) + \frac{\sum_{\mathbf{z}_i < t}\mathbf{z}_i}{t} = s + \frac{\sum_{i=s+1}^{n}\mathbf{z}_i}{t}$. Recall that $\mathbf{z}_s \ge t$ and $\mathbf{z}_{s+1} < t$. Increase t satisfying $\mathbf{z}_s \ge t$, then $\mathrm{num}(\mathbf{z}_i \ge t)$ and $\sum_{\mathbf{z}_i < t}\mathbf{z}_i$ do not change, so $h(t)$ strictly decrease. Further increase t to t' satisfying $\mathbf{z}_s < t'$ and $t' \le \mathbf{z}_{s-j}$, where we allow repetition to consider $\mathbf{z}_s = \mathbf{z}_{s-1} = \cdots = \mathbf{z}_{s-j+1} < \mathbf{z}_{s-j}$. Then $h(t') = s - j + \frac{\sum_{i=s-j+1}^{n}\mathbf{z}_i}{t'} = s - j + \frac{\sum_{i=s+1}^{n}\mathbf{z}_i + \sum_{i=s-j+1}^{s}\mathbf{z}_i}{t'} = s + \frac{\sum_{i=s+1}^{n}\mathbf{z}_i}{t'} + \frac{j\mathbf{z}_s}{t'} - j < s + \frac{\sum_{i=s+1}^{n}\mathbf{z}_i}{t'} < s + \frac{\sum_{i=s+1}^{n}\mathbf{z}_i}{t} = h(t)$. So $h(t)$ is strictly decreasing. We also have $h(\mathbf{z}_n) = n$ and $h(\mathbf{z}_1) = \mathrm{num}(\mathbf{z}_i = \mathbf{z}_1) + \frac{\|\mathbf{z}\|_1 - \mathrm{num}(\mathbf{z}_i = \mathbf{z}_1)\times \mathbf{z}_1}{\mathbf{z}_1} = \frac{\|\mathbf{z}\|_1}{\mathbf{z}_1} = \frac{\|\mathbf{z}\|_1}{\|\mathbf{z}\|_\infty}$. So if $\|\mathbf{z}\|_\infty \ge \frac{1}{p}\|\mathbf{z}\|_1$, then $p \in [h(\mathbf{z}_1), h(\mathbf{z}_n)]$ and there exists an unique $t^* \in [\mathbf{z}_n, \mathbf{z}_1]$ such that $h(t^*) = p$. If $\|\mathbf{z}\|_\infty < \frac{1}{p}\|\mathbf{z}\|_1$, then $h(t) \ge h(\mathbf{z}_1) > p, \forall t \le \mathbf{z}_1 = \|\mathbf{z}\|_\infty$. So $st + \sum_{\mathbf{z}_i < t}\mathbf{z}_i > pt$ and case (4b) in the above analysis dose not hold.

We first consider the case of $\|\mathbf{z}\|_\infty < \frac{1}{p}\|\mathbf{z}\|_1$:

(1). If $t \ge \frac{1}{p}\|\mathbf{z}\|_1$, then $g(t) = 0$.

(2). If $\|\mathbf{z}\|_\infty \le t < \frac{1}{p}\|\mathbf{z}\|_1$, then $g(t) = p\theta$.

(3). If $\mathbf{z}_p - \mathbf{z}_{p+1} < t < \|\mathbf{z}\|_\infty$, then $g(t) = \sum_{i=1}^{r}\mathbf{z}_i - rt + (p-r)\theta$.

(4). If $t \le \mathbf{z}_p - \mathbf{z}_{p+1}$, then $g(t) = \sum_{i=1}^{p}\mathbf{z}_i - pt$.

We then consider the case of $\|\mathbf{z}\|_\infty \ge \frac{1}{p}\|\mathbf{z}\|_1$.

Let $v = \mathrm{num}(\mathbf{z}_i \ge \|\mathbf{z}\|_1/p)$, then $\mathbf{z}_v \ge \|\mathbf{z}\|_1/p$, $\mathbf{z}_{v+1} < \|\mathbf{z}\|_1/p$ and $h(\|\mathbf{z}\|_1/p) = v + p\sum_{i=v+1}^{n}\mathbf{z}_i/\|\mathbf{z}\|_1 = v + p(\|\mathbf{z}\|_1 - \sum_{i=1}^{v}\mathbf{z}_i)/\|\mathbf{z}\|_1 = v + p - p\sum_{i=1}^{v}\mathbf{z}_i/\|\mathbf{z}\|_1 \le v + p - pv\mathbf{z}_v/\|\mathbf{z}\|_1 \le p$. So $\|\mathbf{z}\|_1/p \ge t^*$.

If $\mathbf{z}_{p+1} > 0$, then $h(\mathbf{z}_p - \mathbf{z}_{p+1}) = \mathrm{num}(\mathbf{z}_i \ge \mathbf{z}_p - \mathbf{z}_{p+1}) + \frac{\sum_{\mathbf{z}_i < \mathbf{z}_p - \mathbf{z}_{p+1}}\mathbf{z}_i}{\mathbf{z}_p - \mathbf{z}_{p+1}}$. If $\mathbf{z}_{p+1} \ge \mathbf{z}_p - \mathbf{z}_{p+1}$, then $h(\mathbf{z}_p - \mathbf{z}_{p+1}) \ge \mathrm{num}(\mathbf{z}_i \ge \mathbf{z}_p - \mathbf{z}_{p+1}) \ge p+1$. If $\mathbf{z}_{p+1} < \mathbf{z}_p - \mathbf{z}_{p+1}$, since $\mathbf{z}_p > \mathbf{z}_p - \mathbf{z}_{p+1}$, then $h(\mathbf{z}_p - \mathbf{z}_{p+1}) = p + \frac{\sum_{\mathbf{z}_i < \mathbf{z}_p - \mathbf{z}_{p+1}}\mathbf{z}_i}{\mathbf{z}_p - \mathbf{z}_{p+1}} \ge p + \frac{\mathbf{z}_{p+1}}{\mathbf{z}_p - \mathbf{z}_{p+1}} > p$. So $h(\mathbf{z}_p - \mathbf{z}_{p+1}) > p$ and

$\mathbf{z}_p - \mathbf{z}_{p+1} < t^*$. If $\mathbf{z}_{p+1} = 0$, then $h(\mathbf{z}_p - \mathbf{z}_{p+1}) = h(\mathbf{z}_p) = \text{num}(\mathbf{z}_i \geq \mathbf{z}_p) + \frac{\sum_{\mathbf{z}_i < \mathbf{z}_p} \mathbf{z}_i}{\mathbf{z}_p} = p$. So $t^* = \mathbf{z}_p = \mathbf{z}_p - \mathbf{z}_{p+1}$.

Thus $\mathbf{z}_p - \mathbf{z}_{p+1} \leq t^* \leq \|\mathbf{z}\|_1/p$ and we have

(1). If $t \geq \|\mathbf{z}\|_\infty$, then $g(t) = 0$.
(2). If $\frac{1}{p}\|\mathbf{z}\|_1 \leq t < \|\mathbf{z}\|_\infty$, then $g(t) = \sum_{i=1}^{s} \mathbf{z}_i - st$.
(3). If $t^* \leq t < \frac{1}{p}\|\mathbf{z}\|_1$, then $g(t) = \sum_{i=1}^{s} \mathbf{z}_i - st$.
(4). If $\mathbf{z}_p - \mathbf{z}_{p+1} < t < t^*$, then $g(t) = \sum_{i=1}^{r} \mathbf{z}_i - rt + (p-r)\theta$.
(5). If $t \leq \mathbf{z}_p - \mathbf{z}_{p+1}$, then $g(t) = \sum_{i=1}^{p} \mathbf{z}_i - pt$. ■

Lemma 10 *Consider $g(t) = \sum_{i=1}^{s} \mathbf{z}_i - st$, $t \in (0, \|\mathbf{z}\|_\infty]$, where $s = num(\mathbf{z}_i \geq t)$. Let $\mathbf{z}_{n+1} = 0$. then $g(t)$ is continuous, strictly decreasing and piecewise linear, $g^-(\lambda)$ can be expressed as*

$$g^-(\lambda) = \frac{\sum_{i=1}^{k} \mathbf{z}_i - \lambda}{k}, \qquad \lambda \in \left[\sum_{i=1}^{k} \mathbf{z}_i - k\mathbf{z}_k, \sum_{i=1}^{k+1} \mathbf{z}_i - (k+1)\mathbf{z}_{k+1}\right), k = 1, \cdots, n.$$

Proof If $t \in (\mathbf{z}_{k+1}, \mathbf{z}_k]$ with fixed k, then $s = k$ and $g(t) = \sum_{i=1}^{k} \mathbf{z}_i - kt$, so $g(t)$ is continuous, piecewise linear and strictly decreasing. So

$$g^-(\lambda) = \frac{\sum_{i=1}^{k} \mathbf{z}_i - \lambda}{k}, \lambda \in \left[\sum_{i=1}^{k} \mathbf{z}_i - k\mathbf{z}_k, \sum_{i=1}^{k+1} \mathbf{z}_i - (k+1)\mathbf{z}_{k+1}\right), k = 1, \cdots, n,$$

and $g^-(\lambda) \in (0, \|\mathbf{z}\|_\infty]$. ■

Lemma 11 *Consider $g(t) = p\theta$, $t \in \left(0, \frac{1}{p}\|\mathbf{z}\|_1\right]$, where θ and t satisfies $\sum_{\mathbf{c}_i > \theta}(\mathbf{c}_i - \theta) = pt$, then $g(t)$ is continuous, piecewise linear and strictly decreasing, let $\mathbf{z}_{n+1} = 0$, then $g^-(\lambda)$ can be expressed as*

$$g^-(\lambda) = \frac{\sum_{i=1}^{k} \mathbf{z}_i}{p} - \frac{k\lambda}{p^2}, \qquad \lambda \in [p\mathbf{z}_{k+1}, p\mathbf{z}_k), k = 1, 2, \cdots, n.$$

Proof Fix $\theta \in [\mathbf{z}_{k+1}, \mathbf{z}_k)$, we have

$$\begin{aligned} t &= \frac{\sum_{i=1}^{k} \mathbf{z}_i - k\theta}{p} \\ &\in \left(\frac{\sum_{i=1}^{k} \mathbf{z}_i - k\mathbf{z}_k}{p}, \frac{\sum_{i=1}^{k} \mathbf{z}_i - k\mathbf{z}_{k+1}}{p}\right] \\ &= \left(\frac{\sum_{i=1}^{k} \mathbf{z}_i - k\mathbf{z}_k}{p}, \frac{\sum_{i=1}^{k+1} \mathbf{z}_i - (k+1)\mathbf{z}_{k+1}}{p}\right] \end{aligned}$$

and $\theta = \frac{\sum_{i=1}^{k} \mathbf{z}_i - pt}{k}$. So

$$g(t) = p\frac{\sum_{i=1}^{k} \mathbf{z}_i - pt}{k}, \quad t \in \left(\frac{\sum_{i=1}^{k} \mathbf{z}_i - k\mathbf{z}_k}{p}, \frac{\sum_{i=1}^{k+1} \mathbf{z}_i - (k+1)\mathbf{z}_{k+1}}{p}\right].$$

$g(t)$ is continuous, linear function and strictly decreasing.

$$g^{-}(\lambda) = \frac{\sum_{i=1}^{k} \mathbf{z}_i}{p} - \frac{k\lambda}{p^2}, \quad \lambda \in [p\mathbf{z}_{k+1}, p\mathbf{z}_k), k = 1, 2, \cdots, n.$$

And we have $g^{-}(\lambda) \in \left(0, \frac{\|\mathbf{z}\|_1}{p}\right]$. ■

Lemma 12 *Define interval*

$$S(r,d) = \left(\max\left\{\frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1} - \mathbf{z}_i)}{d-p}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i - \mathbf{z}_d)}{p-r}\right\},\right.$$
$$\left.\min\left\{\frac{\sum_{i=r+1}^{d}(\mathbf{z}_r - \mathbf{z}_i)}{d-p}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i - \mathbf{z}_{d+1})}{p-r}\right\}\right]$$

with $r < p < d$. Let $r - j + 1 = \min\{i : \mathbf{z}_i = \mathbf{z}_r\}$, $d + k = \max\{i : \mathbf{z}_i = \mathbf{z}_{d+1}\}$, $p - j + 1 = \min\{i : \mathbf{z}_i = \mathbf{z}_p\}$, $p + k = \max\{i : \mathbf{z}_i = \mathbf{z}_{p+1}\}$, $\mathbf{z}_0 = \infty$ and $\mathbf{z}_{n+1} = 0$. Then we can divide $(\mathbf{z}_p - \mathbf{z}_{p+1}, t^*)$ if $\|\mathbf{z}\|_\infty \geq \frac{1}{p}\|\mathbf{z}\|_1$ and $(\mathbf{z}_p - \mathbf{z}_{p+1}, \|\mathbf{z}\|_\infty)$ if $\|\mathbf{z}\|_\infty < \frac{1}{p}\|\mathbf{z}\|_1$ into several disjoint and connected intervals by the following way: Go right from non-empty $S(p-j, p+k)$, if $S(r,d)$ is non-empty, then for $S(r,d)$ with $r \geq 0$, $d \leq n$,

1. *If $\max\left\{\frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d-p}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_d)}{p-r}\right\} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r}$, then the right hand side of $S(r,d)$ is $S(r-j,d)$ and $S(r-j,d)$ is non-empty.*
2. *If $\max\left\{\frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d-p}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_d)}{p-r}\right\} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p}$, then the right hand side of $S(r,d)$ is $S(r,d+k)$ and $S(r,d+k)$ is non-empty.*
3. *If $\max\left\{\frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d-p}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_d)}{p-r}\right\} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r} = \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p}$, then the right hand side of $S(r,d)$ is $S(r-j,d+k)$ and $S(r-j,d+k)$ is non-empty.*

Proof We begin with $S(p-j, p+k)$. Since $\mathbf{z}_{p-j} > \mathbf{z}_{p-j+1} = \cdots = \mathbf{z}_p$, $\mathbf{z}_{p+1} = \cdots = \mathbf{z}_{p+k} > \mathbf{z}_{p+k+1}$, then $S(p-j, p+k) = \left(\mathbf{z}_p - \mathbf{z}_{p+1}, \min\left\{\frac{(k+j)\mathbf{z}_{p-j} - j\mathbf{z}_p - k\mathbf{z}_{p+1}}{k}, \frac{j\mathbf{z}_p + k\mathbf{z}_{p+1} - (k+j)\mathbf{z}_{p+k+1}}{j}\right\}\right]$ is nonempty and on the most left of $(\mathbf{z}_p - \mathbf{z}_{p+1}, t^*)$ and $(\mathbf{z}_p - \mathbf{z}_{p+1}, \|\mathbf{z}\|_\infty)$. So we should go right from $S(p-j, p+k)$. We will prove that for every nonempty $S(r,d)$, we can find a nonempty interval connected with $S(r,d)$ on its right hand side.

Case 1: If $\max\left\{\frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d-p}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_d)}{p-r}\right\} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r}$, then $S(r,d) = \left(\max\left\{\frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d-p}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_d)}{p-r}\right\}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p}\right]$, $\mathbf{z}_r > \mathbf{z}_{r+1}$. From the definition of $r-j+1$, we have $\mathbf{z}_{r-j} > \mathbf{z}_{r-j+1} = \cdots = \mathbf{z}_r > \mathbf{z}_{r+1}$. Since

$$\begin{aligned}
& \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i - \mathbf{z}_d)}{p-r} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r - \mathbf{z}_i)}{d-p} \\
\Leftrightarrow \quad & (p-r)\mathbf{z}_r + (d-p)\mathbf{z}_d > \sum_{i=r+1}^{d} \mathbf{z}_i \\
\Leftrightarrow \quad & (p-r)\mathbf{z}_r + \sum_{i=r-j+1}^{r} \mathbf{z}_i + (d-p)\mathbf{z}_d > \sum_{i=r-j+1}^{d} \mathbf{z}_i \\
\Leftrightarrow \quad & (p-r+j)\mathbf{z}_{r-j+1} + (d-p)\mathbf{z}_d > \sum_{i=r-j+1}^{d} \mathbf{z}_i \\
\Leftrightarrow \quad & \frac{\sum_{i=r-j+1}^{d}(\mathbf{z}_i - \mathbf{z}_d)}{p-r+j} < \frac{\sum_{i=r-j+1}^{d}(\mathbf{z}_{r-j+1} - \mathbf{z}_i)}{d-p}.
\end{aligned}$$

then

$$\begin{aligned}
S(r-j,d) &= \Bigg(\max\left\{\frac{\sum_{i=r-j+1}^{d}(\mathbf{z}_{r-j+1} - \mathbf{z}_i)}{d-p}, \frac{\sum_{i=r-j+1}^{d}(\mathbf{z}_i - \mathbf{z}_d)}{p-r+j}\right\}, \\
&\quad \min\left\{\frac{\sum_{i=r-j+1}^{d}(\mathbf{z}_{r-j} - \mathbf{z}_i)}{d-p}, \frac{\sum_{i=r-j+1}^{d}(\mathbf{z}_i - \mathbf{z}_{d+1})}{p-r+j}\right\}\Bigg] \\
&= \left(\frac{\sum_{i=r-j+1}^{d}(\mathbf{z}_{r-j+1} - \mathbf{z}_i)}{d-p}, \min\left\{\frac{\sum_{i=r-j+1}^{d}(\mathbf{z}_{r-j} - \mathbf{z}_i)}{d-p}, \frac{\sum_{i=r-j+1}^{d}(\mathbf{z}_i - \mathbf{z}_{d+1})}{p-r+j}\right\}\right] \\
&= \left(\frac{\sum_{i=r+1}^{d}(\mathbf{z}_r - \mathbf{z}_i)}{d-p}, \min\left\{\frac{\sum_{i=r-j+1}^{d}(\mathbf{z}_{r-j} - \mathbf{z}_i)}{d-p}, \frac{\sum_{i=r-j+1}^{d}(\mathbf{z}_i - \mathbf{z}_{d+1})}{p-r+j}\right\}\right].
\end{aligned}$$

is on the right hand side of $S(r,d)$. It can be easily checked that $\frac{\sum_{i=r-j+1}^{d}(\mathbf{z}_{r-j+1} - \mathbf{z}_i)}{d-p} < \frac{\sum_{i=r-j+1}^{d}(\mathbf{z}_{r-j} - \mathbf{z}_i)}{d-p}$. Since

$$\begin{aligned}
& \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r - \mathbf{z}_i)}{d-p} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i - \mathbf{z}_{d+1})}{p-r} \\
\Leftrightarrow \quad & (p-r)\mathbf{z}_r + (d-p)\mathbf{z}_{d+1} < \sum_{i=r+1}^{d} \mathbf{z}_i \\
\Leftrightarrow \quad & (p-r)\mathbf{z}_r + \sum_{i=r-j+1}^{r} \mathbf{z}_i + (d-p)\mathbf{z}_{d+1} < \sum_{i=r-j+1}^{d} \mathbf{z}_i \\
\Leftrightarrow \quad & (p+j-r)\mathbf{z}_{r-j+1} + (d-p)\mathbf{z}_{d+1} < \sum_{i=r-j+1}^{d} \mathbf{z}_i \\
\Leftrightarrow \quad & \frac{\sum_{i=r-j+1}^{d}(\mathbf{z}_{r-j+1} - \mathbf{z}_i)}{d-p} < \frac{\sum_{i=r-j+1}^{d}(\mathbf{z}_i - \mathbf{z}_{d+1})}{p-r+j}.
\end{aligned}$$

So $S(r-j,d)$ is not empty.

Case 2: If $\max\left\{\frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d-p},\frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_d)}{p-r}\right\} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p}$, so $S(r,d) = \left(\max\left\{\frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d-p},\frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_d)}{p-r}\right\},\frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r}\right]$ and $\mathbf{z}_{d+1} < \mathbf{z}_d$. From the definition of $d+k$, we have $\mathbf{z}_{d+k+1} < \mathbf{z}_{d+k} = \cdots = \mathbf{z}_{d+1} < \mathbf{z}_d$. Since

$$\begin{aligned}
& \frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d-p} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r} \\
\Leftrightarrow\ & (p-r)\mathbf{z}_{r+1} + (d-p)\mathbf{z}_{d+1} < \sum_{i=r+1}^{d}\mathbf{z}_i \\
\Leftrightarrow\ & (p-r)\mathbf{z}_{r+1} + (d-p)\mathbf{z}_{d+1} + \sum_{i=d+1}^{d+k}\mathbf{z}_i < \sum_{i=r+1}^{d+k}\mathbf{z}_i \\
\Leftrightarrow\ & (p-r)\mathbf{z}_{r+1} + (d+k-p)\mathbf{z}_{d+k} < \sum_{i=r+1}^{d+k}\mathbf{z}_i \\
\Leftrightarrow\ & \frac{\sum_{i=r+1}^{d+k}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d+k-p} < \frac{\sum_{i=r+1}^{d+k}(\mathbf{z}_i-\mathbf{z}_{d+k})}{p-r}.
\end{aligned}$$

then

$$\begin{aligned}
S(r,d+k) &= \left(\max\left\{\frac{\sum_{i=r+1}^{d+k}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d+k-p},\frac{\sum_{i=r+1}^{d+k}(\mathbf{z}_i-\mathbf{z}_{d+k})}{p-r}\right\}\right. \\
&\qquad \left.\min\left\{\frac{\sum_{i=r+1}^{d+k}(\mathbf{z}_r-\mathbf{z}_i)}{d+k-p},\frac{\sum_{i=r+1}^{d+k}(\mathbf{z}_i-\mathbf{z}_{d+k+1})}{p-r}\right\}\right] \\
&= \left(\frac{\sum_{i=r+1}^{d+k}(\mathbf{z}_i-\mathbf{z}_{d+k})}{p-r},\min\left\{\frac{\sum_{i=r+1}^{d+k}(\mathbf{z}_r-\mathbf{z}_i)}{d+k-p},\frac{\sum_{i=r+1}^{d+k}(\mathbf{z}_i-\mathbf{z}_{d+k+1})}{p-r}\right\}\right] \\
&= \left(\frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r},\min\left\{\frac{\sum_{i=r+1}^{d+k}(\mathbf{z}_r-\mathbf{z}_i)}{d+k-p},\frac{\sum_{i=r+1}^{d+k}(\mathbf{z}_i-\mathbf{z}_{d+k+1})}{p-r}\right\}\right].
\end{aligned}$$

is on the right hand side of $S(r,d)$. Since

$$\begin{aligned}
& \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p} > \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r} \\
\Leftrightarrow\ & \sum_{i=r+1}^{d}\mathbf{z}_i < (p-r)\mathbf{z}_r + (d-p)\mathbf{z}_{d+1} \\
\Leftrightarrow\ & \sum_{i=r+1}^{d+k}\mathbf{z}_i < (p-r)\mathbf{z}_r + (d-p)\mathbf{z}_{d+1} + \sum_{i=d+1}^{d+k}\mathbf{z}_i
\end{aligned}$$

$$
\begin{aligned}
\Leftrightarrow\quad & \sum_{i=r+1}^{d+k} \mathbf{z}_i < (p-r)\mathbf{z}_r + (d+k-p)\mathbf{z}_{d+k} \\
\Leftrightarrow\quad & \frac{\sum_{i=r+1}^{d+k}(\mathbf{z}_i - \mathbf{z}_{d+k})}{p-r} < \frac{\sum_{i=r+1}^{d+k}(\mathbf{z}_r - \mathbf{z}_i)}{d+k-p}.
\end{aligned}
$$

So $S(r, d+k)$ is not empty.

Case 3: If $\max\left\{\frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d-p}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_d)}{p-r}\right\} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r} = \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p}$, then $\mathbf{z}_{r+1} < \mathbf{z}_r$ and $\mathbf{z}_{d+1} < \mathbf{z}_d$. Since

$$
\begin{aligned}
& \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i - \mathbf{z}_{d+1})}{p-r} = \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r - \mathbf{z}_i)}{d-p} \\
\Leftrightarrow\quad & (p-r)\mathbf{z}_r + (d-p)\mathbf{z}_{d+1} = \sum_{i=r+1}^{d} \mathbf{z}_i \\
\Leftrightarrow\quad & (p-r)\mathbf{z}_r + \sum_{i=r-j+1}^{r} \mathbf{z}_i + (d-p)\mathbf{z}_{d+1} + \sum_{i=d+1}^{d+k} \mathbf{z}_i = \sum_{i=r-j+1}^{d+k} \mathbf{z}_i \\
\Leftrightarrow\quad & (p-r+j)\mathbf{z}_{r-j+1} + (d+k-p)\mathbf{z}_{d+k} = \sum_{i=r-j+1}^{d+k} \mathbf{z}_i \\
\Leftrightarrow\quad & \frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_i - \mathbf{z}_{d+k})}{p-r+j} = \frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_{r-j+1} - \mathbf{z}_i)}{d+k-p}.
\end{aligned}
$$

and

$$
\sum_{i=r+1}^{d} \mathbf{z}_i - (d-r)\mathbf{z}_{d+1} = (p-r)\mathbf{z}_r + (d-p)\mathbf{z}_{d+1} - (d-r)\mathbf{z}_{d+1} = (p-r)(\mathbf{z}_r - \mathbf{z}_{d+1}),
$$

then we have

$$
\begin{aligned}
& \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i - \mathbf{z}_{d+1})}{p-r} = \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r - \mathbf{z}_i)}{d-p} = \mathbf{z}_r - \mathbf{z}_{d+1}, \\
& \frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_i - \mathbf{z}_{d+k})}{p-r+j} = \frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_{r-j+1} - \mathbf{z}_i)}{d+k-p} = \mathbf{z}_{r-j+1} - \mathbf{z}_{d+k} = \mathbf{z}_r - \mathbf{z}_{d+1}.
\end{aligned}
$$

So

$$
S(r,d) = \left(\max\left\{\frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1} - \mathbf{z}_i)}{d-p}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i - \mathbf{z}_d)}{p-r}\right\}, \mathbf{z}_r - \mathbf{z}_{d+1}\right],
$$

and

$$\begin{aligned} S(r-j,d+k) &= \Bigg(\max\left\{\frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_{r-j+1}-\mathbf{z}_i)}{d+k-p},\frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_i-\mathbf{z}_{d+k})}{p-r+j}\right\}, \\ &\qquad \min\left\{\frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_{r-j}-\mathbf{z}_i)}{d+k-p},\frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_i-\mathbf{z}_{d+k+1})}{p-r+j}\right\}\Bigg] \\ &= \Bigg(\mathbf{z}_r-\mathbf{z}_{d+k},\min\left\{\frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_{r-j}-\mathbf{z}_i)}{d+k-p},\frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_i-\mathbf{z}_{d+k+1})}{p-r+j}\right\}\Bigg]. \end{aligned}$$

is on the right hand side of $S(r,d)$. It can be easily checked that $\frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_{r-j+1}-\mathbf{z}_i)}{d+k-p} < \frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_{r-j}-\mathbf{z}_i)}{d+k-p}$ and $\frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_i-\mathbf{z}_{d+k})}{p-r+j} < \frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_i-\mathbf{z}_{d+k+1})}{p-r+j}$. Since $\mathbf{z}_r-\mathbf{z}_{d+1} = \frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_i-\mathbf{z}_{d+k})}{p-r+j} = \frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_{r-j+1}-\mathbf{z}_i)}{d+k-p}$, thus $S(r-j,d+k)$ is not empty.

Next, we consider two special cases: $r=0$ or $d=n$.

If $r=0$, consider $S(0,d) = \left(\max\left\{\frac{\sum_{i=1}^{d}(\mathbf{z}_1-\mathbf{z}_i)}{d-p},\frac{\sum_{i=1}^{d}(\mathbf{z}_i-\mathbf{z}_d)}{p}\right\},\frac{\sum_{i=1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p}\right]$. From the analysis of case 2, the right hand side of $S(0,d)$ is $S(0,d+k)$ and $S(0,d+k)$ is nonempty. Moreover, $S(0,n) = \left(\max\left\{\frac{\sum_{i=1}^{n}(\mathbf{z}_1-\mathbf{z}_i)}{n-p},\frac{\sum_{i=1}^{n}(\mathbf{z}_i-\mathbf{z}_n)}{p}\right\},\frac{\sum_{i=1}^{n}\mathbf{z}_i}{p}\right]$. If $\|\mathbf{z}\|_1 \le p\|\mathbf{z}\|_\infty$, then $t^* \le \|\mathbf{z}\|_1/p$ and $S(0,n)$ reaches the right hand side of $[\mathbf{z}_p-\mathbf{z}_{p+1},t^*]$. If $\|\mathbf{z}\|_1 > p\|\mathbf{z}\|_\infty$, then $S(0,n)$ reaches the right hand side of $[\mathbf{z}_p-\mathbf{z}_{p+1},\|\mathbf{z}\|_\infty]$.

If $d=n$, consider $S(r,n) = \left(\max\left\{\frac{\sum_{i=r+1}^{n}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{n-p},\frac{\sum_{i=r+1}^{n}(\mathbf{z}_i-\mathbf{z}_n)}{p-r}\right\},\min\left\{\frac{\sum_{i=r+1}^{n}(\mathbf{z}_r-\mathbf{z}_i)}{n-p},\frac{\sum_{i=r+1}^{n}\mathbf{z}_i}{p-r}\right\}\right]$.

If $\|\mathbf{z}\|_1 > p\|\mathbf{z}\|_\infty$, then $\sum_{i=r+1}^{n}\mathbf{z}_i > p\mathbf{z}_1 - \sum_{i=1}^{r}\mathbf{z}_i = (p-r)\mathbf{z}_1 + r\mathbf{z}_1 - \sum_{i=1}^{r}\mathbf{z}_i \ge (p-r)\mathbf{z}_1 \ge (p-r)\mathbf{z}_r$, which is equivalent to $\frac{\sum_{i=r+1}^{n}(\mathbf{z}_r-\mathbf{z}_i)}{n-p} < \frac{\sum_{i=r+1}^{n}\mathbf{z}_i}{p-r}$, so we have $\max\left\{\frac{\sum_{i=r+1}^{n}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{n-p},\frac{\sum_{i=r+1}^{n}(\mathbf{z}_i-\mathbf{z}_n)}{p-r}\right\} < \frac{\sum_{i=r+1}^{n}(\mathbf{z}_r-\mathbf{z}_i)}{n-p} < \frac{\sum_{i=r+1}^{n}\mathbf{z}_i}{p-r}$ and then it reduces case 1, the right hand side of $S(r,n)$ is $S(r-j,n)$ and $S(r-j,n)$ is not empty.

If $\|\mathbf{z}\|_1 \le p\|\mathbf{z}\|_\infty$, if $\max\left\{\frac{\sum_{i=r+1}^{n}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{n-p},\frac{\sum_{i=r+1}^{n}(\mathbf{z}_i-\mathbf{z}_n)}{p-r}\right\} < \frac{\sum_{i=r+1}^{n}(\mathbf{z}_r-\mathbf{z}_i)}{n-p} < \frac{\sum_{i=r+1}^{n}\mathbf{z}_i}{p-r}$, then the right hand side of $S(r,n)$ is $S(r-j,n)$ and $S(r-j,n)$ is nonempty. Otherwise, we claim that $S(r,n)$ reaches the right hand side of $(\mathbf{z}_p-\mathbf{z}_{p+1},t^*)$ by proving $\frac{\sum_{i=r+1}^{n}\mathbf{z}_i}{p-r} = t^*$. Let $t = \frac{\sum_{i=r+1}^{n}\mathbf{z}_i}{p-r}$. Since

$$\begin{aligned} \frac{\sum_{i=r+1}^{n}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{n-p} < \frac{\sum_{i=r+1}^{n}\mathbf{z}_i}{p-r} &\Leftrightarrow (p-r)\mathbf{z}_{r+1} < \sum_{i=r+1}^{n}\mathbf{z}_i, \\ \frac{\sum_{i=r+1}^{n}(\mathbf{z}_r-\mathbf{z}_i)}{n-p} \ge \frac{\sum_{i=r+1}^{n}\mathbf{z}_i}{p-r} &\Leftrightarrow (p-r)\mathbf{z}_r \ge \sum_{i=r+1}^{n}\mathbf{z}_i, \end{aligned}$$

so $\mathbf{z}_r \ge t > \mathbf{z}_{r+1}$. Thus $\mathrm{num}(\mathbf{z}_i \ge t) + \frac{\sum_{\mathbf{z}_i<t}\mathbf{z}_i}{t} = r + \frac{\sum_{i=r+1}^{n}\mathbf{z}_i}{t} = p$, from lemma 7 we know $h(t) = \mathrm{num}(\mathbf{z}_i \ge t) + \frac{\sum_{\mathbf{z}_i<t}\mathbf{z}_i}{t}$ is strictly decreasing and $h(t^*) = p$, so $t = t^*$. ■

Lemma 13 *Consider* $g(t)=\sum_{i=1}^{r}\mathbf{z}_i-rt+(p-r)\theta$ *with* $t\in(\mathbf{z}_p-\mathbf{z}_{p+1},t^*)$ *if* $\|\mathbf{z}\|_\infty\geq\frac{1}{p}\|\mathbf{z}\|_1$ *and* $t\in(\mathbf{z}_p-\mathbf{z}_{p+1},\|\mathbf{z}\|_\infty)$ *if* $\|\mathbf{z}\|_\infty<\frac{1}{p}\|\mathbf{z}\|_1$*, then for each interval* $S(r,d)$ *constructed in Lemma 12, we have*

$$g(t)=\sum_{i=1}^{r}\mathbf{z}_i-rt+(p-r)\frac{\sum_{i=r+1}^{d}\mathbf{z}_i-(p-r)t}{d-r},\quad\forall t\in S(r,d),$$

and

$$g^{-}(\lambda)=\frac{d-r}{dr+p^2-2pr}\left(\sum_{i=1}^{r}\mathbf{z}_i+\frac{p-r}{d-r}\sum_{i=r+1}^{d}\mathbf{z}_i-\lambda\right),\quad\forall\lambda\in g\left(S(r,d)\right),$$

where $g\left(S(r,d)\right)$ *means the function value* $g(t)$ *on the interval* $S(r,d)$*. Moreover,* $g(t)$ *and* $g^{-}(\lambda)$ *is continuous, piecewise linear and strictly decreasing.*

Proof $t\in S(r,d)\neq\emptyset$ is equivalent to

$$\frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p}\geq t>\frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d-p},$$
$$\frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r}\geq t>\frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_d)}{p-r}.$$

which can be further written as

$$\mathbf{z}_r-t\geq\frac{\sum_{i=r+1}^{d}\mathbf{z}_i-(p-r)t}{d-r},\quad\mathbf{z}_{r+1}-t<\frac{\sum_{i=r+1}^{d}\mathbf{z}_i-(p-r)t}{d-r},$$
$$\mathbf{z}_{d+1}\leq\frac{\sum_{i=r+1}^{d}\mathbf{z}_i-(p-r)t}{d-r},\quad\mathbf{z}_d>\frac{\sum_{i=r+1}^{d}\mathbf{z}_i-(p-r)t}{d-r}.$$

On the other hand, fix t,r,d, consider θ satisfying

$$\mathbf{z}_r-t\geq\theta,\quad\mathbf{z}_{r+1}-t<\theta,\quad\mathbf{z}_{d+1}\leq\theta,\quad\mathbf{z}_d>\theta,\quad r<p<d,\tag{19}$$

then

$$h(\theta)\;=\;\mathrm{num}(\mathbf{z}_i-\theta\geq t)\times t+\sum_{0<\mathbf{z}_i-\theta<t}(\mathbf{z}_i-\theta)=rt+\sum_{i=r+1}^{d}(\mathbf{z}_i-\theta)$$

is strictly decreasing. So $\theta=\frac{\sum_{i=r+1}^{d}\mathbf{z}_i-(p-r)t}{d-r}$ is the unique solution for $h(\theta)=pt$ satisfying (19). Thus we have

$$g(t)=\sum_{i=1}^{r}\mathbf{z}_i-rt+(p-r)\frac{\sum_{i=r+1}^{d}\mathbf{z}_i-(p-r)t}{d-r},\quad\forall t\in S(r,d),$$

and $g(t)$ is a linear strictly decreasing function in $S(r,d)$.

Now we prove that $g(t)$ is continuous.

Case 1:

If $\max\left\{\frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d-p}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_d)}{p-r}\right\} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r}$, then $S(r,d) = \left(\max\left\{\frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d-p}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_d)}{p-r}\right\}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p}\right]$, and
$S(r-j,d) = \left(\frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p}, \min\left\{\frac{\sum_{i=r-j+1}^{d}(\mathbf{z}_{r-j}-\mathbf{z}_i)}{d-p}, \frac{\sum_{i=r-j+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r+j}\right\}\right]$ is on the right hand side of $S(r,d)$. Consider the interval $S(r,d)$, we have $g\left(\frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p}\right) = \sum_{i=1}^{r}\mathbf{z}_i + \frac{p-r}{d-r}\sum_{i=r+1}^{d}\mathbf{z}_i - \left(r + \frac{(p-r)^2}{d-r}\right)\frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p}$. Consider $S(r-j,d)$, we have

$$
\begin{aligned}
& \lim_{t \to \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p}} g(t) \\
= & \sum_{i=1}^{r-j}\mathbf{z}_i + \frac{p-r+j}{d-r+j}\sum_{i=r-j+1}^{d}\mathbf{z}_i - \left(r-j+\frac{(p-r+j)^2}{d-r+j}\right)\frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p} \\
= & \sum_{i=1}^{r}\mathbf{z}_i - j\mathbf{z}_r + \frac{p-r+j}{d-r+j}\left(\sum_{i=r+1}^{d}\mathbf{z}_i + j\mathbf{z}_r\right) - \left(r-j+\frac{(p-r+j)^2}{d-r+j}\right)\frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p} \\
= & g\left(\frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p}\right) - j\mathbf{z}_r + \frac{p-r+j}{d-r+j}j\mathbf{z}_r + \sum_{i=r+1}^{d}\mathbf{z}_i\left(\frac{p-r+j}{d-r+j} - \frac{p-r}{d-r}\right) \\
& - \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p}\left(\frac{(p-r+j)^2}{d-r+j} - j - \frac{(p-r)^2}{d-r}\right) \\
= & g\left(\frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p}\right) - \frac{d-p}{d-r+j}j\mathbf{z}_r + \frac{j(d-p)\sum_{i=r+1}^{d}\mathbf{z}_i}{(d-r+j)(d-r)} + \frac{j(d-p)\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{(d-r+j)(d-r)} \\
= & g\left(\frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p}\right).
\end{aligned}
$$

Case 2: If $\max\left\{\frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d-p}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_d)}{p-r}\right\} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p}$, then $S(r,d) = \left(\max\left\{\frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d-p}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_d)}{p-r}\right\}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r}\right]$ and
$S(r,d+k) = \left(\frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r}, \min\left\{\frac{\sum_{i=r+1}^{d+k}(\mathbf{z}_r-\mathbf{z}_i)}{d+k-p}, \frac{\sum_{i=r+1}^{d+k}(\mathbf{z}_i-\mathbf{z}_{d+k+1})}{p-r}\right\}\right]$ is on the right hand side of $S(r,d)$. Consider the interval $S(r,d)$, we have $g\left(\frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r}\right) = \sum_{i=1}^{r}\mathbf{z}_i +$

$\frac{p-r}{d-r}\sum_{i=r+1}^{d}\mathbf{z}_i - \left(r+\frac{(p-r)^2}{d-r}\right)\frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r}$. Consider $S(r,d+k)$, we have

$$\begin{aligned}
&\lim_{t\to\frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r}} g(t)\\
=\;& \sum_{i=1}^{r}\mathbf{z}_i + \frac{p-r}{d+k-r}\sum_{i=r+1}^{d+k}\mathbf{z}_i - \left(r+\frac{(p-r)^2}{d+k-r}\right)\frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r}\\
=\;& \sum_{i=1}^{r}\mathbf{z}_i + \frac{p-r}{d+k-r}\left(\sum_{i=r+1}^{d}\mathbf{z}_i + k\mathbf{z}_{d+1}\right) - \left(r+\frac{(p-r)^2}{d+k-r}\right)\frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r}\\
=\;& g\left(\frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r}\right) + \frac{(p-r)k\mathbf{z}_{d+1}}{d+k-r} + \sum_{i=r+1}^{d}\mathbf{z}_i\left(\frac{p-r}{d+k-r}-\frac{p-r}{d-r}\right)\\
&-\frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r}\left(\frac{(p-r)^2}{d+k-r}-\frac{(p-r)^2}{d-r}\right)\\
=\;& g\left(\frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r}\right) + \frac{(p-r)k\mathbf{z}_{d+1}}{d+k-r} - \frac{k(p-r)\sum_{i=r+1}^{d}\mathbf{z}_i}{(d+k-r)(d-r)} + \frac{k(p-r)\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{(d+k-r)(d-r)}\\
=\;& g\left(\frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r}\right).
\end{aligned}$$

Case 3: If $\max\left\{\frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d-p}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_d)}{p-r}\right\} < \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r} = \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p}$, then $\frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_{d+1})}{p-r} = \frac{\sum_{i=r+1}^{d}(\mathbf{z}_r-\mathbf{z}_i)}{d-p} = \mathbf{z}_r - \mathbf{z}_{d+1}$,
$S(r,d) = \left(\max\left\{\frac{\sum_{i=r+1}^{d}(\mathbf{z}_{r+1}-\mathbf{z}_i)}{d-p}, \frac{\sum_{i=r+1}^{d}(\mathbf{z}_i-\mathbf{z}_d)}{p-r}\right\}, \mathbf{z}_r-\mathbf{z}_{d+1}\right]$,
$S(r-j,d+k) = \left(\mathbf{z}_r-\mathbf{z}_{d+1}, \min\left\{\frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_{r-j}-\mathbf{z}_i)}{d+k-p}, \frac{\sum_{i=r-j+1}^{d+k}(\mathbf{z}_i-\mathbf{z}_{d+k+1})}{p-r+j}\right\}\right]$ is on the right hand side of $S(r,d)$. Consider the interval $S(r,d)$, we have

$$\begin{aligned}
g(\mathbf{z}_r-\mathbf{z}_{d+1}) &= \sum_{i=1}^{r}\mathbf{z}_i + \frac{p-r}{d-r}\sum_{i=r+1}^{d}\mathbf{z}_i - \left(r+\frac{(p-r)^2}{d-r}\right)(\mathbf{z}_r-\mathbf{z}_{d+1})\\
&= \sum_{i=1}^{r}\mathbf{z}_i + \frac{p-r}{d-r}((p-r)\mathbf{z}_r + (d-p)\mathbf{z}_{d+1}) - \left(r+\frac{(p-r)^2}{d-r}\right)(\mathbf{z}_r-\mathbf{z}_{d+1})\\
&= \sum_{i=1}^{r}\mathbf{z}_i - r\mathbf{z}_r + p\mathbf{z}_{d+1},
\end{aligned}$$

and consider $S(f-j,d+k)$ we have

$$\begin{aligned}
& \lim_{t\to \mathbf{z}_r-\mathbf{z}_{d+1}} g(t) \\
= & \sum_{i=1}^{r-j}\mathbf{z}_i + \frac{p-r+j}{d+k-r+j}\sum_{i=r-j+1}^{d+k}\mathbf{z}_i - \left(r-j+\frac{(p-r+j)^2}{d+k-r+j}\right)(\mathbf{z}_r-\mathbf{z}_{d+1}) \\
= & \sum_{i=1}^{r}\mathbf{z}_i - j\mathbf{z}_r + \frac{p-r+j}{d+k-r+j}\left(\sum_{i=r+1}^{d}\mathbf{z}_i + j\mathbf{z}_r + k\mathbf{z}_{d+1}\right) - \left(r-j+\frac{(p-r+j)^2}{d+k-r+j}\right)(\mathbf{z}_r-\mathbf{z}_{d+1}) \\
= & \sum_{i=1}^{r}\mathbf{z}_i - j\mathbf{z}_r + \frac{p-r+j}{d+k-r+j}\left((p-r)\mathbf{z}_r + (d-p)\mathbf{z}_{d+1} + j\mathbf{z}_r + k\mathbf{z}_{d+1}\right) \\
& -\left(r-j+\frac{(p-r+j)^2}{d+k-r+j}\right)(\mathbf{z}_r-\mathbf{z}_{d+1}) \\
= & \sum_{i=1}^{r}\mathbf{z}_i + \mathbf{z}_r\left(-j+\frac{(p-r+j)^2}{d+k-r+j} - \left(r-j+\frac{(p-r+j)^2}{d+k-r+j}\right)\right) \\
& +\mathbf{z}_{d+1}\left(\frac{(p-r+j)(d-p+k)}{d+k-r+j} + \left(r-j+\frac{(p-r+j)^2}{d+k-r+j}\right)\right) \\
= & \sum_{i=1}^{r}\mathbf{z}_i - r\mathbf{z}_r + p\mathbf{z}_{d+1} \\
= & g\left(\mathbf{z}_r-\mathbf{z}_{d+1}\right).
\end{aligned}$$

So $g(t)$ is continuous, piecewise linear and strictly decreasing in $(\mathbf{z}_p-\mathbf{z}_{p+1},t^*)$ if $\|\mathbf{z}\|_\infty \geq \frac{1}{p}\|\mathbf{z}\|_1$ and in $(\mathbf{z}_p-\mathbf{z}_{p+1},\|\mathbf{z}\|_\infty)$ if $\|\mathbf{z}\|_\infty < \frac{1}{p}\|\mathbf{z}\|_1$.

We can easily get that

$$g^-(\lambda) = \frac{d-r}{dr+p^2-2pr}\left(\sum_{i=1}^{r}\mathbf{z}_i + \frac{p-r}{d-r}\sum_{i=r+1}^{d}\mathbf{z}_i - \lambda\right), \quad \forall\lambda\in g\left(S(r,d)\right).$$

and $g^-(\lambda)$ is continuous, piecewise linear and strictly decreasing. ∎

Theorem 14 $g(t)=\sum_{i=1}^{p}(\mathbf{z}_i-\mathbf{x}_i)$ *with* $t\in[0,\max\{\|\mathbf{z}\|_\infty,\|\mathbf{z}\|_1/p\}]$ *and its inverse function* $g^-(\lambda)$ *with* $\lambda\in[0,\sum_{i=1}^{p}\mathbf{z}_i]$ *are continuous, strictly decreasing and piecewise linear.*

Proof Based on Lemma 10, 11 and 13, we only need to prove that $g(t)$ is continuous at point $t=\|\mathbf{z}\|_\infty$, $\|\mathbf{z}\|_1/p$, t^* and $\mathbf{z}_p-\mathbf{z}_{p+1}$. We first consider $\|\mathbf{z}\|_\infty \geq \|\mathbf{z}\|_1/p$.

When $t \xrightarrow{+} \|\mathbf{z}\|_\infty = \mathbf{z}_1$, then $s\equiv \mathrm{num}(\mathbf{z}_i\geq t)\to k^*\equiv\max\{i,\mathbf{z}_i=\mathbf{z}_1\}$ and $\lim_{t\to\mathbf{z}_1}h(t)=\sum_{i=1}^{k^*}\mathbf{z}_i-k^*\mathbf{z}_1=0$.

When $t\xrightarrow{+}t^*$, then $st+\sum_{\mathbf{z}_i<t}\mathbf{z}_i\xrightarrow{-}pt$. We claim that $\theta\to 0$. Otherwise, $pt=\sum_{i=1}^{n}\mathbf{x}_i\leq st+\sum_{i=s+1}^{n}\mathbf{z}_i = st+\sum_{\mathbf{z}_i<t}\mathbf{z}_i$, where we use $\mathbf{x}_i\leq t$ and $\mathbf{x}_i\leq\mathbf{z}_i$. So $\mathbf{x}_i\to t,\forall i\leq s$ and $\mathbf{x}_i\to\mathbf{z}_i,\forall i>s$. On the other hand, From case 4 and case 2 of Theorem 4, we have $\mathbf{x}_j=t$ if $\mathbf{z}_j-\theta\geq t$; $\mathbf{x}_j=\mathbf{z}_j-\theta$ if $0<\mathbf{z}_j-\theta<t$; $\mathbf{x}_j=0$ if $\mathbf{z}_j\leq\theta$. Thus $\theta\to 0$. So $r\equiv\mathrm{num}(\mathbf{z}_i-\theta\geq t)\to s$ and $\lim_{t\to t^*}h(t)=\sum_{i=1}^{s}\mathbf{z}_i-st$.

When $t \xrightarrow{-} \mathbf{z}_p - \mathbf{z}_{p+1}$, from case 4.3 and 4.1, we know $pt = \sum_{i=1}^n \mathbf{x}_i$. Thus there are two cases: $\mathbf{x}_p = t$, $\mathbf{x}_{p+1} = 0$; $\mathbf{x}_p < t$, $0 < \mathbf{x}_{p+1} < t$. For the first case, we have $\mathbf{z}_p - \theta \geq t$ and $\mathbf{z}_{p+1} \leq \theta$, thus $\mathbf{z}_p - \mathbf{z}_{p+1} \geq t$. Thus we have $\mathbf{z}_p - \theta \to t$ and $\mathbf{z}_{p+1} \to \theta$. For the second case, we have $\mathbf{z}_p - \theta < t$ and $0 < \mathbf{z}_{p+1} - \theta$, thus $\mathbf{z}_p - \mathbf{z}_{p+1} < t$. So we also have $\mathbf{z}_p - \theta \to t$ and $\mathbf{z}_{p+1} \to \theta$. So $r \to p$ and $\lim_{t \xrightarrow{-} \mathbf{z}_p - \mathbf{z}_{p+1}} g(t) = \sum_{i=1}^p \mathbf{z}_i - pt$.

Then we consider $\|\mathbf{z}\|_\infty < \|\mathbf{z}\|_1/p$. When $t \xrightarrow{+} \|c\|_1/p$, from $\sum_{\mathbf{z}_i > \theta}(\mathbf{z}_i - \theta) = pt \to \|\mathbf{z}\|_1$ we have $\theta \to 0$ and $\lim_{t \xrightarrow{+} \|\mathbf{z}\|_1/p} g(t) = 0$. When $t \xrightarrow{+} \mathbf{z}_1$, then from the analysis in Lemma 9 we have $st + \sum_{\mathbf{z}_i < t} \mathbf{z}_i > pt$. We claim $\theta > 0$. Otherwise, if $\theta \to 0$, then from case 4.3 and case 3 in Theorem 4, we have $\mathbf{x}_i \to t$ if $\mathbf{z}_i \geq t$ and $\mathbf{x}_i \to \mathbf{z}_i$ if $\mathbf{z}_i < t$. Then $\sum_{i=1}^n \mathbf{x}_i \to st + \sum_{\mathbf{z}_i < t} \mathbf{z}_i > pt$, which contradicts with $\sum_{i=1}^n \mathbf{x}_i \leq pt$. So $\theta > 0$. Then $r \equiv \mathrm{num}(\mathbf{z}_i - \theta \geq t) \to 0$ when $t \xrightarrow{+} \mathbf{z}_1$. So $\lim_{t \xrightarrow{+} \mathbf{z}_1} g(t) = p\theta$.

■

5. Numerical Experiments

In this section, We verify the convergence of the proposed methods: the Augmented Lagrangian Multiplier method with direct Babel Function minimization (ALM-BF) and the Alternating Projection method (APM, Algorithm 2). We take Φ to be a $d \times n$ random Gaussian matrix and test on three settings with varying sizes of Φ: (1) $d = 400$, $n = 500$; (2) $d = 800$, $n = 1000$; (3) $d = 1200$, $n = 1500$. We fix $m = 50$ and $p = 20$ in model (7). Thus the redundancy of the effective dictionary $\mathbf{D}$, n/m, varies on the three settings. In ALM-BF we set $\gamma = 1.2$, $\varpi = 0.9$, $\underline{\Lambda} = 10^{-20}$, $\overline{\Lambda} = 10^{20}$ and $\tau = 10^{-5}$. We run the inner loop of ALM-BF for 10 iterations and 100 iterations respectively and note the method as ALM-BF-5 and ALM-BF-100. We set the threshold t as the Welch bound $\sqrt{\frac{n-m}{m(n-1)}}$ in Algorithm 2. Figure 1 plot the curves of the mutual coherence $\max_{1 \leq i,j \leq n} \frac{|\langle \mathbf{d}_i, \mathbf{d}_j \rangle|}{\|\mathbf{d}_i\|_2 \|\mathbf{d}_j\|_2}$, Babel function $\max_{\mathbf{\Lambda}, |\mathbf{\Lambda}| = p} \max_{j \notin \mathbf{\Lambda}} \sum_{i \in \mathbf{\Lambda}} \frac{|\langle \mathbf{d}_i, \mathbf{d}_j \rangle|}{\|\mathbf{d}_i\|_2 \|\mathbf{d}_j\|_2}$, constraint violations $\|\mathbf{X} - \mathbf{Y}\|_F^2$ and $\|\mathbf{Y} - \mathbf{V}\mathbf{W}\mathbf{V}^T + \mathbf{I}\|_F^2$ vs. iteration respectively for ALM-BF-10, ALM-BF-100 and APM. We run Algorithm 2 for 50 (100; 200) iterations as the initialization procedure for ALM-BF on the setting of $d = 400$, $n = 500$ ($d = 800$, $n = 1000$; $d = 1200$, $n = 1500$). We can see that both ALM-BF and APM converge well. Since ALM-BF minimizes the Babel function directly while APM only uses an approximated threshold, ALM-BF produces a solution with much lower mutual coherence and Babel function. ALM-BF-5 performs a little worse than ALM-BF-100. In applications with large size matrix $\mathbf{D}$, too many inner iterations are not affordable and we can still obtain a good solution with only a few inner iterations. We should mention that the initialization is critical for ALM-BF. Otherwise, it may get stuck at a bad saddle point or local minimum, especially when d and n are large.

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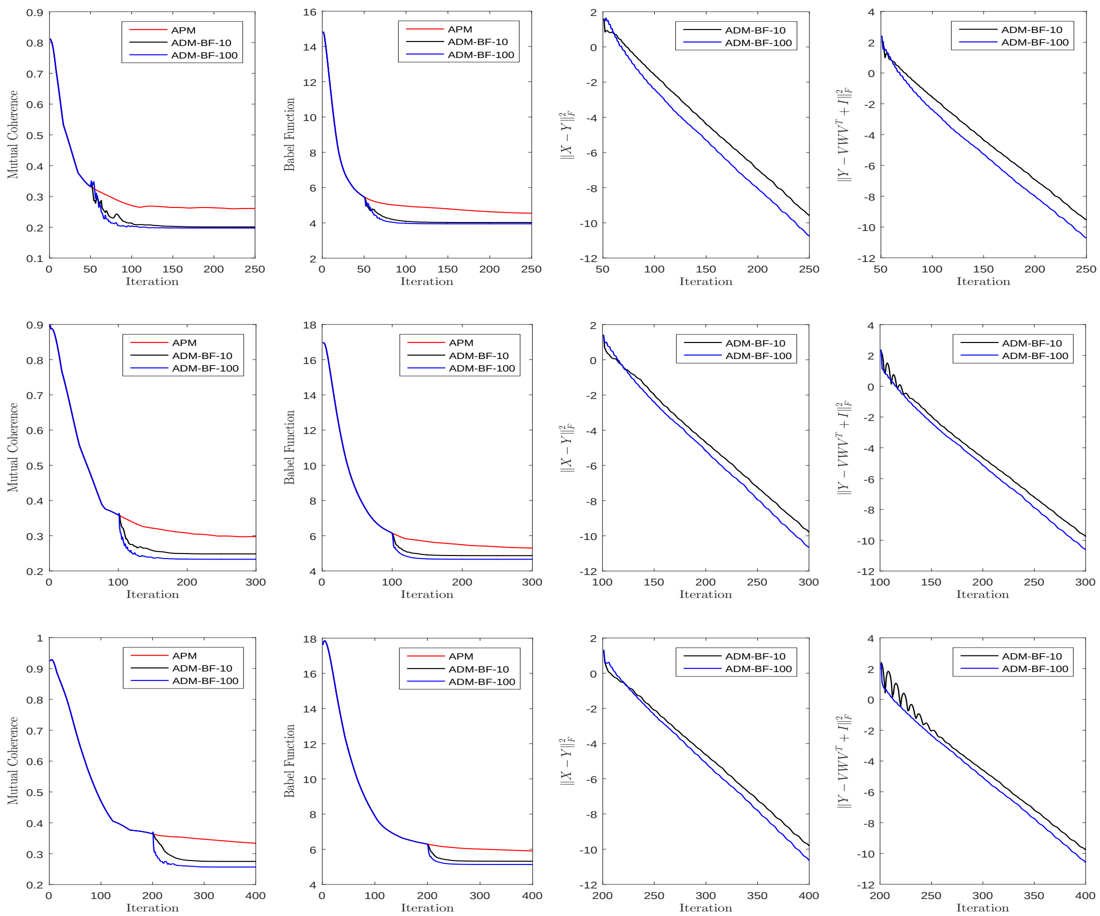

Figure 1: The mutual coherence and Babel function of ADM-BF and APM. The constraint violations of ADM-BF. Top: $d = 400$, $n = 500$. Middle: $d = 800$, $n = 1000$. Bottom: $d = 1200$, $n = 1500$

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